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Articles

Simple pendulum in a rotating reference frame

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Describing the harmonic motion of a simple pendulum observed from a rotating reference frame necessitates solving the differential equations governing three-dimensional motion in a non-inertial reference frame, which can be a challenging endeavor for newcomers to this subject. However, in this study, we propose a highly didactic and visually accessible approach by considering the rotating reference frame as a matrix transformation. This approach enables a deeper understanding of the origin of Coriolis and centrifugal accelerations, which are responsible for the curved trajectories exhibited by the simple pendulum in a rotating reference frame.
Keywords: Harmonic motion, simple pendulum, non-inertial reference frame, foucault pendulum, bravais pendulum.

1. Introduction

Simple pendulums have been of great historical importance for humanity [1] by allowing time intervals to be measured with reasonable precision, being the fundamental piece to count a second of lapse in ancient mechanical clocks [2]. In the middle of the 19th century, a pair of French scientists, Léon Foucault and Augustine Bravais, independently developed experiments related to pendulums in Paris to prove the Earth's rotation motion. On the one hand, Foucault observed how the plane of oscillation of the pendulum in its back-and-forth movement changed subtly as the hours passed [3]. On the other hand, Bravais analyzed the motion of a conical pendulum in clockwise or counterclockwise circular paths, and observed a difference in the period of oscillation between these two situations [4]. In summary, both experiments reveal the existence of the terrestrial rotational motion, for this reason Newtonian mechanics considers our planet as a non-inertial frame of reference, or rotating reference frame.

Thus, when someone starts this study for the first time, one is faced with the need to seek the solution of the differential equations associated with the dynamics of the three-dimensional motion of the simple pendulum in a rotating reference frame. According to the initial conditions of motion, one could have the situation of the pendulum of Foucault [5–8] or the pendulum of Bravais [9, 10].

For that reason, in this work, we present an analytical and didactic approach that avoids the need for intricate solutions to describe the pendulum motion in the rotating frame. Instead, we focus on analyzing the motion of the simple pendulum within an inertial reference frame, utilizing the small angle approximation 2. Subsequently, we introduce the concept of the rotating frame 3 to describe the pendulum's motion relative to this frame of reference. Finally, we demonstrate how to correlate the motion observed in the rotating frame with the motion of the pendulum in the inertial frame 4. To enhance comprehension, we provide visual examples and concise videos that are easily accessible and shareable.

2. Simple Pendulum

In this section, the motion of a simple pendulum is analyzed under the small angle approximation, considering it to be a point-shaped object, with mass m, and suspended by a rigid rod of length L of negligible mass. To describe its motion, a three-dimensional inertial reference frame expressed in the Cartesian coordinates xyz will be assumed, whose origin will be located on the mass when it is in its equilibrium position, and the axis z represents the vertical direction. Thus, in the xy plane, its oscillatory back and forth motion will be registered.

If initially we consider the pendulum deviated from its equilibrium orientation along the z axis, so that it only exhibits a horizontal displacement along the x axis, related to the angle θ as we can observe in Figure 1, due to there is not imbalance along the y axis, its position in relation to time can be described by instantly knowing the angle θ, then its coordinates can be expressed as $(x, y, z) = \big(L\sin\theta, 0, L(1 - \cos\theta)\big)$. Thus, when we analyze the dynamics of the pendulum without frictional forces or air drag, we have that the force responsible to return it to its equilibrium position corresponds to one of the components of the pendulum's weight perpendicular to the direction of the rod, $F = -mg\sin\theta$, where the negative sign is related to the decrease of the angular deviation, allowing the pendulum to return always to the

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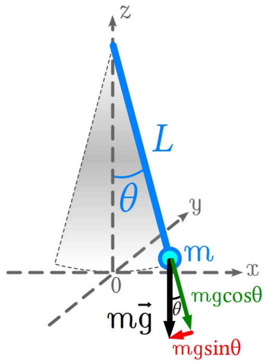

Figure 1: Decomposition of the weight force according to the orientation of the simple pendulum that moves in the plane of oscillation according to the inertial reference.

equilibrium position, for this reason this class of force is known as restoring force.

Based on that, Newton's second law is re-expressed after a few algebraic steps as,

$$-mg\sin\theta = mL\frac{d^2\theta}{dt^2}, \quad \Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta. \tag{1}$$

This is a second order nonlinear differential equation, it independents of the mass of the suspended object, corresponding to the angular motion of the pendulum, whose analytical solution is quite difficult to obtain [11]. However, if initially it is considered that the horizontal deviation is small in relation to the length of the pendulum itself, related to small angular deviation, smaller than a radian $\theta \ll 1$ rad. Then, you can apply the trigonometric approximation for small angles [12], letting us to re-write the sine function of eq. (1) to its own argument, $\sin\theta \approx \theta$. Thus, the second time derivative of the angular orientation function $\theta(t)$ is related to itself through the negative of the constant $\omega_0^2 = g/L$, which depends inversely proportional by the pendulum's length. Therefore, the differential equation related to angular motion for small deviations is expressed as,

$$\frac{d^2\theta}{dt^2} = -\omega_0^2\theta, \tag{2}$$

whose solution $\theta(t) = A\cos(\omega_0 t + \phi)$ represents a simple harmonic motion with angular frequency ω_0, and period $T = 2\pi\sqrt{L/g}$. Where A is the angular amplitude of the oscillation expressed in radians, and a phase factor ϕ related to the initial conditions of the motion. Since the previous pair of terms will depend on the initial conditions, below we will briefly present three possible cases of the initial motion of the simple pendulum.

Case 1: If initially the pendulum starts from an inclination denoted by the angle $\theta_0 \neq 0$, and with zero speed $v_0 = 0$, which implies the phase factor $\phi = 0$, then the angular orientation function and its time derivative denoted by a dot, are expressed as,

$$\begin{aligned} \theta(t) &= \theta_0 \cos\omega_0 t, \\ \dot{\theta}(t) &= -\theta_0\omega_0 \sin\omega_0 t. \end{aligned} \tag{3}$$

Where $\dot{\theta}$ is the function that represents the angular velocity of the periodic motion of the simple pendulum.

Case 2: If initially the pendulum starts its motion from the equilibrium position $\theta_0 = 0$, with an initial speed $v_0 \neq 0$. The maximum angular velocity is related to the initial tangential velocity of the pendulum $v_0 = L\dot{\theta}_{\max}$, as a consequence the phase factor must be $\phi = \pi/2$. So, after a brief simplification we obtain that the angular deviation and its angular velocity as,

$$\begin{aligned} \theta(t) &= \frac{(v_0/L)}{\omega_0}\sin\omega_0 t, \\ \dot{\theta}(t) &= (v_0/L)\cos\omega_0 t. \end{aligned} \tag{4}$$

Case 3: When initially the angle of the pendulum is $\theta_0 \neq 0$ and the initial speed $v_0 \neq 0$, then the amplitude of angular motion and the phase factor will be,

$$A = \frac{\theta_0}{\cos\phi}, \quad \phi = \arctan\left(\frac{-(v_0/L)}{\theta_0\omega_0}\right). \tag{5}$$

Regardless of the case in question for the initial conditions of the pendulum, its coordinates as a function of time are:

$$\begin{aligned} x &= L\sin\theta, \\ y &= 0, \\ z &= L\big(1-\cos\theta\big). \end{aligned} \tag{6}$$

The velocity vector can be obtained by taking the time derivative of every coordinates,

$$\begin{aligned} \dot{x} &= L\dot{\theta}\cos\theta, \\ \dot{y} &= 0, \\ \dot{z} &= L\dot{\theta}\sin\theta. \end{aligned} \tag{7}$$

The acceleration vector can be obtained by taking the second time derivative of the position coordinates, which is denoted by a double dot in its notation,

$$\begin{aligned} \ddot{x} &= L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta, \\ \ddot{y} &= 0, \\ \ddot{z} &= L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta. \end{aligned} \tag{8}$$

Thus, we explored the dynamics of a simple pendulum in an inertial reference frame, revealing the intricate interplay between weight and angular motion. In the next section, we delve into rotating reference frames to understand pendulum motion, particularly in the context of Foucault and Bravais experiments. Through this transition, we unify theoretical concepts with experimental findings, elucidating the intricate effects of rotation on pendulum dynamics.

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3. Rotating Reference Frame

A rotating reference frame is defined as a reference system that presents only a rotational motion with a constant angular velocity Ω, in relation to some previously established constant direction. Based on this definition, a three-dimensional Cartesian rotating reference frame could be algebraically described by assuming the particular case in which z is the axis of rotation, and it is coaxial with an inertial reference frame at the instant $t = 0$. Therefore, the description of the events that happen in the plane xy for any particular subsequent instant t, will be related to the counterclockwise deviation, $\alpha = \Omega t$, of the rotating reference frame in relation to the inertial frame. Based on this definition, we will consider any fixed point P whose coordinates in the inertial reference frame plane xy are (x_P, y_P), as shown in Figure 2.

In contrast, in the rotating reference frame the point is observed at coordinates (X_P, Y_P), whose relationship with the coordinates of the inertial reference frame can be found when we consider an independent information on rotations, the distance from the coordinate origin to the object P, defined as $r = \sqrt{x_P^2 + y_P^2}$. In addition, the orientation of the object relative to the x-axis of the inertial frame is defined using the angle $\varphi = \arctan\big(y_P/x_P\big)$. These last two pieces of information, r and φ, allow us to define the curvilinear polar coordinates, enabling us to re-express the Cartesian coordinates of the object P as $x_P = r\cos\varphi$ and $y_P = r\sin\varphi$. In this way, if we consider the orientation difference of the rotating reference frame α in relation to the inertial one, then we can express the coordinates of the object P in the rotating reference frame as,

$$
\begin{aligned}
X_P &= r\cos(\varphi - \alpha) = x_P\cos\alpha + y_P\sin\alpha,\\
Y_P &= r\sin(\varphi - \alpha) = y_P\cos\alpha - x_P\sin\alpha.
\end{aligned} \tag{9}
$$

The previous pair of coordinates can be condensed using matrix notation,

$$
\begin{pmatrix} X_P \\ Y_P \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_P \\ y_P \end{pmatrix}. \tag{10}
$$

Thus, we can define a three-dimensional transformation matrix that allows us to express the coordinates of

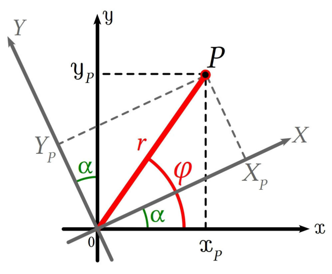

Figure 2: The point P in relation to the inertial frame (x_P, y_P), and according to the rotating frame (X_P, Y_P), where α is the orientation difference between both references.

any object in the rotating frame with respect to its coordinates in the inertial frame when the rotation occurs along the z-axis,

$$
\hat{R}_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11}
$$

For instance, if the rotating frame with a constant angular velocity Ω observes a simple pendulum whose coordinates in the inertial frame are given by eq. (6), its position will be expressed as:

$$
\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L\sin\theta \\ 0 \\ L\big(1-\cos\theta\big) \end{pmatrix}, \tag{12}
$$

thus, its coordinates are,

$$
\begin{aligned}
X &= L\sin\theta\cos\Omega t,\\
Y &= -L\sin\theta\sin\Omega t,\\
Z &= L\big(1-\cos\theta\big).
\end{aligned} \tag{13}
$$

Then, the pendulum's velocity vector components in this frame exhibit an compound motion,

$$
\begin{aligned}
\dot{X} &= L\dot{\theta}\cos\theta\cos\Omega t - L\Omega\sin\theta\sin\Omega t,\\
\dot{Y} &= -L\dot{\theta}\cos\theta\sin\Omega t - L\Omega\sin\theta\cos\Omega t,\\
\dot{Z} &= \dot{z}.
\end{aligned} \tag{14}
$$

Additionally, the components of its acceleration vector are,

$$
\begin{aligned}
\ddot{X} &= L\ddot{\theta}\cos\theta\cos\Omega t - L\dot{\theta}^2\sin\theta\cos\Omega t\\
&\quad - 2L\dot{\theta}\Omega\cos\theta\sin\Omega t - L\Omega^2\sin\theta\cos\Omega t,\\
\ddot{Y} &= -L\ddot{\theta}\cos\theta\sin\Omega t + L\dot{\theta}^2\sin\theta\sin\Omega t\\
&\quad - 2L\dot{\theta}\Omega\cos\theta\cos\Omega t + L\Omega^2\sin\theta\sin\Omega t,\\
\ddot{Z} &= \ddot{z}.
\end{aligned} \tag{15}
$$

Consequently, this description reveals a more complex swing pendulum motion from a non-inertial reference frame. Therefore, to have a good understanding of every acceleration term obtained, we should change the description by using Polar coordinates and expressing the unitary vectors of this coordinate system as functions of the Cartesian coordinates, as follows,

$$
\begin{aligned}
\hat{r} &= \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y},\\
\hat{\varphi} &= -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}.
\end{aligned} \tag{16}
$$

In Polar Coordinates, as shown in Figure 3, the vector position of an object is given by,

$$
\vec{r} = r\,\hat{r}, \tag{17}
$$

where r is the distance from the origin coordinate system to the object, and $\hat{r}$ represent its radial orientation. If both information changes in function of time, when

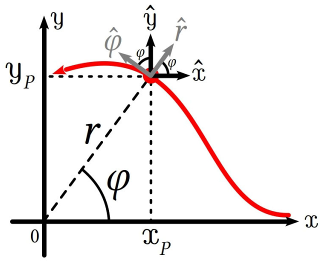

Figure 3: The solid curve represent the trajectory of an object in the inertial frame. Over its position at any instant of time, we see represent the Cartesian unitary vectors and the Polar unitary vector.

we calculate the time derivative of position vector taking into account the product derivative rule and chain rule, we have,

$$\dot{\vec{r}} = \dot{r}\,\hat{r} + r\frac{d\hat{r}}{d\varphi}\frac{d\varphi}{dt}, \tag{18}$$

Looking at the eq. (16), we can get that $d\hat{r}/d\varphi = \hat{\varphi}$,

$$\dot{\vec{r}} = \dot{r}\,\hat{r} + r\dot{\varphi}\,\hat{\varphi}. \tag{19}$$

When the motion of an object is described in polar coordinates, the velocity vector has two components, $\dot{r}$ is the radial velocity and $r\dot{\varphi}$ is the tangential velocity [13].

The second time derivative of position vector represented the acceleration vector of the object. Additionally, alone the derivation process and taking into account the product rule and chain rule, we will get that $d\hat{\varphi}/d\varphi = -\hat{r}$. After some steps we get,

$$\ddot{\vec{r}} = \left(\ddot{r} - r\dot{\varphi}^2\right)\hat{r} + \left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right)\hat{\varphi}. \tag{20}$$

Where the first term alone the radian orientation represents the radial acceleration $\ddot{r}$, and the second one $r\dot{\varphi}^2$ represents the centripetal acceleration. The first term alone angular orientation $2\dot{r}\dot{\varphi}$ is equivalent to Coriolis acceleration, and the second one $r\ddot{\varphi}$ is the tangential acceleration [14]. This representation of the acceleration vector could be re-expressed in a general way for a non-inertial frame by the vectorial notation [15],

$$\vec{a} = \ddot{\vec{r}} + \dot{\vec{\varphi}} \times \left(\dot{\vec{\varphi}} \times \vec{r}\right) + 2\left(\dot{\vec{\varphi}} \times \dot{\vec{r}}\right) + \left(\ddot{\vec{\varphi}} \times \vec{r}\right). \tag{21}$$

For the particular case in which the rotating frame have a constant angular velocity Ω, the acceleration vector in eq. (20) is expressed by,

$$\ddot{\vec{r}} = \left(\ddot{r} - r\Omega^2\right)\hat{r} + 2\dot{r}\Omega\,\hat{\varphi}. \tag{22}$$

Alternatively, we can re-express the acceleration vector in the Polar coordinates (22) to the Cartesian coordinates by using the unitary vectors (16) and applying the kinematics of the pendulum in which there is a relation one to one between r and x position of the pendulum and its derivatives. Then we have,

$$\begin{aligned}\ddot{\vec{r}} = &\left(L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta + L\Omega^2\sin\theta\right)\\ &\times\left(\cos\Omega t\,\hat{x} + \sin\Omega t\,\hat{y}\right)\\ &+ 2L\dot{\theta}^2\Omega\cos\theta\left(-\sin\Omega t\,\hat{x} + \cos\Omega t\,\hat{y}\right),\end{aligned} \tag{23}$$

The acceleration alone each coordinate is,

$$\begin{aligned}\ddot{X} = &\,L\ddot{\theta}\cos\theta\cos\Omega t - L\dot{\theta}^2\sin\theta\cos\Omega t\\ &- 2L\dot{\theta}\Omega\cos\theta\sin\Omega t - L\Omega^2\sin\theta\cos\Omega t,\end{aligned} \tag{24}$$

and,

$$\begin{aligned}\ddot{Y} = &\,L\ddot{\theta}\cos\theta\sin\Omega t - L\dot{\theta}^2\sin\theta\sin\Omega t\\ &+ 2L\dot{\theta}\Omega\cos\theta\cos\Omega t - L\Omega^2\sin\theta\sin\Omega t.\end{aligned} \tag{25}$$

Is worth to be clarified that the rotating frame spins at clockwise direction, $-\Omega$, this end up to visualize the pendulum at the opposite Y direction. Thus, throughout this description using Polar coordinates, we can understand the reason why the acceleration vector component appears complicated when applying a matrix transformation eq. (15). Despite this complexity, this reasoning allows us to avoid solving the differential equation of motion obtained from Newton's second law for a simple pendulum in a rotating frame.

In the next section, we will analyze the motion perception of the simple pendulum according to the angular velocity of the rotating frame as the Foucault and Bravais pendulum.

4. Pendulum in the Rotating Frame

In this section, we scrutinize the motion of the pendulum from a rotating reference frame, which maintains a consistent angular velocity. Unlike Earth's rotational angular velocity, Ω_E, which is constrained by geographical latitude λ as $\Omega = \Omega_E \sin\lambda$, our examination does not adhere to this constraint. Instead, we explore a rotating frame with a predefined angular velocity, which serves as a multiple of the angular frequency of the simple pendulum oscillation. Denoting this constant angular velocity as $\Omega = k\omega_0$, where $0 < k \leq 1$ is a real constant, aligns with the experimental setup observed by Foucault and Bravais. This approach enables us to delve into the interplay between the pendulum's motion and the predetermined rotational dynamics, shedding light on the fundamental dynamics of rotating frames and pendulum systems.

Prior to this, we established the X and Y coordinates of any object within the rotating frame relative to the inertial frame (10). Now, we delve into the specifics of the pendulum's motion as observed from this rotating reference frame. By using the relationship between the angular velocity of the rotating frame and the simple pendulum's angular frequency, we can gain valuable insights into the system's behavior. Note that the choice of Ω allows us to explore a broad range of scenarios beyond a direct association with Earth's rotational parameters, enabling us to study the pendulum's dynamics under various conditions and external influences.

For instance, in the particular case $k = 1$, the rotating frame will return to the initial orientation after a complete pendulum oscillation. Then, if we assume that, in the inertial frame when the pendulum starts its motion satisfying the initial conditions (*case 1*), it will be restricted to the plane defined by the xz axes. However, when we observe it in the rotating reference frame, contrary to our intuition, the pendulum will follow a distinct path along the positive X-axis, with a shape that closely resembles a circle, especially under the small angle approximation. Analyzing its speed, we find a constant value, which is indicative of circular motion,

$$\begin{aligned} \dot{X}^2 + \dot{Y}^2 &= L^2\big[\dot{\theta}^2\cos^2\theta + \Omega^2\sin^2\theta\big], \\ &= L^2\omega_0^2\big[\theta_0^2\sin^2\omega t\cos^2\theta + \sin^2\theta\big], \\ &\approx L^2\omega_0^2\big[\theta_0^2\sin^2\omega_0 t + \theta_0^2\cos^2\omega_0 t\big] = L^2\omega_0^2\theta_0^2. \end{aligned} \tag{26}$$

In this particular case, the pendulum displays a distinct behavior, exhibiting an oscillation with an angular frequency precisely twice as high as its characteristic frequency, denoted by ω_0. This intriguing phenomenon arises due to the synchronization between the rotating reference frame and the pendulum's motion. As a consequence of this synchronized motion, the pendulum undergoes a unique circular trajectory along the positive region of the X-axis, as illustrated in Figure 4. This fascinating behavior highlights the profound influence of the tuned interaction between the rotating frame and the pendulum system, leading to the observed doubling of its angular frequency during oscillation. For a better visualization of the pendulum motion observed by the rotating frame in the previous condition, we recommend to watch the videos by clicking here [16] and here [17].

To gain insights into the pendulum's motion relative to the rotating reference frame, we will explore cases where this frame exhibits an angular frequency smaller than the oscillation frequency of the pendulum, with values in the range $0 < k < 1$. For instance, when the angular frequency of rotation is exactly half the angular frequency of the pendulum, $k = 1/2$, it takes two complete oscillations of the pendulum for the rotating reference frame to return to its initial orientation. As a result, the observed trajectory takes on the unique shape of a *flower* with four distinct petals, a visually intriguing pattern indicative of the dynamic interplay between the rotating frame and the pendulum system.

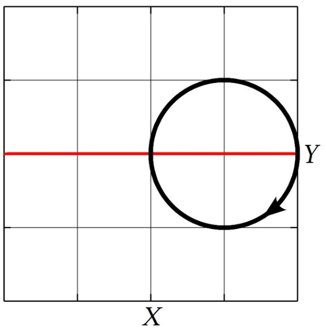

Figure 4: The thick curve represents the trajectory of the pendulum according to the rotating reference frame, while the thin line shows the plane oscillatory motion of the pendulum in the inertial frame of reference.

Similarly, for fractional values of k, such as one-third, one-fourth, one-fifth, and so on, the pendulum traces trajectories that form *flower*-like patterns with a distinct number of petals, as depicted in Figure 5. For a better visualization of the *flower*-like trajectories of the pendulum in the rotating reference frame, we recommend watching the video by clicking here [18].

Notably, trajectories corresponding to these fractional values of k have finite lengths. However, when k is expressed as an irrational fractional number, such as $1/\sqrt{2}$, it does not represent a natural number partition of unity as in the previous examples, leading to trajectories with infinite lengths, see Figure 6.

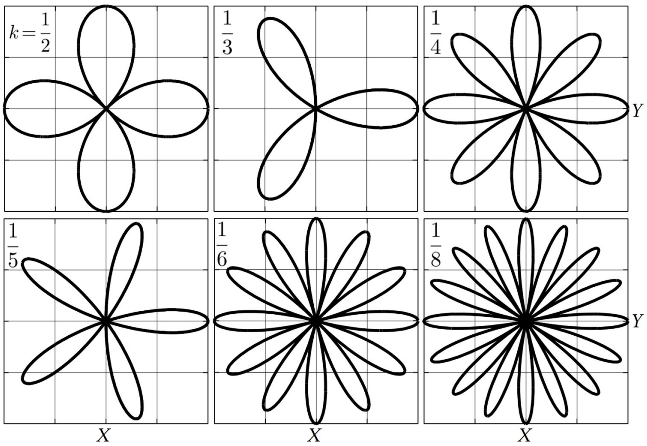

Figure 5: Trajectories in the shape of *flower* performed by the simple pendulum recorded in the rotating reference frame when its angular velocity of rotation is a fraction of the characteristic angular frequency of the pendulum.

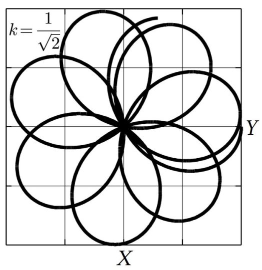

Figure 6: When $k = 1/\sqrt{2}$, the rotating frame observes a segment of the open pendulum's trajectory.

In contrast to the previous analysis, from now on, we will consider the pendulum attached to the X-axis of the rotating reference frame, so both rotate together. When the position of the pendulum coincides with the orientation of the positive x-axis of the inertial reference frame, it is released at the instant t_0. According to the inertial reference, the pendulum that initially exhibited uniform circular motion with a translational velocity $v_t = \Omega L \sin\theta_0$, it presents a motion that decomposed in relation to xy-axis shows two oscillatory motions with equal angular frequency $\omega_0 = \sqrt{g/L}$, but with different amplitudes. In other words, along the x-axis the motion will be that of a pendulum released from rest from an initial angular opening θ_0, satisfying the initial condition (*case 1*). However, along the y-axis, the pendulum will start its motion from the zero angle with velocity v_t, reaching a maximum amplitude $\theta_{\max} = k\sin\theta_0$, satisfying the initial condition (*case 2*). Thus, the angular functions associated with each axis are,

$$\begin{aligned} \theta_x &= \theta_0 \cos\omega_0 t, \\ \theta_y &= (k\sin\theta_0)\sin\omega_0 t. \end{aligned} \tag{27}$$

In this manner, the displacement of the pendulum from its equilibrium position on the plane can be described by considering its motion in both horizontal directions, given by $L\sqrt{\sin^2\theta_x + \sin^2\theta_y}$. By recognizing that the vertical distance from the pendulum's support to its position is $L\cos\theta_z$, we can use the Pythagorean theorem to relate this horizontal displacement to the angular orientation of the pendulum with respect to the vertical z-axis. Thus, the cosine of the pendulum's angular orientation can be expressed as $\cos\theta_z = \sqrt{1-\sin^2\theta_x - \sin^2\theta_y}$. Consequently, the coordinates of the pendulum in the inertial frame are given by,

$$\begin{aligned} x &= L\sin\theta_x, \\ y &= L\sin\theta_y, \\ z &= L\big(1-\cos\theta_z\big). \end{aligned} \tag{28}$$

In this case, the pendulum's motion will no longer be restricted to a vertical plane, and it will follow a curved trajectory whose projection on the xy-plane is similar to the shape of an ellipse. But this is not exactly a plane ellipse because the pendulum motion is tree-dimensional. In other words, this happens over a spherical surface of radius equal to the pendulum's length. Thus, we have this inequality,

$$\frac{x^2}{[L\sin\theta_0]^2} + \frac{y^2}{[L\sin(k\sin\theta_0)]^2} \neq 1. \tag{29}$$

In which one of its 'semi-axes' will depend on the initial tangential velocity, while the other 'semi-axes' will depend on the initial position along the X-axis. Consequently, the rotating reference frame will register the pendulum motion in another way, where the pendulum coordinates in this reference frame are easily obtained through the action of the rotation matrix on the vector of the pendulum coordinates in the inertial reference frame,

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos(k\omega_0 t) & \sin(k\omega_0 t) & 0 \\ -\sin(k\omega_0 t) & \cos(k\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L\sin\theta_x \\ L\sin\theta_y \\ L(1-\cos\theta_z) \end{pmatrix}, \tag{30}$$

thus, its coordinates are,

$$\begin{aligned} X &= L\sin\theta_x\cos(k\omega_0 t) + L\sin\theta_y\sin(k\omega_0 t), \\ Y &= -L\sin\theta_x\sin(k\omega_0 t) + L\sin\theta_y\cos(k\omega_0 t), \\ Z &= L\big(1-\cos\theta_z\big). \end{aligned} \tag{31}$$

As in the previous situation, we will assume the rotation frequency of the rotating frame as being a fraction of the oscillation frequency of the simple pendulum, $k = 1/2, 1/3, 1/4, \ldots$; So, the trajectory that the pendulum performs in the rotating reference frame exhibits a shape similar to a *star*. Because at the instants when the pendulum reaches the maximum angular deviation, its tangential velocity allows it to follow the motion of the rotating frame, and for this reason it is seen momentarily at rest. Furthermore, it is worth mentioning that the circular spacing observed in the central region of the *star*-shaped trajectory is related to the 'elliptical' motion of the pendulum in the inertial reference frame, whose radius is related to $L\sin(k\sin\theta_0)$, as shown in Figure 7. For a better understanding of the *star*-shaped trajectories of the pendulum in the rotating reference frame, we recommend to watch the video by clicking here [19].

In contrast, this is not solely a theoretical explication of pendulum motion in a rotating frame. Empirical evidence can verify some of the preceding findings through experimental observation of a simple pendulum's movement recorded in a rotating reference frame. These observations reveal distinctive trajectories resembling

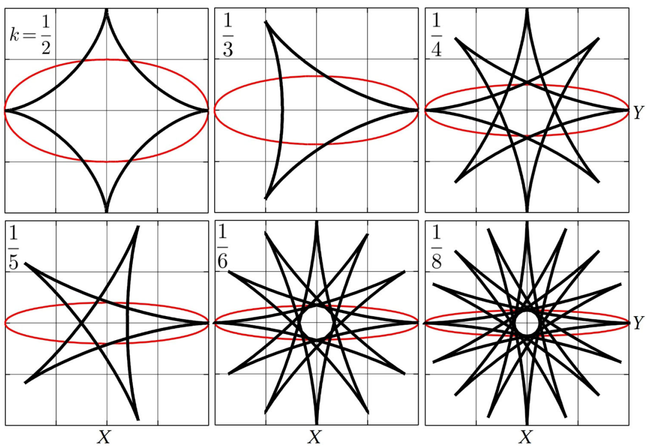

Figure 7: The thick curves represent the *star*-shaped trajectories performed by the simple pendulum released in the rotating reference frame, while the thin curves show the 'elliptical' motion of the pendulum in the inertial reference frame.

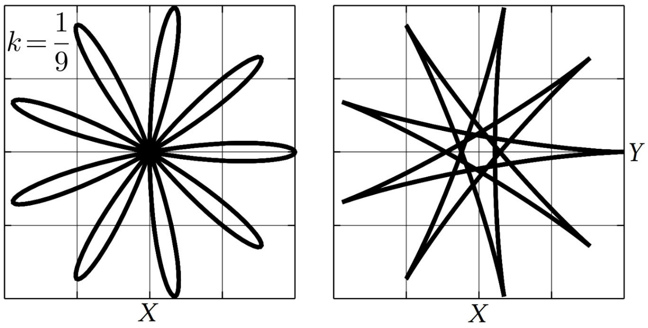

Figure 8: Theoretical pendulum's trajectories when $k = 1/9$, as we can see in the experiment shown in [21].

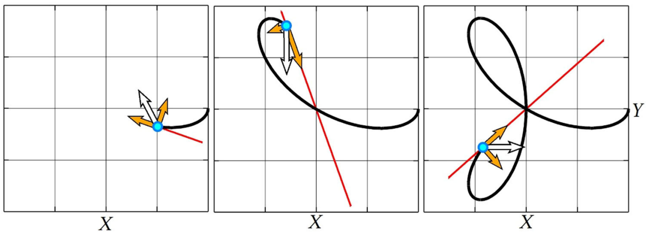

Figure 9: Horizontal vector acceleration and its projections of the simple pendulum observed from the rotating frame when $k = 1/3$ (*black curve*), and the back-and-forth motion in the inertial frame (*red line*).

a *flower* or *star* shape, contingent upon the initial motion conditions [20], as well as [21] in which we can depict these trajectories using the rotating frame with an angular velocity nine times lower than the pendulum's angular frequency, as illustrated in Figure 8.

Finally, considering the discussion in the previous section 3 regarding two-dimensional kinematics based on polar coordinates, we derived the expression for the acceleration experienced by an object observed from a rotating reference frame (20). In the particular cases when we observing the oscillatory motion of a simple pendulum from a rotating reference frame with an angular velocity equal to one-third of the pendulum's angular frequency, as shown in Figure 9, it becomes evident that the pendulum experiences a fictitious acceleration perpendicular to the direction of its oscillation. This component is directly associated with the Coriolis acceleration, a consequence of the rotating reference frame, resulting in relative deviations in the pendulum's trajectory. This type of description could help us understand the origin of the fictitious forces experienced by an object observed from a non-inertial reference frame. For a better visualization of the pendulum's acceleration observed by the rotating frame, we recommend to watch the videos by clicking here [22].

5. Considerations

Through the exposition of the motion of the simple pendulum in a rotating reference frame, modeled by a matrix transformation of rotation in this study, we were able to replicate scenarios similar to the Foucault pendulum and the Bravais pendulum. It is essential to highlight that the primary aim of this description is to facilitate the comprehension of the relative deviation in pendulum motion within a rotating reference frame for individuals approaching this subject for the first time, particularly those interested in the renowned experiments conducted by Léon Foucault and Augustine Bravais.

Consequently, there is no necessity to involve oneself in the intricate task of solving equations of motion for non-inertial reference frames or grappling with the appearance of fictitious forces on the object. Rather, the emphasis lies in understanding the origin of the fictitious acceleration contribution, which governs the trajectory observed by the pendulum within the rotating frame.

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