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Lissajous-like figures with triangular and square waves

(Generalização de figuras de Lissajous com ondas triangulares e quadradas)

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We show a generalization of the well-known Lissajous figures, changing the two orthogonal simple harmonic oscillations to triangular and square-wave-type oscillations. All figures cross the origin of axes when there is no phase difference between the oscillations. Aside from this common feature, square-wave-type figures show a very different behavior, with discontinuous phase changes and impossibility of using a simple formula to get the frequency ratio between the two component oscillations.
Keywords: Lissajous figures, triangular and square waves, Fourier analysis.

Mostramos uma generalização das bem conhecidas figuras de Lissajous, trocando-se as duas oscilações harmônicas simples ortogonais por oscilações tipo onda triangular e onda quadrada. Todas as figuras cruzam a origem dos eixos quando não existe diferença de fase entre as oscilações. Excetuando-se esta característica em comum, as figuras formadas com ondas quadradas mostram um comportamento bastante distinto, com mudanças descontínuas destas figuras conforme se varia a fase e a impossibilidade de utilizar uma fórmula simples para obter a razão entre as frequências das componentes oscilatórias.
Palavras-chave: figuras de Lissajous, ondas triangulares e quadradas, análise de Fourier.

1. Introduction

Lissajous figures are formed when two simple harmonic vibrations are coupled at right angle to each other [1]. Nathaniel Bowditch seems to have been the first (1815) to discuss such curves and Jules Antoine Lissajous studied them on a deeper level in 1857-58 [2]. Besides their aesthetic beauty, Lissajous figures are used in undergraduate teaching laboratories to obtain the frequency of a signal (like sound, radio waves, etc.) by combining them with another signal of known frequency. Also, from the eccentricity of an ellipse (a typical Lissajous figure), one can precisely measure the phase difference between two waves of the same frequency. Using this technique the speed of sound or light may be obtained by studying the phase difference between a direct modulated signal and another one which has traveled a precise distance [3, 4] or measure phase delays between currents and voltages among components on an RLC circuit [5]. There are also recent research applications of Lissajous figures as, for instance, commensurateness and phase between quantities relevant to helicopter flight [6], light polarization structures created by using second-harmonic generation from lasers [7], Michelson interferometry to measure micro-vibration displacements [8], oscillatory deformation in strongly nonlinear materials [9] and satellite trajectories around Lagrange points in the outer space [10].

Harmonic vibrations are the most usual scenario but triangular and square wave functions are also important periodic functions that one may come across. They are useful in digital circuits and frequency and time-interval measurements [11], besides other situations [12, 13]. Here we show what happens when one combines two orthogonal vibrations of these two latter types. These important cases seem to be left untouched, although some other variations of Lissajous figures have already been studied [14].

2. Simulation and discussion

One may use several graphical software to trace those curves or, alternatively, an oscilloscope in $X-Y$ mode and two function generators. The results, illustrated in Fig. 1, were obtained with "Curvay" software [15] and were named Lissajous-like figures. Some true Lissajous curves, with harmonic vibrations, are shown in the first three columns (from left to right) of Fig. 1. They obey simple parametric equations, with time t as a parameter [1, 14]

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$$X = \cos(\omega_1 t - \alpha), \quad (1)$$
$$Y = \cos(\omega_2 t - \beta), \quad (2)$$

where ω_1/ω_2 must be a rational number. For the sake of simplicity, we supposed unitary amplitudes for both vibrations. Several examples are shown, with variations of ratio between the two angular frequencies ω_1/ω_2 and their phase difference $\delta = \alpha - \beta$, as indicated respectively on the left and upper side of Fig. 1.

Combinations of two orthogonal triangular and square-type vibrations are shown in the last six columns (from left to right) of the same figure, for phase differences of zero, $\pi/4$ and $\pi/2$. The curves for phase differences of $3\pi/4$ and π (not shown) are symmetric and produce the same graphs as those of $\pi/4$ and zero respectively, except for a reflection about the vertical axis. For true Lissajous figures the ratio between the two frequencies is equal to the ratio of maximum intersections of a vertical line and a horizontal one with the figure. In other words we have

$$\frac{f_x}{f_y} = \frac{N_y}{N_x}, \quad (3)$$

where f is the frequency in each axis and N the maximum number of intersections of the true Lissajous figure with a straight line parallel to the respective axis. For instance, in the fourth row of Fig. 1 the vertical frequency is 2/3 times the horizontal one, because one has vertical to horizontal intersections on the ratio 3/2 (for any phase difference). This basic feature of the true Lissajous figures also appears in the triangular case, which can be checked through the same example. So, for the triangular case, we may apply the same simple formula above to calculate the frequency in one axis if one knows the frequency in the other axis. On the other hand, this rule does not apply for the square Lissajous-like figures, as one can easily verify through the same example.

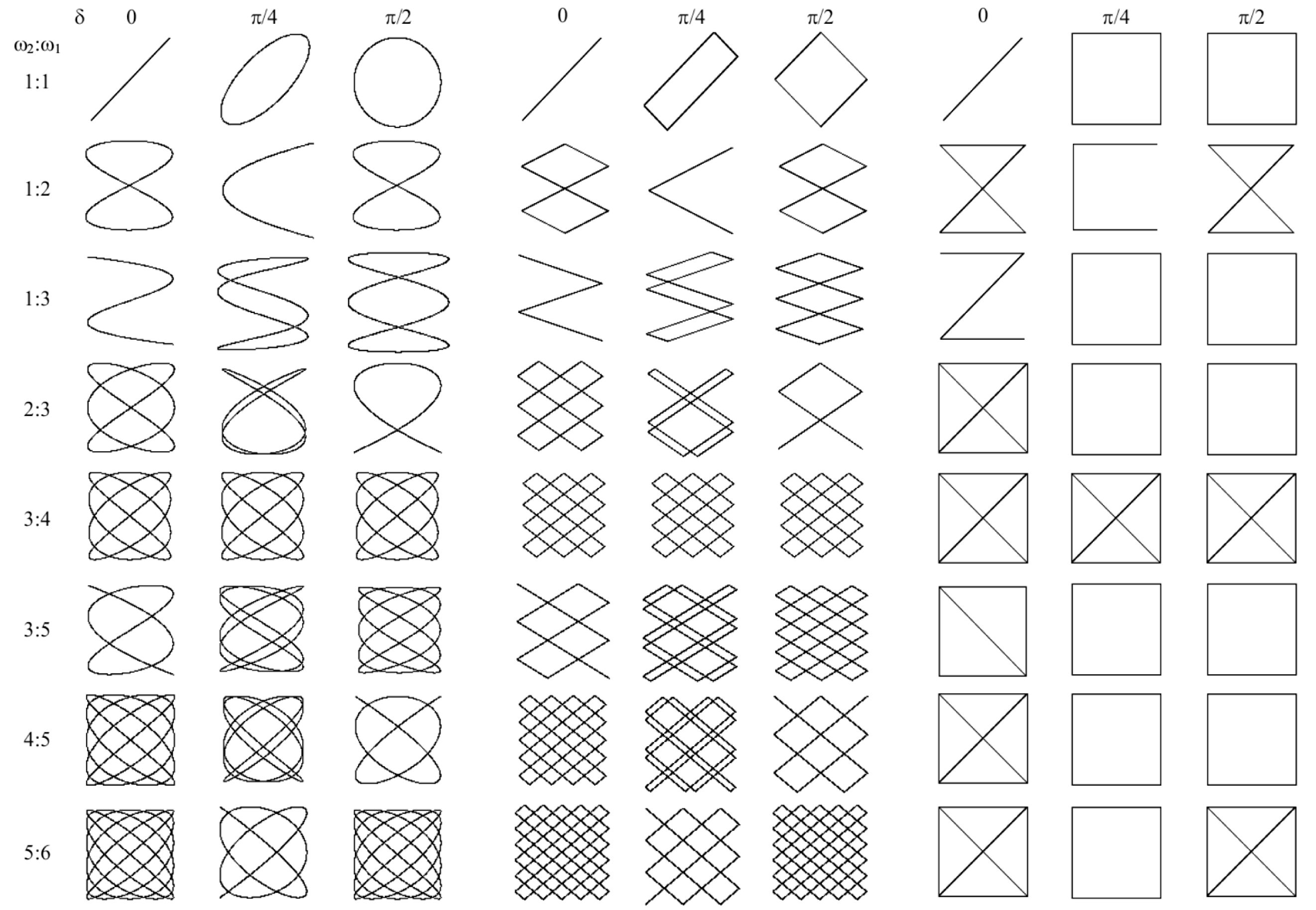

Figure 1 - Lissajous-like figures. From left to right we have harmonic, triangular and square wave components, with three columns set for each case.

Another interesting point to mention is the lack of continuity of the square Lissajous-like figures as a function of the phase difference between components. For example, in the harmonic case with equal frequencies, one has a continuous change from a straight line segment for $\delta = 0$ to a circle for $\delta = \pi/2$, through ellipses of different eccentricities. Something similar is valid for the triangular case. On the other hand, for the square type, we have a straight line segment for $\delta = 0$ and then an abrupt change to a square figure with an infinitesimal δ increase. This square remains immutable until $\delta = \pi$ is reached, when it changes again abruptly to a straight line segment (oriented perpendicular to the first one). It is easy to see that this behavior comes from the fact that the square wave is a discontinuous function. For any frequency ratio, true Lissajous curves always pass through the origin of the $X - Y$ plane when $\delta = 0$, which means no phase difference between the two harmonic waves [16]. As one can check by the examples in Fig. 1, this is a common characteristic also shared by the other two types of Lissajous-like figures studied here.

All periodic functions, as the square and triangular waves, can be written as a Fourier series. One may argue, on such basis, that it is a straightforward mathematical procedure to outguess the Lissajous-like figures reported here and computer simulations are needless. Although one can use Lissajous figures to Fourier analyze a given signal [17] the other way around (use Fourier analysis to forecast Lissajous-like figures) is not as easy. To see this let us expand, for instance, the square wave in Fourier series. If the wave has angular frequency ω and unitary amplitude, we have the following expression [18]

$$f(t) = \frac{4}{\pi} \sum_{n\ odd}^{\infty} \frac{1}{n} \sin n\omega t. \tag{4}$$

Suppose we combine two square waves with angular frequencies ω_1 and ω_2 at right angle to each other, with unitary amplitudes and no phase difference between them. To forecast the Lissajous-like figure that would emerge from that, assume we truncate the Fourier expansion at the pth term, for both waves. One must, then, sum up graphically p^2 true Lissajous figures, coming from the combinations of each term from one series with each term of the other. As the square wave shows discontinuity points, the series above is not uniformly convergent and one needs several terms to get a good approximation of the function, something known as the Gibb's phenomenon [18]. If there is a phase difference between the two square waves the situation is even worst, due to the greater number of variables involved. Of course, except for the Gibb's phenomenon, the same arguments would apply for the construction of the Lissajous-like figures with triangular waves. Preliminary results from this manuscript were reported elsewhere [19].

3. Conclusions

In summary, we show what happens when one couples two triangular or square waves at right angle to each other. These are important periodic functions, along with the harmonic case. We get the formation of what we named Lissajous-like figures, due to their resemblance to true Lissajous figures, formed with harmonic waves. Some aspects are common to all three types of figures, such as the origin crossing when there is no phase difference between the two orthogonal waves. Other features, as the possibility of calculating the frequency ratio between the waves by manipulating the figure, are not shared by the square-wave case. We generated these curves through graphical software. The need for simulations (or, alternatively, the use of an oscilloscope) is justified from the fact that an algebraic Fourier forecast of these Lissajous-like figures is not a straightforward task. One may choose the particular free graphical software used here or different widespread options available nowadays, in order to easily reproduce these figures and introduce them in teaching laboratories.

References

[1] J.B. Marion and S.T. Thornton, *Classical Dynamics of Particles and Systems* (Harcourt, Forth Worth, 1988), 3^{rd} ed.

[2] Encyclopædia Britannica, "Curves, Special" (Encyclopædia Britannica Inc., London, 1953), 14^{th} ed.

[3] R.E. Berg and D.R. Brill, Phys. Teach. **43**, 36 (2005).

[4] "Measurement of the Speed of Light" `http://uni-leipzig.de/~prakphys/pdf/VersucheIPSP/Optics/O-14e-AUF.pdf`, access in March, 2013.

[5] Phase Displacement in AC Circuits" `http://uni-leipzig.de/~prakphys/pdf/VersucheIPSP/Electricity/E-11e-AUF.pdf`, access in March, 2013.

[6] N. Sarigulklijn and M.M. Sarigulklijn, J. Aircraft **34**, 20 (1997).

[7] D.A. Kessler and I. Freund, Opt. Lett. **28**, 111 (2003).

[8] Z.J. Li, S.L. Zhen, B. Chen, M. Li, R.Z. Liu and B.L. Yu, Opt. Comm. **281**, 4744 (2008).

[9] R.H. Ewoldt and G.H. Mckinley, Rheol. Acta **49**, 213 (2010).

[10] D. Romagnoli and C. Circi, Celestial Mech. Dyn. Astron. **107**, 409 (2010).

[11] P. Horowitz and W. Hill, *The Art of Electronics* (Cambridge University Press, Cambridge, 1989).

[12] S.M. Morgan and R.H. Victora, Appl. Phys. Lett. **97**, 093705:1 (2010).

[13] K. Miyasato, S. Abe, H. Takezoe, A. Fukuda and E. Kuze, Jap. J. Appl. Phys. **22**, L661 (1983).

[14] D. von Seggern, *Practical Handbook of Curve Design and Generation* (CRC Press, Boca Raton, 1994).

[15] "Curvay" software can be downloaded for free at `http://www.spelunkcomputing.com/curvay/download.html`, access in March, 2013.

[16] M.S. Wu and W.H. Tsai, Am. J. Phys. **52**, 657 (1984).

[17] P.P. Ong and S. H. Tang, Am. J. Phys. **53**, 252 (1985).

[18] P.H. Hwei, *Fourier Analysis* (Simon & Schuster, New York, 1967).

[19] A. Hornes and J. Flemming, *Anais do XI Encontro Anual de Iniciação Científica*, Maringá, PR, Brazil (2002).