Revista Brasileira de Ensino de Física, vol. 45, e20230270 (2023)
www.scielo.br/rbef
DOI: https://doi.org/10.1590/1806-9126-RBEF-2023-0270

Articles

Harmony as an underlying ingredient in the numerical search for suitable equal temperaments

Harmonia como um ingrediente subjacente na busca numérica por temperamentos iguais apropriados

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Received on September 14, 2023. Revised on October 16, 2023. Accepted on October 24, 2023.

This work investigates whether there are harmonic sets which are universally prioritized when fitting the chromatic just scale with equal-tempered tunings. We design a method to find which temperaments properly and non-ambiguously fit a set of pure intervals and show that: (a) Within range $N \in [9, 54]$, temperaments with $12, 19, 22, 24, 31, 34, 41$ and 53 divisions are suitable for adjusting the twelve-tone chromatic just scale – in particular, 53 divisions provide the most accurate fit, (b) For M-element subsets of the chromatic universe with $M \in [3, 7]$, these solutions are always more faithfully reproduced when the target set is a major scale rather than a minor one, and (c) Sequences of notes uniformly distributed over the octave are disadvantageous references for finding suitable temperaments numerically. The latter observation suggests a mathematical background for understanding the preference for non-uniform scales in world music revealed by recent studies in ethnomusicology.
Keywords: Music theory, equal-temperament, harmony, pure intervals.

Este trabalho investiga se existem conjuntos harmônicos que são universalmente priorizados ao se ajustar a escala cromática pura com uma afinação igualmente temperada. Nós criamos um método para encontrar quais temperamentos ajustam de forma não-ambígua um conjunto de intervalos puros e mostramos que: (a) No intervalo $N \in [9, 54]$ temperamentos com $12, 19, 22, 24, 31, 41$ e 53 divisões são os adequados para ajustar a escala cromática pura de 12 notas – em particular, 53 divisões proporcionam o ajuste mais preciso, (b) Para subconjuntos com $M \in [3, 7]$ notas, essas soluções são sempre mais fielmente reproduzidas por escalas maiores, em comparação com as menores, e (c) Sequências de notas uniformemente distribuídas pela oitava são referências desvantajosas para a busca numérica de bons temperamentos. A última observação sugere um contexto matemático para entender a preferência por escalas não-uniformes na música mundial revelada por estudos recentes em etnomusicologia.
Palavras-chave: Teoria musical, temperamento igual, harmonia, intervalos puros.

1. Introduction

An intimate connection between physics and music underlies a statement of beauty in combining sound waves into harmonies. The fact that these cannot be perfectly adjusted in a frequency structured scheme is a curious and intriguing feature of nature, mixing wave properties, our sound perceptions, and consequently the way we make art. Investigations into universal features of music were popularized with the increasing statistical data on ethnomusicology, revealing patterns across different cultures [1, 2]. For example, non-uniform scales, harmonic sequences built with different step sizes between notes, are recurrent and have been shown to provide cognitive benefits for learning and memorizing melodies – a possible explanation for their prevalence throughout history [3, 4].

The choice of a tuning system, or *temperament*, reflects requirements of the pieces and instruments to be played alongside physics and psychoacoustic observations [5–9]. It is a known fact that the most pleasing harmonies for the human ear are combinations of notes with frequency ratios that can be written as small integer fractions, which are called *pure intervals*, a natural result of systems like air columns and vibrating strings [10–12]. One important example is the octave, defined by two notes with a ratio of 2/1 between its frequencies. Other intervals within this range can be used to build harmony, like the fifth (3/2) and the major third (5/4). Many tuning systems were designed in antiquity and the medieval period in order to produce such intervals, which referred to beauty on the philosophical background of the time, but none of them was able to be key independent – as music of that period used primarily the **C** major key these discrepancies were little noticed [13, 14].

Alternatively, *equal temperament* can reasonably adjust pure intervals while allowing for pieces to be played in any key [15, 16]. Over this setting, the only pure interval is the octave whereas intermediate notes are determined by frequencies in a geometric progression [17, 18]. In particular, the 12-tone equal temperament (or 12-TET), for which the octave is divided into

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12 intervals, became the mainstay of Western music. This arrangement is very criticized for not reproducing pure harmonies well enough, given that some musicians claim to hear annoying beatings in some chords [19]. Evidently, only very good ears will actually hear the difference. Nevertheless, the search for feasible alternatives received special attention in subsequent years [20–22, 24]. According to the numerical approach in Ref. [23], temperaments with 19 and 29 divisions would also provide good approximations; indeed, the use of 19-TET was brilliantly advocated by Joel Mandelbaum in his P.h.D. thesis [24]. However, other N-TETs are highlighted in literature for providing remarkable fits of pure intervals like the 24-TET [25, 26], the 31-TET [27] and even the 53-TET [28, 29]. An universal measure for comparing temperaments performances does not seem to be uniquely accepted in literature, each reported strategy is commonly advocated for benefiting specific groups of notes; then some preferences (like for the fifth interval) leads to a temperament choice. Here we address the following question: Are there universal harmonic sets which are prioritized when setting an N-TET? What subsets of the chromatic universe are more strategic references for identifying which tuning systems are good?

In this work, we design a method to find which temperaments properly and non-ambiguously fit a set of pure intervals within an error tolerance which is chosen to be the comma (an important partition of the semi-tone) and show that: (a) By addressing the twelve-tone chromatic just scale as target set, we find within range $N \in [9, 54]$ that temperaments with $12, 19, 22, 24, 31, 34, 41$ and 53 divisions are suitable, which is confirmed by literature [25–30] – in particular, 53-TET is shown to provide the most accurate fit, (b) For M-element subsets of the chromatic universe with $M \in [3, 7]$, these solutions are always more faithfully reproduced when the target set is a major scale rather than a minor one, and (c) Sequences of notes uniformly distributed over the octave, like the whole-tone scale, are the most disadvantageous reference sets for setting proper equal temperaments. The latter observation suggests a mathematical background for understanding the universal preference for non-uniform harmonic sets throughout music cultures, as revealed by statistical data in recent ethnomusicology studies [1–4].

This paper is organized as follows: In Sec. 2 we overview required concepts of music theory, in Sec. 3 we present the method and our numerical results, and in Sec. 4 we conclude this paper.

2. An Overview of Music Theory

The concept of harmonicity is derived from wave physics. The strong consonance between a fundamental frequency, say f_0, and its integer multiples, nf_0 where $n \in \mathbb{N}$, is the resource for building harmonic combinations in music, these are called *just intonation intervals*. These properties were observed throughout history with the use of vibrating strings and air columns, promoting the development of musical instruments.

Table 1: Pure intervals alongside their 12-TET counterpart frequency ratios [11].

	Label	Just intonation	12-TET	Difference (Hz)
Unison	**C**	1	1	0
Minor Second	**C#/D♭**	16/15	$2^{1/12}$	−4.66
Major Second	**D**	9/8	$2^{2/12}$	+0.66
Minor Third	**D#/E♭**	6/5	$2^{3/12}$	+2.82
Major Third	**E**	5/4	$2^{4/12}$	−2.60
Fourth	**F**	4/3	$2^{5/12}$	−0.39
Diminished Fifth	**F#/G♭**	45/32	$2^{6/12}$	−2.08
Major Fifth	**G**	3/2	$2^{7/12}$	+0.44
Minor Sixth	**G#/A♭**	8/5	$2^{8/12}$	+3.30
Major Sixth	**A**	5/3	$2^{9/12}$	−3.96
Minor Seventh	**A#/B♭**	9/5	$2^{10/12}$	+4.76
Major Seventh	**B**	15/8	$2^{11/12}$	−3.34
Perfect octave	**C**	2	2	0

The simplest harmonic combination, called *perfect octave*, is produced when the higher frequency is the double of the lower one. For that interval, the higher note label is identical to the tonic; for example, the **C** note present on the fourth octave of the piano, or $\mathbf{C}_4$, has a frequency which is the double of the $\mathbf{C}_3$ note. The next integer combination is a pair of pitches with frequency ratio 3/1, which because of the cyclic octave property can be found closest to the tonic by proportion 3/2. If the tonic is **C**, the interval 3/2 is labelled as **G** (see Tab. 1). The harmonic combination of these two notes is called *perfect fifth*, and it was the mainstay of music on antiquity and the building block of the *Pythagorean tuning* [6]. Another important interval is the *major third*, which is built with frequency ratio 5/4. In Fig. 1 we show pictorial representations of each of the aforementioned intervals as standing waves on a string. Other fractions between integers can be built in order to produce pure intervals: In Tab. 1 we show a list of twelve such intervals which is called *twelve-tone chromatic just scale*, or *just intonation scale* [11]. The step between two neighboring frequencies is called *semi-tone* while two steps yields a *tone*.

Addressing the tonic note as **C**, the major third is labelled as **E** and then $\{\mathbf{C}, \mathbf{E}, \mathbf{G}\}$ is called **C** *major* chord. It has a kind of twin: If the middle note is lowered in pitch to the frequency ratio 6/5, the combination $\{\mathbf{C}, \mathbf{E}\flat, \mathbf{G}\}$[1] is called **C** *minor* chord. The major and minor thirds are the nearest consonances from the tonic and the middle elements for constructing the simplest

[1] The label ♭ is read *flat*, representing the preceding semi-tone interval. For the subsequent semi-tone interval it is used symbol #, which is read as *sharp*.

 DOI: https://doi.org/10.1590/1806-9126-RBEF-2023-0270

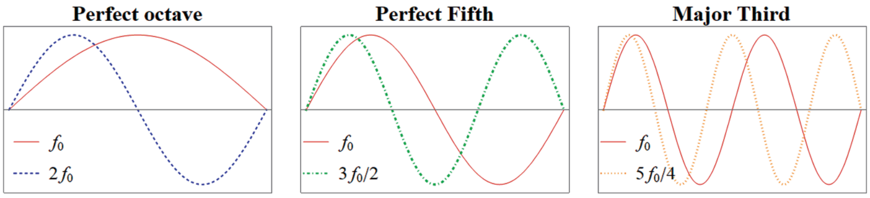

Figure 1: Representation of harmonics as vibrations on a string in terms of either distance or time.

3-element chords, also called *triads*. The notion of harmonicity in music can be extended to larger sets of notes which present some characteristic harmonic behavior. We discuss such special sets in the next subsection.

2.1. Harmonic scales

Subsets of the chromatic universe may be chosen to favor appealing combinations of notes, these are called *harmonic scales*. These special sets have interesting mathematical properties [31, 32] and are used by musicians as a basis for composition and improvisation. The most used harmonic scales in Western music are 5-element sets, so called *pentatonics*, and 7-element sets, so called *heptatonics*. The simplest examples of such scales, the major and minor, are represented in Fig. 2 as inscribed M-side polygons in a dodecagon where each vertex represents a just interval. For having 12 markings, this type of representation is often called *clock diagram* [4, 32, 33], or *Krenek diagram* because of its appearance in [34]. We may also address major and minor sets of different sizes: In Tab. 2 it is shown the *tetratonic*[2] (4 notes *per* octave) and *hexatonic* (6 notes *per* octave) versions of the major and minor scales – note that these are simply subsets of the corresponding heptatonics. Although triads are not considered scales, they may have major/minor harmonic behavior, so for the purposes of this work we have them included in Tab. 2.

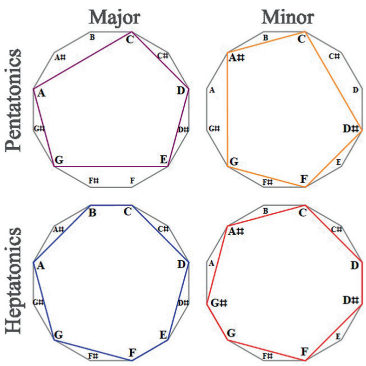

Figure 2: Krenek diagrams for major and minor pentatonic (5 notes *per* octave) and heptatonic (7 notes *per* octave) scales. The addressed heptatonic minor scale is also known as *natural minor scale.*

[2] Tetratonics are not much used in modern music and are commonly associated with prehistoric times [3].

Table 2: Major, minor and uniform sets of sizes $3, 4$ and 6. The uniform hexatonic is also called *whole-tone scale.*

Size	3	4	6
Major	**C, E, G**	**C, E, G, A**	**C, D, E, F, G, A**
Minor	**C, E♭, G**	**C, E♭, G, B♭**	**C, D, E♭, F,G, B♭**
Uniform	**C, E, G#**	**C, E♭, G♭, A**	**C, D, E, G♭, A♭, B♭**

Harmonic scales are usually described by the size of the steps taken between subsequent notes. For example, the major heptatonic scale is built by the following sequence of steps: *tone, tone, semi-tone, tone, tone, tone, semi-tone.* Evidence shows that most scales developed around the world are built with different step sizes, being *non-uniform* [2], as it is the case of the aforementioned major and minor scales. *Uniform*, or *equal-step*, scales are rare across music cultures. Evidently, the chromatic scale is a uniform set built only by semi-tone steps, but because it contains all intervals it is seen as non-harmonic. Within the twelve-tone chromatic universe, one can build uniform subsets of $3, 4$ and 6 elements by varying the step size (see Tab. 2). Commonly used in jazz improvisation, the most famous example of such group of scales is the uniform hexatonic, so called *whole-tone scale* for being built only by tone steps[3].

2.2. Tuning systems

2.2.1. Just intonation

For building musical instruments, it is important to set tuning systems – particular ways of discretizing the sound range of interest. The just intonation scale was designed to have as many pure intervals as possible [5]. Example in Tab. 1 is constructed from ratios between powers of prime numbers $2, 3$ and 5, an arrangement called 5-*limit just intonation* [11]. More ratios can be constructed for building appealing combinations, which is not an easy task[4].

This type of configuration has a big shortcoming: it strongly depends on the key in which a piece is going to be played. For example, the notes **E** and **G#/A♭** should make a major third interval, however the frequency ratio

[3] One of the most famous applications of this scale was made by the french composer Claude Debussy [35].
[4] Also, more prime numbers can be addressed. One usual extension is the 7-limit just intonation [11].

between those two notes in the **C** tuning is 32/25, which is only close to 5/4, but yields an audible difference. The most famous discrepancy was noted by Pythagoras: If one goes around twelve fifth intervals for finding the same note only seven octaves higher, the difference between the ratios $(3/2)^{12}$ and $(2/1)^7$ yields an interval that is approximately a quarter of semitone higher than pure, which was called *Pythagorean comma* [5]. So making all octaves pure guarantees that all fifths cannot be [16], which is a direct consequence of the impossibility of finding equal powers of different prime numbers [36].

2.2.2. Equal temperament

The desire of having pieces played in different keys alongside musical developments over the centuries, like the invention of the piano, ended up selecting equal temperament as the widely used fit. This fancy arrangement is structured as a geometric progression of frequencies constrained to reproduce the perfect octave's ratio 2/1 [6]. Formally, a general N-TET is a sequence of frequencies $\mathcal{F}^N = \{\mathcal{F}_k^N\}_{k=0}^N$ described by:

$$\mathcal{F}_k^N = \sqrt[N]{2^k} f_0, \quad k = 0, 1, \ldots, N, \tag{1}$$

where f_0 is a reference frequency. The conventional temperament is made of 12 intervals, for which neighboring frequencies are related by the factor $\sqrt[12]{2}$ (see Tab. 1). It circumvents the Pythagorean comma by fixing *all* fifth intervals at frequency ratio $2^{7/12}$, being slightly lower than pure ones.

Aiming for better adjustments of other intervals (like the major and minor sevenths, for instance), alternative TETs like the 19-TET [24] were seriously proposed. In Fig. 3 we see a comparison between a 12-TET and a possible 19-TET piano octave [37]. For building 19 divisions, we split each of the 5 semi-tones represented by the black keys of a twelve-tone piano octave into two distinct intervals: So pairs **C#/D♭** and **G#/A♭** are

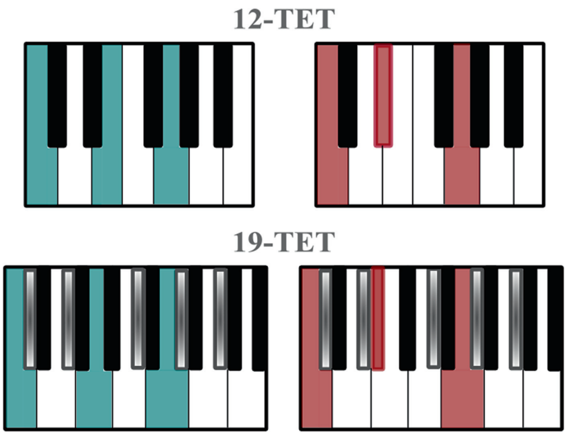

Figure 3: Piano octaves of 12-TET and 19-TET. In colors we show the corresponding representations of the **C** major (left) and minor (right) chords.

mapped into different notes (represented by black and gray keys in Fig. 3). In addition, a semi-tone interval is added between pairs **E/F** and **B/C**, totaling 19 divisions.

From a practical point of view, the 24-TET is the most direct extension of 12-TET: Each interval is splitted into two distinct notes – for that, it is also called *quarter-tones scale* [25, 26]. Another worth-mentioning temperament for the benefit of our discussion is the 53-TET. There are records of a theoretical interest in this arrangement since ancient China – the music theorist Jing Fang (78–37 BCE) calculated the proximity between a sequence of 53 pure fifths and 31 octaves, $(3/2)^{53} \approx 2^{31}$ [28] – for that, 53-TET is seen as a microtonal extension of the Pythagorean tuning[5] [8]. For other temperaments, see Refs. [13, 27, 30].

TETs may produce just intonation intervals only approximately and the number of partitions N must be chosen to favor such adjustments. Note that the number of intermediate notes between target intervals increases with N. This feature allows for greater harmonic exploration in music and for different sensations of tone.

3. Finding N-TETs Using M-element Sets of Pure Intervals

3.1. Method

In this section, we built a method for finding which N-TETs, letting N be other than 12, suitably fit a given M-element subset of the chromatic just scale. Firstly, we set the error we are willing to take by addressing specific partitions of the semi-tone which are called *commas*. For example, in 12-TET each semi-tone has 4.5 commas, yielding a total of 54 such partitions *per* octave [15]. N-TETs have different densities of commas *per* interval, $\mu = \mu(N)$, while maintaining the total number of such partitions within the octave. We choose to use the minimum of those partitions as a measure of frequency resolution[6]:

$$\Delta f(\mu) = \left(2^{\frac{1}{N\mu}} - 1\right) f_0, \tag{2}$$

which is slightly lower than the Pythagorean comma [14]. Throughout this paper, without loss of generality, f_0 is chosen to be the frequency of the $\mathbf{C}_4$ which is approximately 261.63 Hz, so $\Delta f(\mu) \approx 3.38$ Hz. Note in Tab. 1 that some frequency differences between the 12-TET and just intonation are higher than (2) yielding out-of-tune notes, but it stands as a good fit in average.

[5] In late seventeenth-century, William Holder (1616–1698) also observed how well this temperament would reproduce major thirds [29].

[6] Note that this measure depends on the reference octave, fixed by the frequency f_0 [38]. For sufficiently lower frequency ranges it is used $\Delta f \approx 3$ Hz, which stands as a lower bound called *just noticeable difference* [39].

 DOI: https://doi.org/10.1590/1806-9126-RBEF-2023-0270

We now design a measure for quantifying the average performance of an N-TET in producing a certain target set of pure intervals. Consider an ordered set of frequencies corresponding to just intervals $\mathcal{P}^M = \{\mathcal{P}_k^M\}_{k=0}^M$ with $M \leq 12$, where $\mathcal{P}_{k+1}^M > \mathcal{P}_k^M \; \forall \; k$. Using definitions (1) and (2) we address the following quantity:

$$\beta^M(N) = \frac{1}{\Delta f}\sqrt{\sum\nolimits_{k=0}^{M} \frac{\min\limits_{\ell \leq N} \left(\mathcal{F}_\ell^N - \mathcal{P}_k^M\right)^2}{M}}. \quad (3)$$

Then if $\beta^M(N) < 1$ frequency deviations are less than Δf in average, indicating that the corresponding N-TET properly fits the target set, whereas the $(N - M)$ extra intervals are labelled as intermediate notes. Note that if $\left||\mathcal{F}_{\ell+1}^N - \mathcal{P}_k^M| - |\mathcal{F}_\ell^N - \mathcal{P}_k^M|\right| < \Delta f$ for a certain $\ell \in [0, N]$, an evident ambiguity would take place when labelling which frequency corresponds to the target interval and a potentially useful intermediate note would be lost. Therefore, we sum the two differences in (3) as a measure of a non-desirable ambiguity.

For being an unique set, the chromatic just scale will be denoted by $\mathcal{P}^{12}$. With the intent of testing different target sets $\mathcal{P}^M$ with $M < 12$ we use subscripts as useful labels: Major and minor scales will be denoted by $\mathcal{P}_{\text{major}}^M$ and $\mathcal{P}_{\text{minor}}^M$ respectively, whereas uniform scales will be denoted by $\mathcal{P}_{\text{unif}}^M$. The corresponding parameter (3) of each addressed set will carry the same label (like β_{major}^M and so on).

The test whether $\beta^M(N) < 1$ or $\beta^M(N) > 1$ is read as *it is a good fit* or *it is not a good fit* respectively. For comparing within range $N \in [N_{\min}, N_{\max}]$ the average similarity between the results when addressing $\mathcal{P}^{12}$ and $\mathcal{P}^M$ as target sets, we design the following quantity:

$$\mathcal{F}\left(\mathcal{P}^M\right) = 1 - \sum_k \frac{\left|\Theta(\beta^M(k) - 1) - \Theta(\beta^{12}(k) - 1)\right|}{N_{\max} - N_{\min}}, \quad (4)$$

where $k \in [N_{\min}, N_{\max}]$ and Θ is the Heaviside function. Equation (4) then computes the coincidences between $\beta^M(N) < 1$ and $\beta^{12}(N) < 1$, and returns a value between 0 and 1. Evidently, $\mathcal{F}\left(\mathcal{P}^{12}\right) = 1$. For that, we call Eq. (4) *fidelity*. It allows us to compare which target sets most faithfully reproduce the results of the chromatic just scale.

3.2. Numerical results

We start by addressing the simplest divisions of the chromatic universe, which constitute the chromatic just scale $\mathcal{P}^{12}$, built only by semi-tone steps, and the whole-tone scale $\mathcal{P}_{\text{unif}}^6$, built only by tone steps (see Tab. 2). The corresponding values of parameter (3) are shown in Fig. 4(a) within range $N \in [9, 54]$. By addressing the chromatic just scale as target set we find TETs with $12, 19, 22, 24, 31, 34, 41$ and 53 intervals as suitable and non-ambiguous[7]. As for the parameter β_{unif}^6, corresponding to the uniform hexatonic (see Fig. 4(b)), the correct identifications of the proper N-TETs are not reproduced. This means that some configurations can reasonably adjust the whole-tone intervals, like the 18-TET for example, while making other intervals audibly out-of-tune.

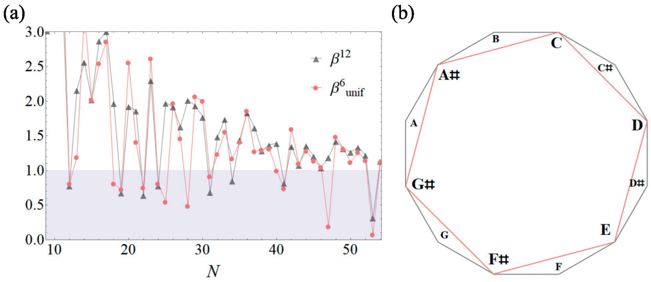

Figure 4: (a) Plots of Eq. (3) within range $N \in [9, 54]$ when addressing as target sets the chromatic just scale, built only by semi-tone steps, and the whole-tone scale, built only by tone steps, yielding β^{12} and β_{unif}^6 respectively. (b) Krenek diagram of the whole-tone scale.

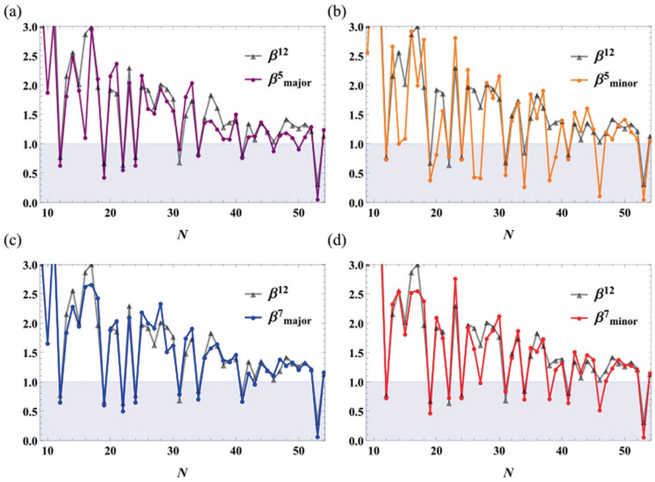

Figure 5: Comparisons between values of (3) for the chromatic scale (gray triangles) and: (a) the major pentatonic (purple circles), (b) the minor pentatonic (orange circles), (c) the major heptatonic (blue circles) and (d) the natural minor heptatonic (red circles).

In Figs. 5(a) and 5(b) (Figs. 5(c) and 5(d)) we plot parameter (3) for major and minor pentatonic (heptatonic) scales, respectively (see Fig. 2). Note that, when compared with the results achieved by addressing the chromatic just scale, major scales present more faithful answers about suitable N-TETs than the minor counterparts; the similarity increases with the scales

[7] Although it is a known fact that 29 is the lowest number of equal divisions that produces a better fifth than the 12-TET [18, 23], it does not turn into a good fit by our numerical search because of ambiguities created by neighboring equal-tempered frequencies around certain target pure intervals.

sizes as expected. Note that the 53-TET is the most accurate fit for all addressed subsets of the chromatic universe.

We computed fidelities (4) within range $N_{\min} = 9$ and $N_{\max} = 54$ by taking major, minor and uniform scales of different sizes as targets (see Fig. 2 and Tab. 2). The corresponding plot in Fig. 6 shows the difference between the convergences for each harmonic set: Major scales provide more faithful answers about proper N-TETs than minor and uniform ones of corresponding sizes, the latter being the most disadvantageous reference set. For example, the fidelities achieved when addressing the *major* pentatonic ($M = 5$) and the *minor* heptatonic ($M = 7$) as targets are comparable, both standing above the fidelity achieved when addressing the whole-tone scale ($M = 6$). Note in Fig. 6 that as we increase the scales sizes, thus addressing more information about pure intervals, we get more faithful identifications of the proper N-TETs for fitting the chromatic just scale with exception of the uniform case: The decreasing fidelity from $M = 3$ to $M = 4$ is legitimate because the uniform triad is not a subset of the uniform tetratonic.

Moreover, two observations suggest that the major and minor thirds are the most relevant ingredients for the convergence: (a) The main difference between major and minor scales of any size is the third interval, indicating that the extra major/minor sevenths are not crucial for the convergence, and (b) The uniform tetratonic, which produces the lower fidelity in Fig. 6, is the only uniform set which contains the minor third instead of the major one (see Tab. 2). The reason for a major thirds preference when setting a suitable equal-temperament, if not just a mathematical issue involving prime numbers, is quite mysterious but it seems to dialog with the fact that lower integer fractions correspond to the most appealing harmonies: The major and minor thirds are the nearest consonances from the tonic, moreover the 5/4 ratio is made of lower integers than the 6/5 ratio.

The observation that strategic harmonic subsets of the chromatic universe can store sufficient information for setting proper equal-temperaments for the whole chromatic just scale indicates that extensions of the 5-limit just intonation to more than twelve tones, which exhibits notable difficulty in the construction of more and more fractions between powers of prime numbers, may not be necessary. Our results suggest that suitable fits for major scales are preferable in advance. Evidently, the average faithfulness will depend on the range of interest: Note in Fig. 5(a) that if we wish to look for suitable N-TETs within range $N \in [9, 42]$ then the set of pure intervals corresponding to a major pentatonic would be sufficient for finding the proper solutions, i.e. $\mathcal{F}\left(\mathcal{P}^5_{\text{major}}\right) = 1$.

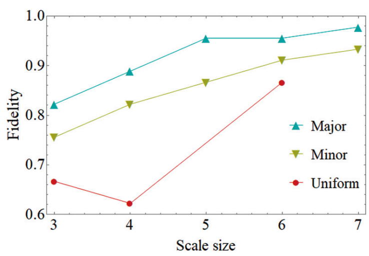

Figure 6: Comparison between fidelities for $N_{\min} = 9$ and $N_{\max} = 54$ when taking major, minor and uniform scales as target sets.

4. Conclusion

This work investigated whether there are harmonic sets which are universally prioritized when setting a suitable tuning system through an equal-tempered configuration. We designed a measure for finding which N-TETs properly and non-ambiguously fit a target set of pure intervals and investigated the role played by harmony in the convergence. By addressing the twelve-tone chromatic just scale as target set, we find within range $N \in [9, 54]$ that temperaments with $12, 19, 22, 24, 31, 34, 41$ and 53 divisions are suitable fits, all confirmed by literature [25–30] – for that, we believe that our method is a proper candidate for an universal measure of suitability of equal-tempered systems. On the other hand, by using M-element subsets of the chromatic universe with $M \in [3, 7]$, these solutions are always more faithfully reproduced when the target set is a major scale rather than a minor one. In addition, we find that uniform scales are the most disadvantageous reference sets for finding proper N-TETs numerically, suggesting that the very design of an equal-tempered configuration universally prioritizes the tuning of non-uniform harmonic scales over uniform ones. This feature suggests a mathematical background for understanding the universal preference for non-uniform sequences of notes in world music revealed by recent studies in ethnomusicology [1, 2, 4].

Acknowledgments

This work received financial support from the Brazilian agency Coordenação de Aperfeiçoamento de Pessoal de Nıvel Superior (CAPES) through its program (Grant). The author thanks Juan Carlos Chaves Capella and Irio Menezes Coutinho Abreu for useful discussions on the topics addressed in this manuscript.

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