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Artigos Gerais

Pac-Man Josephson junctions: useful trigonometric puzzles?

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Rather interesting trigonometric equations arise when considering a Josephson junction obtained by embedding a Pac-Man shaped superconducting island in between two superconducting electrodes. In the present work we unfold these equations, written in terms of the superconducting phase difference between the two electrodes, and find the current-phase relation and the maximum superconducting current of the Josephson junction network. The solution of the trigonometric equations defining the superconducting current state of the system can be proposed to advanced high-school students or to undergraduate students in an interdisciplinary lecture.
Keywords: Josephson junction, Quantum mechanics, Trigonometry.

1. Introduction

A Josephson junction is a macroscopic quantum system: its properties can be detected by a classical measuring instrument, even though its superconducting current states possess an intrinsically non-classical nature. B. D. Josephson was the first to describe the properties of superconducting junctions [1] by deriving the dynamical equations that are now named after him. For the "*theoretical predictions of the properties of a supercurrent through a tunnel barrier, in particular those phenomena which are generally known as the Josephson effects*", the scientist received the Nobel prize in 1973.

In more details, a Josephson junction (JJ) is a system made of two weakly coupled superconductors, S_1 and S_2. By defining the macroscopic wave function of the first and second electrode as ψ_1 and ψ_2, respectively, we may write:

$$\psi_1 = |\psi_1| \, e^{i\phi_1} \tag{1a}$$

$$\psi_2 = |\psi_2| \, e^{i\phi_2}, \tag{1b}$$

where ϕ_1 and ϕ_2 are the superconducting phases of the first and the second electrode, respectively, and where the amplitude $|\psi_k|$ $(k = 1, 2)$ is equal to $\sqrt{N_k}$, where N_k is the density of Cooper pairs in the k-th electrode [2]. A more detailed discussion on the properties of superconductors and on the form of Eq. (1a-b) will be given in Section 2.

One could consider a tunnel junction, in which S_1 and S_2 are separated by a very thin insulating barrier, as shown in fig. 1. The superconducting current I flowing in the JJ can be written, according to Josephson equations [2], as follows:

$$I = I_{J0} \sin\phi \tag{2}$$

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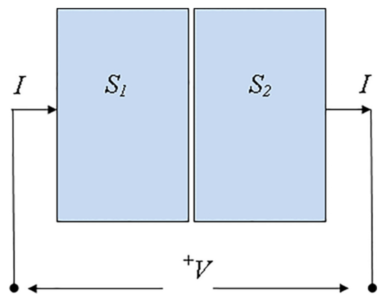

Figure 1: Schematic representation of a tunnel Josephson junction consisting of two superconductors, S_1 and S_2, separated by a very thin insulator. A voltage V can be present across the junction and the current I flows in the two electrodes, which are both connected to the external "classical" world.

where I_{J0} is the maximum value the superconducting current can attain in the JJ, and where $\phi = \phi_2 - \phi_1$ is the superconducting phase difference. The same relation can be written for the supercurrent flowing through a small-area contact point between two superconducting electrodes (point-contact JJ) [3]. Equation (2) is also referred to as current-phase relation (CPR) of the Josephson junction.

Furthermore, in case an explicit time-dependence of the superconducting phase is present, a voltage V is detectable between the two electrodes, given by [3]:

$$V = \frac{\Phi_0}{2\pi}\frac{d\phi}{dt}, \tag{3}$$

where $\Phi_0 = h/2e$ is the elementary flux quantum, expressed as the ratio of Planck's constant h and of the absolute value of the Cooper-pair charge *2e*.

The main physical properties of the JJ can be captured by means of a rather immediate analytical approach was first proposed by Feynman [4] and successively refined by H. Ohta [5]. Ohta's model has been proven useful in deriving the Josephson equations for a tri-layer system (double-barrier Josephson junction), in which the sandwiched superconductor is considered to be a pure quantum system [6].

The physics of Josephson junctions can be presented at high-school and undergraduate level by means of simple demonstrations or by mechanical analogs [7-9] as it will discussed in Section 3. In this way, the beauty of quantum mechanics can be appreciated by direct measurements or by referring to the familiar properties of pendula. Rarely, however, it has been possible to present simple networks of Josephson junctions showing analytic properties that can be proposed in high-school physics courses.

In the present work, a brief account of the basic properties of superconductors is given in Section 2. A short discussion on the analogies between simple pendula and overdamped Josephson junctions is given in section in Section 3. The constitutive equations for the Pac-Man Josephson junction (PMJJ), obtained by embedding a Pac-Man shaped superconducting island (see fig. 2) in between two superconducting electrodes, are written in Section 4. Starting from the derived equations, in Section 5 we see that they give rise to rather interesting expressions for the CPR, which can be obtained by solving basic trigonometric identities. To the best of the author's knowledge, PMJJ have never been studied in the past. In Section 6, a generalization of the findings of the preceding section to multi-Pac-Man systems is made. Conclusions are dawn in Section 7.

2. Superconductivity in Brief

In 1911 Kamerlingh Onnes, in his laboratory in Leiden, first noticed that the resistivity of mercury (Hg) vanished completely below $4.2\,K$. Some other metals and compounds were also observed to make the same transition from a "normal" state to a "superconducting" state below a critical temperature T_c which depended on the particular substance considered [10]. Years later, Meissner and Ochsenfeld [11] noticed that superconductors are perfect diamagnets; i.e., the magnetic induction B in these materials is exactly zero for temperatures $T < T_c$. In this way, the magnetization M can be written as $M = -H$, which means that the magnetic susceptibility is $\chi = -1$. The Bardeen, Copper and Schrieffer (BCS) theory of superconductivity [12], published in 1957, finally established that condensation of electron pairs (Cooper pairs) in a coherent macroscopic state would explain most of the experimental properties of superconductors. A Cooper pair consists of two electrons with opposite spin and opposite momenta coupled via an effective electron-electron interaction mediated by lattice vibrations. Because of the bosonic character of the zero-spin Cooper pairs, all these may condensate in a macroscopic state whose wave-function is characterized by a complex number whose phase plays an important role in determining the superconducting properties, as we shall see, referring to Josephson junction.

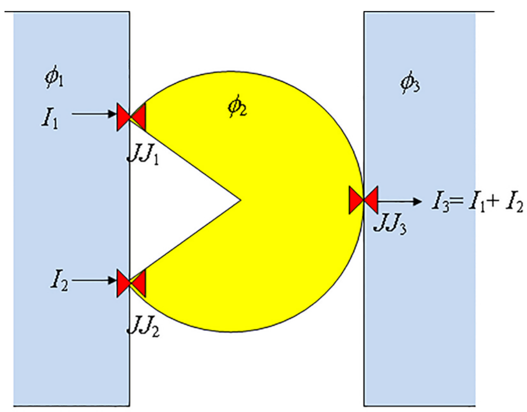

Figure 2: A tri-layer Josephson junction obtained by embedding a Pac-Man shaped superconducting island in between two superconducting electrodes.The three small Josephson junctions are indicated as reddish butterflies.

Early attempts to grasp the essential properties of the superconducting state were made by F. London and H. London in 1935 [13] and by Ginzburg and Landau in 1950 [14]. The former authors relied essentially on classical physics and on a two-fluid model, one normal one superconducting, in order to describe the behaviour of a superconductor in the presence of an electromagnetic field. The Ginzburg-Landau theory of superconductivity, on the other hand, took into account quantum effects. In order to do this, the authors considered an order parameter $\Psi(\vec{r})$, a complex function depending on the position $\vec{r}$, for describing the two "phases", the normal (N) phase and the more ordered superconducting (S) phase. They also hypothesized that the systems would suffer a second-order phase transition at $T = T_c$. According to a previously developed theory by Landau [15], in a second-order phase transition the order parameter changes continuously, while its underlying symmetry does not, so that the system goes abruptly from one less ordered state to a more ordered at $T = T_c$ as the temperature decreases from $T > T_c$ to $T < T_c$. An example of a second-order phase transition is given by the ferromagnetic transition at the Curie point. However, how would one reconcile this feature with the quantum mechanics requirement, by which the system of superconducting electrons has to possess an effective wave-function? The most simple way to do this is to consider the same order parameter $\Psi(\vec{r})$ as the effective wave-function of superconducting electrons. By having hypothesized this function to be complex, we can express $\Psi(\vec{r})$ in its trigonometric form, so that we may write:

$$\Psi(\vec{r}) = |\Psi(\vec{r})|\, e^{i\phi}, \tag{4}$$

which is just what we wrote for the two superconductors in Eq. (1a-b), where we had specified that the supercon-

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ducting phase ϕ and the modulus $|\Psi(\vec{r})|$ of the order parameter can attain different values in two adjacent superconductors. In Eq. (1a-b) we have also specified that, for a homogenous superconductor, we have:

$$|\Psi(\vec{r})| = \sqrt{\frac{N_s}{2}} \quad , \tag{5}$$

where N_s is the density of Cooper pairs. Therefore, since $\Psi(\vec{r})$ must vary continuously with the temperature, N_s is non-null for $T < T_c$ and zero for $T \geq T_c$. Furthermore, we may notice that the intuition of Ginzburg and Landau, i.e., that an effective single wave-function for all electrons can be adopted for the superconducting state, is consistent with the observation made at a later time by Bardeen, Cooper, and Schrieffer in the BSC theory, in which the zero-spin Cooper pairs condensate in just one macroscopic quantum state. We may finally state that, for the purposes of the present work, it is not necessary to dwell further into the theory of superconductivity.

3. Pendula and Josephson Junctions

Let us now consider the simplest circuital model for a Josephson junction and let us see how mechanical analogs [7-9] can be justified by means of the Resistively Shunted Junction (RSJ) model [16]. Let us then may imagine to bias our JJ by means of a current I. When this current is lower than the maximum Josephson current I_{J0} of the junction, a Zero-Voltage State (ZVS) is realized: current may flow in the JJ without energy loss, so that a zero voltage is measured at the junction electrodes. This situation can be described by considering a circuital analog in which an ideal element is present in one branch. In this element a Josephson current I_J, described by means of Eq. (2), flows. In this way, for $I < I_{J0}$, we have:

$$I = I_J = I_{J0} \sin\phi \tag{6}$$

In Eq. (6) we can see that the current I cannot exceed I_{J0}, having the sine function an upper bound equal to $+1$. Notice also that, by Eq. (6), a constant superconducting phase across the JJ, $\phi = \sin^{-1}\left(\frac{I}{I_{J0}}\right)$, can be measured. A constant phase gives, according to Eq. (3), $V = 0$ across the junction electrodes, as hypothesized from the beginning.

However, we could as well choose to inject a current higher that I_{J0} in the junction. What would happen in that case? The ideal element would not be able to absorb all the injected current, as we may see from Eq. (6), being now $I > I_{J0}$. Therefore, at least a second branch must be present in the circuital analog. In addition, this branch should mimic a Resistive State (RS), i.e. should provide a voltage $V \neq 0$ across the junction electrodes for $I > I_{J0}$. The simplest way to realize this condition consists in having a resistive branch in parallel with the ideal Josephson element, as in fig. 3. Therefore, for $I > I_{J0}$, a residual current flows in the resistive branch,

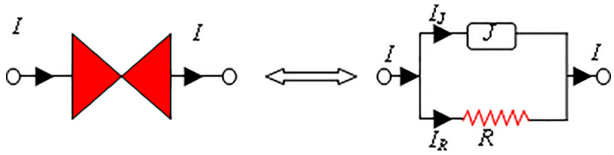

Figure 3: Resistively Shunted Junction model: a Josephson junction is seen as a circuit containing a resistive branch in parallel with an ideal Josephson element.

a voltage $V \neq 0$ appears across the junction electrodes, and a time dependent superconducting phase must be present across the JJ.

By now applying Kirchhoff current law to the circuit analog, we may write:

$$I = I_J + I_R = I_{J0} \sin\phi + \frac{V}{R} \quad , \tag{7}$$

where we have used the second identity in Eq. (6). By now making use of Eq. (3), we may write Eq. (7) in the following form:

$$\frac{\Phi_0}{2\pi R}\frac{d\phi}{dt} + I_{J0}\sin\phi = I. \tag{8}$$

The above equation is similar to the equation of an overdamped pendulum [17-18] in which ϕ represents the angular displacement. In order to make a rather simple correspondence between mechanical and electromagnetic quantities, let us consider the overdamped pendulum of mass m and length l represented in Fig. 4, in which we take the angular displacement to be equal to θ. Let the pendulum be subject to an external torque M_0. By neglecting the inertial term of the system, assuming that the damping and the gravitational effects are predominant, and by not considering buoyancy for the sake of simplicity, we may write the torque equation as follows:

$$F_R l + mgl\sin\theta = M_0, \tag{9}$$

where $F_R = \beta l \frac{d\theta}{dt}$ is the viscous force, β being related to the coefficient of viscosity η of the liquid through Stoke's

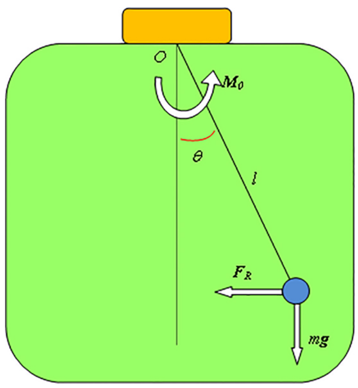

Figure 4: A pendulum in a viscous fluid. Buoyancy effects are neglected, while the weight mg and the viscous force F_R are shown. A torque M_0 is applied to the system.

law [19], so that $\beta = 6\pi\eta r$, with r the radius of the sphere of mass m. By substituting the expression of $F_R = \beta l \frac{d\theta}{dt}$ in Eq. (9), we can finally write:

$$\beta l^2 \frac{d\theta}{dt} + mgl \sin\theta = M_0, \tag{10}$$

in such a way that a direct correspondence between the properties of the mechanical system and the electrodynamic quantities in Eq. (8) can be easily found by inspection. By Eq. (8) or, alternatively, by Eq. (10), the dynamic properties of a current-biased overdamped Josephson junction can be found. Furthermore, the static properties of these systems can be recovered by setting $\frac{d\phi}{dt} = 0$ in Eq. (8) and $\frac{d\theta}{dt} = 0$ in Eq. (10).

The analogy here illustrated can be useful in understanding the properties a superconducting quantum device by means of a rather simple and well-known mechanical system.

4. The Pac-Man Junction: The Equations

Let us derive the equations for the stationary superconducting currents in the Pac-Man JJ in fig. 2. In the latter figure, the three small overdamped JJs [16] are taken to be identical for simplicity and are indicated as reddish butterflies, labelled as JJ_k ($k = 1, 2, 3$). In the junctions JJ_1 and JJ_2 the same current flows. As a matter of fact, according to Eq. (2) we may write:

$$I_1 = I_2 = I_{J0} \sin(\phi_2 - \phi_1). \tag{11}$$

The current I_3 flowing in JJ_3 is given by:

$$I_3 = I_{J0} \sin(\phi_3 - \phi_2). \tag{12}$$

By current conservation, we may set $I_3 = I_1 + I_2$, so that, by Eq. (11) and (12), we have:

$$\sin(\phi_3 - \phi_2) = 2\sin(\phi_2 - \phi_1). \tag{13}$$

In order to define experimentally meaningful quantities, we set:

$$\theta = \phi_3 - \phi_1; \; \phi = \phi_2 - \phi_1, \tag{14}$$

where θ is the superconducting phase difference across the whole PMJJ and ϕ is the superconducting phase difference between the middle and the first electrode. While the first quantity θ determines the CPR of the overall PMJJ, the quantity ϕ is to be determined by means of the current-conservation relation in Eq. (13) and by the condition of minimum energy for the system. In this respect, we recall that, as it happens in a overdamped pendulum [18], the energy related to a small JJ, in which the superconducting phase difference across the two electrodes is γ, can be written as follows [16]:

$$E_J = -E_{J0} \cos\gamma, \tag{15}$$

where $E_{J0} = \frac{I_{J0}\Phi_0}{2\pi}$. In the case of the PMJJ system, we have the sum of three terms which, according to Eq. (15), together with the definition of Eq. (14), can be written as follows:

$$E_J = -E_{J0} \cos(\theta - \phi) - 2E_{J0} \cos\phi. \tag{16}$$

The conditions for the existence of a minimum of E_J, by taking θ fixed, are thus the following:

$$\frac{\partial E_J}{\partial \phi} = 0 \to 2\sin\phi - \sin(\theta - \phi) = 0. \tag{17a}$$

$$\frac{\partial^2 E_J}{\partial \phi^2} > 0 \to \cos(\theta - \phi) + 2\cos\phi > 0. \tag{17b}$$

Eq. (17a-b) are the conditions to be satisfied, in order to obtain the supercurrent flowing in the PMJJ. In our case, we would like to determine the CPR of the system, namely, the relation $I_3 = I(\theta)$, where $I(\theta)$ is an unknown function of the variable θ.

Finally notice that, while Eq. (17a) corresponds exactly to the current conservation condition already written in Eq. (13), Eq. (17b) represents an additional constraint which we should consider. Solutions of Eq. (17a-b) will be sought in the following section.

5. The Current-Phase Relation of the Pac-Man Junction

Let us now find the current-phase relation of the PMJJ by solving Eq. (17a) under the condition given by Eq. (17b). By trigonometric identities, we may rewrite Eq. (17a) as follows:

$$(2 + \cos\theta)\sin\phi = \sin\theta\cos\phi. \tag{18}$$

By now squaring both members of Eq. (18), we get the following relation:

$$\sin^2\phi = \frac{\sin^2\theta}{5 + 4\cos\theta}. \tag{19}$$

From Eq. (19) we can thus write:

$$\sin\phi = \frac{|\sin\theta|}{\sqrt{5 + 4\cos\theta}} \operatorname{sign}(\sin\phi), \tag{20a}$$

$$\cos\phi = \frac{2 + \cos\theta}{\sqrt{5 + 4\cos\theta}} \operatorname{sign}(\cos\phi), \tag{20b}$$

where the sign function $\operatorname{sign}(x)$ is equal to $+1$, if $x > 0$, or to -1, if $x < 0$. In order to get a first condition on the signs of $\sin\phi$ and of $\cos\phi$, we may substitute the above relations back into Eq. (18), finding:

$$\operatorname{sign}(\sin\phi)\operatorname{sign}(\cos\phi) = \operatorname{sign}(\sin\theta). \tag{21}$$

We are now ready to complete our analysis by considering Eq. (17b), which can be written in the following way:

$$(2 + \cos\theta)\cos\phi + \sin\theta\sin\phi > 0. \tag{22}$$

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By substituting the solutions given in Eq. (20a) and (20b) in the above expression, we get:

$$(2+\cos\theta)^2 \,\mathrm{sign}(\cos\phi) + \sin^2\theta \,\mathrm{sign}(\sin\theta)\,\mathrm{sign}(\sin\phi) > 0. \tag{23}$$

Taking into account Eq. (21), we may thus write:

$$\left[(2+\cos\theta)^2 + \sin^2\theta\right] \mathrm{sign}(\cos\phi) > 0. \tag{24}$$

The above relation can be satisfied only for $\mathrm{sign}(\cos\phi) > 0$. Therefore, by Eq. (21) we have $\mathrm{sign}(\sin\phi) = \mathrm{sign}(\sin\theta)$. In this way, Eq. (20a) and Eq. (20b) can be written as follows:

$$\sin\phi = \frac{\sin\theta}{\sqrt{5+4\cos\theta}}, \tag{25a}$$

$$\cos\phi = \frac{2+\cos\theta}{\sqrt{5+4\cos\theta}}. \tag{25b}$$

Having obtained the solution for $\sin\phi$ and $\cos\phi$ in terms of θ, we can write the CPR of the system by means of Eq.(12-14) as follows:

$$i(\theta) = \frac{2\sin\theta}{\sqrt{5+4\cos\theta}}, \tag{26}$$

where $i(\theta) = I_3/I_{J0}$. Representation of the CPR of a PMJJ is given in fig. 5, where a comparison with the CPR of a single JJ, namely $i(\theta) = \sin\theta$, is shown. From fig. 5 we notice that the CPR of a PMJJ exhibits a maximum in the interval $\frac{\pi}{2} < \theta < \pi$. In fact, calculating the derivative of $i(\theta)$ in Eq. (26) and by setting it equal to zero, we have:

$$2\cos^2\theta + 5\cos\theta + 2 = 0. \tag{27}$$

By solving the above second degree algebraic equation, both extrema (maximum and minimum) are found to be given by $\cos\theta = -1/2$. Therefore, by considering Eq. (26), we see that $\theta = \frac{2\pi}{3}$ is the maximum point and $\theta = -\frac{2\pi}{3}$ the minimum point. By substituting the values $\theta = \pm\frac{2\pi}{3}$ in Eq. (26) we finally find that the maximum and the minimum of the function $i(\theta)$ are $+1$ and -1, respectively.

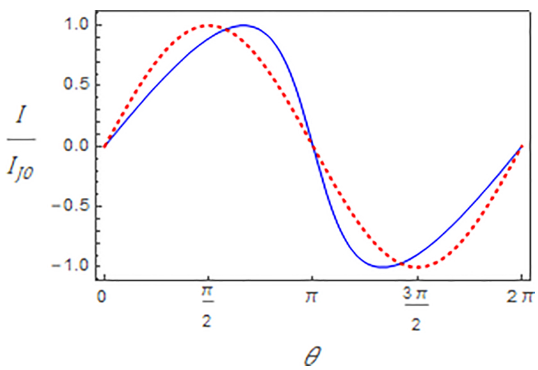

Figure 5: Current-phase relation (CPR) of a Pac-Man Josephson junction (full blue line). The red dotted line represents the CPR of a single Josephson junction.

6. The MultiPac-Man System

Let us now generalize the findings in the previous section by considering the multi-Pac-Man system shown in fig. 6. In this system, N Josephson junctions are present in between the electrode with superconducting phase ϕ_1 and the middle electrode, characterized by a superconducting phase ϕ_2. Therefore, by current conservation, we write, as done for Eq. (13):

$$\sin(\phi_3 - \phi_2) = N\sin(\phi_2 - \phi_1). \tag{28}$$

Proceeding as in Section 2, the energy of the system is written as follows:

$$E_J = -E_{J0}\cos(\theta-\phi) - NE_{J0}\cos\phi, \tag{29}$$

where the variables θ and ϕ are defined in Eq. (14). The conditions for the existence of a minimum of E_J, by taking θ fixed, in this case are the following:

$$\frac{\partial E_J}{\partial\phi} = 0 \rightarrow N\sin\phi - \sin(\theta-\phi) = 0. \tag{30a}$$

$$\frac{\partial^2 E_J}{\partial\phi^2} > 0 \rightarrow \cos(\theta-\phi) + N\cos\phi > 0. \tag{30b}$$

By operating in the same way as in the previous section, we therefore find:

$$\sin\phi = \frac{\sin\theta}{\sqrt{N^2+1+2N\cos\theta}}, \tag{31a}$$

$$\cos\phi = \frac{N+\cos\theta}{\sqrt{N^2+1+2N\cos\theta}}. \tag{31b}$$

The CPR of the system can now be found by means of Eq. (28), together with the definition in Eq. (14), so that:

$$i(\theta) = N\sin\phi = \frac{N\sin\theta}{\sqrt{N^2+1+2N\cos\theta}}, \tag{32}$$

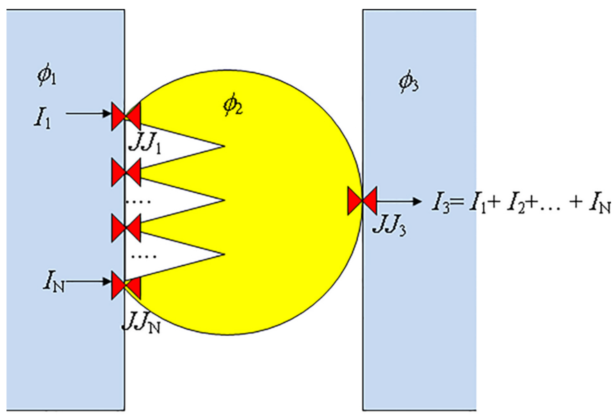

Figure 6: A tri-layer Josephson junction obtained by embedding a multi-tentacle Pac-Man shaped superconducting island in between two superconducting electrodes.The N small Josephson junctions are indicated as reddish butterflies

where, again, $i(\theta) = I_3/I_{J0}$. A graphical representation of the CPR of a multi-PMJJ is given in fig. 7 for $N = 1$ (dashed line), $N = 3$ (full line), and $N = 100$ (dotted line). The case in which $N = 1$ has also been treated in ref. [6], so that the CPR represented by the dashed line in fig. 7 is similar to the CPR of a tri-layer system in the case the direct interaction between the outer electrodes is neglected, and reads:

$$i(\theta) = \sin\left(\frac{\theta}{2}\right) \text{sign}[\cos\left(\frac{\theta}{2}\right)]. \quad (33)$$

The appearance of a sharp discontinuity in the CPR of the PMJJ for $N = 1$ is a rather interesting feature, arising mainly from the properties of the sign function in Eq. (33).

In fig. 7 we also notice that the maximum points of all curves tend to move toward $\theta = \frac{\pi}{2}$, as N increases. In fact, by calculating the derivative of $i(\theta)$ in Eq. (32) and by setting it equal to zero, we have:

$$N\cos^2\theta + \left(N^2 + 1\right)\cos\theta + N = 0. \quad (34)$$

By solving the above second degree algebraic equation, both extrema (maximum and minimum) are found, for $N > 1$, to be given by $\cos\theta = -1/N$. Therefore, considering Eq. (32), we see that $\theta = \cos^{-1}(-1/N)$ is the maximum point and $\theta = -\cos^{-1}(-1/N)$ the minimum point. In the limit for $N \to \infty$, this solution tends to $\theta = \frac{\pi}{2}$, as specified before. Therefore the CPR, for increasing N, tend to get closer and closer to the $\sin\theta$ function, as if the complex network would behave just like a single JJ in the limit for $N \to \infty$.

By substituting the values of the maximum and minimum points found from Eq. (34) in Eq. (32) we find that, for $N > 1$, the maximum and the minimum of the function $i(\theta)$ are $+1$ and -1, respectively. Of course, the case $N = 2$ treated in the previous section can be considered as a particular case of the present more general analysis. However, we feel that the simplicity of the analysis given in the previous section can be used as a propaedeutic approach to this interdisciplinary topic.

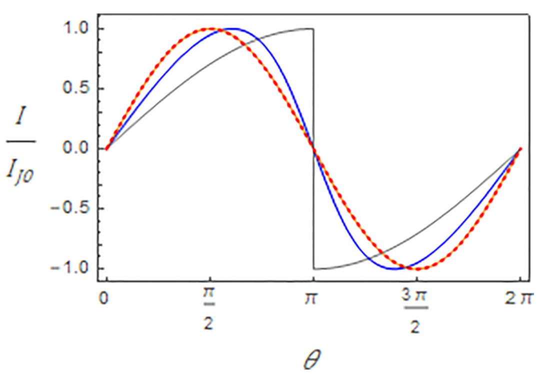

Figure 7: Current-phase relation (CPR) of a multi-Pac-Man Josephson junction with N=1 (gray line), N=3 (blue line), N=100 (orange line). The red dotted line represents the trigonometric function sinθ, which is superimposed to the orange line.

7. Conclusions

The current-phase relation of a Pac-Man Josephson junction (PMJJ), consisting of a Pac-Man shaped superconducting island embedded in two superconducting electrodes, has been found. In studying the system, we realized that the trigonometric equations defining the supercurrent state of the PMJJ are interesting either for their physical meaning, either for the particular care one needs to use to solve them analytically. Therefore, in the present work a gradual approach to the solution of these equations is presented. Starting from the case the Pac-Man has only two contact point on the left electrode, we found the analytic expression of the current-phase relation (CPR) of the PMJJ, by following a strict step-to-step solution. Successively, generalization to the case of a multi-Pac-Man JJ is treated, in analogy to what previously done for the simpler PMJJ. The current-phase relations of the junction network show significant deviations from the usual sine function characterizing the CPR of single Josephson junctions. Extension of these results to a multi-Pac-Man Josephson junction has been given.

Therefore, the present work, besides presenting some interesting features of superconducting devices, may also be proposed to advanced high-school students or to undergraduate college students in an interdisciplinary lecture, provided some introductory remarks on superconductivity and Josephson devices are given, as suggested in Sections 2 and 3. Finally, we remark that the mathematical rigour in treating trigonometric functions is coupled, in this particular work, to an original application in the field of Josephson junction networks, so that possible future applications of these systems may be sought.

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