Determination of the kinematic viscosity
by the liquid rise in a capillary tube
(Determinação da viscosidade cinética através da subida de um líquido num tubo capilar)

J.L.G. Santander[1], G. Castellano

Departamento de Ciencais Experimentales y Matemáticas, Universidad Católica de Valencia, Valencia, Spain

We discuss first the valid time interval of a formula found in the literature which gives the theoretical distance run by a liquid which rises along a capillary tube as a function of time. A non-linear least square fitting of experimental data to this theoretical curve allows the measurement of the kinematic viscosity of the liquid. The goodness of the fitting for three different sets of experimental data given in the literature is above 99%.
Keywords: viscosity, kinematic viscosity.

Discutimos inicialmente a validade do intervalo de tempo fornecido por uma fórmula encontrada na literatura, que dá a distância percorrida em função do tempo para um líquido que sobe num tubo capilar. Um ajuste não linear de mínimos quadrados dos dados experimentais com essa fórmula permite a medida da viscosidade cinética do fluido. A qualidade do ajuste para dois conjuntos de dados experimentais daliteraura é acima de 99%.
Palavras-chave: viscosidade, viscosidade cinética.

1. Introduction

The rise rate of a liquid in a capillary tube depends on its viscosity. However, the familiar laboratory experiments involving the rise of liquids in capillary tubes ignores this fact and the attention is focused on the stationary regime. Despite it is described in the literature a very simple method for the experimental determination of the kinematic viscosity using a capillary tube [1], this method does not take into account the effect of surface tension and does not give a statistical procedure for the determination of the experimental error. The falling sphere viscosimeter, based on the Stokes' law, allows the determination of the kinematic viscosity from the measurement of the terminal velocity of a sphere falling into a liquid [2, Eq. (4.9.20)], but we need to assure that the Reynolds number is small enough to apply the Stokes' law and to know how much time is needed for the falling sphere to reach the stationary regime. Other classical experimental setups for the determination of the kinematic viscosity are principally based on the liquid motion into a capillary [3], or the friction that exerts a liquid over a rotating cylinder [4]. Nevertheless, all these classical methods assume implicitly a stationary regime in the motion of the liquid, but there is a lack of discussion over this point.

In this work, we offer a method for the experimental determination of the kinematic viscosity of a liquid which is rising through a capillary tube, taking into account the transient regime. In order to perform this measurement, firstly we discuss the valid time interval of the theoretical distance run by the liquid along the capillary as a function of time [5,6] and its approximation to the Lucas-Washburn equation [7,8]. Secondly, the theoretical curve for the liquid rise allows us to perform a non-linear least square fitting to experimental data, so that we may determine the kinematic viscosity of the liquid. The goodness of the non-linear fitting for three different sets of experimental data given in the literature is above 99%.

2. Time-dependent theoretical rise

Let us consider a liquid of constant density ρ which is rising through a capillary tube of radius r inclined an angle β with respect to the vertical, as shows Fig. 1. The force F that the liquid experiments along the capillary is due to the surface tension γ upwards, and the weight of the liquid downwards, so that

$$F = 2\pi r\gamma \cos\theta - \rho g\pi r^2 \cos\beta \ s(t),\qquad(1)$$

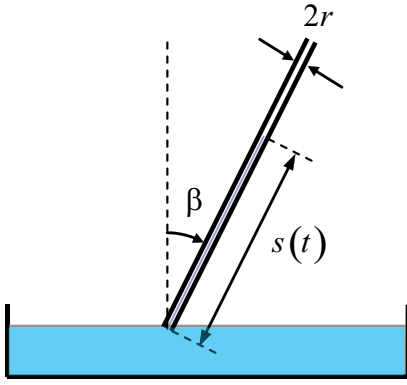[1]E-mail: martinez.gonzalez@ucv.es.

Figure 1 - Liquid rising through a capillary tube of radius r inclined an angle β with respect to the vertical.

where θ is the contact angle between the liquid and the capillary, and $s(t)$ is the distance run by the liquid along the capillary as a function of time t.

Therefore, according to Eq. (1), there is a pressure difference which makes the liquid rise along the capillary

$$\Delta P = \frac{F}{\pi r^2} = \frac{2\gamma \cos\theta}{r} - \rho g \cos\beta \, s(t). \qquad (2)$$

Poiseuille's law [9, Eq. (17.10)] gives an expression for the volume discharge Q of a liquid of dynamic viscosity η flowing in laminar regime through a pipe of circular cross section of radius r and length s, in which extrema we apply a pressure difference ΔP. Applying Poiseuille's law, according to Eq. (2), we have

$$Q = \frac{\pi r^4 \Delta P}{8\eta s} = \frac{\pi r^3}{4\eta}\left(\frac{\gamma\cos\theta}{s(t)} - \frac{\rho g r \cos\beta}{2}\right). \qquad (3)$$

Since most liquids in usual conditions are incompressible, the volume discharge Q is

$$Q = \pi r^2 \, s'(t). \qquad (4)$$

Equating Eq. (3) with Eq. (4), we have the following ODE,

$$s'(t) = \frac{r}{4\eta}\left(\frac{\gamma\cos\theta}{s(t)} - \frac{\rho g r \cos\beta}{2}\right). \qquad (5)$$

We may find Eq. (5) in Ref. [10] for a contact an inclination angles both null, $\theta = \beta = 0$. Assuming that the lower end of the capillary is initially just beneath the surface of the liquid, we have to take in Eq. (5) as initial condition,

$$s(0) = 0. \qquad (6)$$

In the stationary regime, the liquid does not rise any more, so that the distance covered by it is maximum, $s = s_{\max}$, and the velocity of the liquid is null, $s'(t) = 0$. Thus, according to Eq. (5), we have,

$$s_{\max} = \frac{2\gamma\cos\theta}{\rho g r \cos\beta}. \qquad (7)$$

Equation (7) is known as Jurin's law [11]. If we neglect the gravity term in Eq. (5), we have

$$\frac{\gamma\cos\theta}{s(t)} \gg \frac{\rho g r \cos\beta}{2},$$

so, according to Eq. (7),

$$s(t) \ll \frac{2\gamma\cos\theta}{\rho g r \cos\beta} = s_{\max}, \qquad (8)$$

and Eq. (5) becomes

$$\frac{1}{2}\frac{d}{dt}\left[s^2(t)\right] = s(t)\,s'(t) \approx \frac{\gamma r \cos\theta}{4\eta}. \qquad (9)$$

Taking into account the initial condition (6) we may solve Eq. (9), arriving to

$$s(t) \approx \sqrt{\frac{\gamma r \cos\theta}{2\eta}t}, \qquad (10)$$

which is known as Lucas-Washburn equation [7,8]. Notice that Eq. (10) is a good approximation when Eq. (8) is satisfied, that is, at the first stage of the liquid rise. Therefore, substituting Eq. (10) in Eq. (8), Lucas-Washburn equation is a good approximation when

$$t \ll \tau_0 := \frac{8\eta\gamma\cos\theta}{\rho^2 g^2 r^3 \cos^2\beta}. \qquad (11)$$

In the literature [5,6], we may find the explicit solution of Eq. (5)

$$s(t) = s_{\max}\left\{1 + W\left[-\exp\left(-1 - \frac{g r^2 \cos\beta}{8\nu s_{\max}}t\right)\right]\right\}, \qquad (12)$$

where ν is the kinematic viscosity,

$$\nu := \frac{\eta}{\rho}.$$

and the Lambert W function is the inverse function of ze^z [12]. Straightforward from the Lambert W function definition, we have $W\left(-e^{-1}\right) = -1$ and $W(0) = 0$ [12], so Eq. (12) satisfies the initial condition (6)

$$s(0) = s_{\max}\left[1 + W\left(-e^{-1}\right)\right] = 0,$$

and the stationary regime (7)

$$\lim_{t\to\infty} s(t) = s_{\max}\left[1 + W\left(-e^{-\infty}\right)\right] = s_{\max}.$$

There is a very simple connection between Eq. (10) and Eq. (12) that, as far as we know, is absent in the literature. If we define the function $\alpha(t)$ as the height proportion reached at time t,

$$\alpha(t) := \frac{s(t)}{s_{\max}}, \qquad (13)$$

and the parameter

$$\kappa := \frac{gr^2 \cos\beta}{8\nu s_{\max}}, \qquad (14)$$

we may rewrite Eq. (12) as

$$\alpha(t, \kappa) = 1 + W\left(-e^{-1-\kappa t}\right). \qquad (15)$$

Remembering the definition of the Lambert W function as the inverse function of ze^z, we may invert Eq. (15) obtaining

$$t(\alpha) = -\frac{1}{\kappa}\left[\alpha + \log(1-\alpha)\right]. \qquad (16)$$

In the first stage of the liquid rise, we have $\alpha \approx 0$, so we may take the following approximation in Eq. (16) $\log(1-\alpha) \approx -\alpha - \alpha^2/2$, thus,

$$t(\alpha) \approx \frac{\alpha^2}{2\kappa}. \qquad (17)$$

Inverting in Eq. (17) and taking into account Eq. (13), we arrive to

$$s(t) \approx s_{\max}\sqrt{2\kappa t}, \qquad t \gtrsim 0. \qquad (18)$$

Substituting now Eq. (7) and Eq. (14) in Eq. (18), we recover the Lucas-Washburn equation (10).

In order to know the valid time range for Lucas-Washburn equation, notice that from Eq. (12) we can define the following characteristic time for the liquid rise,

$$t_0 := \frac{1}{\kappa} = \frac{8\nu s_{\max}}{gr^2 \cos\beta} = \frac{16\eta\gamma\cos\theta}{\rho^2 g^2 r^3 \cos^2\beta}, \qquad (19)$$

so, according to Eq. (12), the proportion of distance covered with respect to $s_{\max}$ at the relaxation time t_0 is

$$\alpha(t_0) = 1 + W\left(-e^{-2}\right) \approx 84.14\%.$$

Therefore, the relaxation time t_0 defined in Eq. (19) gives us an idea of how rapid is the liquid rise within the capillary tube. Comparing Eq. (11) with Eq. (19), we may rewrite Eq. (11) as

$$t \ll t_0 = 2\tau_0,$$

that is, Lucas-Washburn equation is a good approximation when is much lesser than the characteristic time for the liquid rise, as we have considered before.

2.1. Limitations of the model

Notice that we have assumed Poiseuille's law during the whole rise of the liquid along the capillary. In fact, there is a transient regime for the liquid, which is initially at rest

$$s'(0) = 0, \qquad (20)$$

and the steady flow regime given by Poiseuille's law. Moreover, the model described by the differential equation (5) does not work at $t = 0$ since the substitution of the initial condition (6) in Eq. (5) does not give Eq. (20) but

$$\lim_{t \to 0} s'(t) = \infty. \qquad (21)$$

For a more detailed discussion about the ill-posedness of the model see Ref. [13]. Therefore, at the beginning of the liquid rise, there is a transient regime whose characteristic time is given by [2, Eq. (4.3.19)],

$$t^* = \frac{r^2}{\nu\lambda_1^2}, \qquad (22)$$

where $\lambda_1 \approx 2.41$ is the first positive root of the Bessel function of the first kind of order zero J_0. Therefore, in our results we must check out if $t_0 \gg t^*$, in order to be consistent in the use of Eq. (12).

It is worth noting as well that we have made the implicit assumption that the contact angle θ given in Eq. (7) for equilibrium (static contact angle) is equal to the contact angle during the liquid rise given in Eq. (5) (dynamic contact angle). In fact, this is not full satisfied, as it is could be found in the literature [14].

3. Curve fitting

Let us rewrite Eq. (16) as

$$y = \log(1-\alpha) + \alpha = -\kappa t. \qquad (23)$$

We may perform a linear fitting using Eq. (23) in order to obtain κ, and therefore ν according to Eq. (14). However, the linearization performed in Eq. (23) is not always advisable because violates the implicit assumption that the distribution of errors is normal [15]. Therefore, we may determine the parameter κ minimizing the quadratic residuals between the theoretical curve $\alpha(t, \kappa)$ Eq. (15), and the experimental data α_i Eq. (13),

$$F(\kappa) := \sum_{i=1}^{n} \left[\alpha(t_i, \kappa) - \alpha_i\right]^2,$$

so that

$$F'(\kappa_{\mathrm{fit}}) = 0. \qquad (24)$$

Equation (24) may be solved numerically, using as starting iteration the result obtained in the linear fitting. The uncertainty in κ may be obtained from the residual standard deviation of the function $\alpha(t, \kappa)$

$$\Delta\alpha = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left[\alpha(t_i, \kappa_{\mathrm{fit}}) - \alpha_i\right]^2},$$

If we consider that the variability of α is mainly due to the variability of κ

$$\Delta\alpha = \left|\frac{\partial\alpha}{\partial\kappa}\right|\Delta\kappa,$$

we have

$$\Delta\kappa_{\text{fit}} = \Delta\alpha \left|\frac{\partial\alpha}{\partial\kappa}\right|_{\text{fit}}^{-1}, \qquad (25)$$

where we can take the average of all the experimental points t_i,

$$\left|\frac{\partial\alpha}{\partial\kappa}\right|_{\text{fit}} = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{\partial\alpha(t_i,\kappa_{\text{fit}})}{\partial\kappa}\right|.$$

From Eq. (14), we may give an experimental measure of the kinematic viscosity, $\nu_{\text{exp}} \pm \Delta\nu_{\text{exp}}$, where

$$\nu_{\text{exp}} = \frac{g\,r^2\cos\beta}{8\,\kappa_{\text{fit}}\,s_{\text{max}}^{\text{exp}}}, \qquad (26)$$

and

$$\Delta\nu_{\text{exp}} = \left|\frac{d\nu_{\text{exp}}}{d\kappa}\right|\Delta\kappa_{\text{fit}} = \frac{g\,r^2\cos\beta}{8\,\kappa_{\text{fit}}^2\,s_{\text{max}}^{\text{exp}}}\,\Delta\kappa_{\text{fit}}. \qquad (27)$$

being $s_{\text{max}}^{\text{exp}}$ the experimental maximum distance run by the liquid. Notice that by using $s_{\text{max}}^{\text{exp}}$, we avoid using an experimental contact angle (7), because it is quite easier to measure $s_{\text{max}}^{\text{exp}}$ than the contact angle between the liquid and the capillary.

4. Experimental method

Figure 2 depicts the experimental setup that can be used for an undergraduate laboratory experience. The experimental procedure is as follows. First, assure that the glass strip attached to the clamp stand is vertical. Then, fix a capillary tube to the glass strip with plasticine. Take a tampered capillary tube, disinfect and pre-wet it. Knowing the glass strip length and using the graduated rules at both ends of it, measure the inclination angle β of the capillary. Prepare the liquid in a petri dish and put it on a stand of adjustable height. Align a high speed CCD camera and focus it on the base of the capillary tube. In order to preserve the initial condition (6), begin recording with the CCD camera, and then rise the Petri dish very slowly until the liquid starts rising into the capillary tube. In order to have enough experimental data points ($n \approx 20$), adjust the camera speed (frames per second) and extract stills from the recorded movies (1 frame out of every 20 up to 100) depending on the liquid speed. Analyze the stills with some commercial software, converting pixel distances into mm. For this purpose it can be used as a reference the graduated rule recorded during the liquid rise. To find the inner radius of the capillary, remove the capillary from the glass strip and cut it at where it was the meniscus. Use a caliber to measure the external radius of the capillary. Place it horizontally and zoom the camera to focus the cut end of the capillary. The image of the cross section of the capillary is then analyzed converting pixel distances into mm, with reference to the external radius of the capillary previously measured.
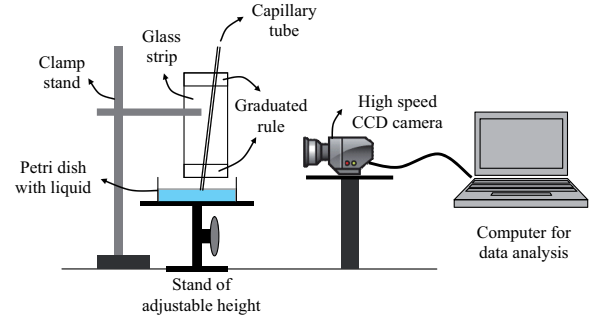

Figure 2 - Experimental setup for observing the liquid rise in a capillary tube.

5. Experimental evidence

Figure 3 shows the experimental data ($n = 10$) of a 55% sugar solution at 25 °C rising through a vertical capillary tube of radius $r = 10^{-4}$ m [10], and the error band within the theoretical curves $\alpha(t, \kappa_{\text{fit}} + \Delta\kappa_{\text{fit}})$ and $\alpha(t, \kappa_{\text{fit}} - \Delta\kappa_{\text{fit}})$. The experimental maximum height obtained was $s_{\text{max}}^{\text{exp}} = 0.123$ m. According to Eqs. (24) and (25)

$$\begin{aligned}\kappa_{\text{fit}} &= 5.352 \times 10^{-3}\ \text{s}^{-1}, \\ \Delta\kappa_{\text{fit}} &= 2.859 \times 10^{-4}\ \text{s}^{-1}.\end{aligned}$$

According to Eqs. (26) and (27), the experimental kinematic viscosity is

$$\nu_{\text{exp}} \pm \Delta\nu_{\text{exp}} = (1.862 \pm 0.099) \times 10^{-5}\ \text{m}^2\,\text{s}^{-1},$$

which agrees with the value given by Ref. [10], $\nu = 1.984 \times 10^{-5}$ m^2 s^{-1}. The adjusted coefficient of determination for the non-linear fitting is [16, Eq. (3.18)]

$$\bar{R} = 0.999625,$$

which indicates that the fitting is quite good. The characteristic time for the liquid rise (19) is

$$t_0^{\text{exp}} = \frac{8\nu_{\text{exp}}s_{\text{max}}^{\text{exp}}}{gr^2} = 188.2\ \text{s},$$

which is much more greater than the characteristic time for Poiseuille's law (22),

$$t_{\text{exp}}^* = \frac{r^2}{\nu_{\text{exp}}\lambda_1^2} = 9.22 \times 10^{-5}\ \text{s}.$$
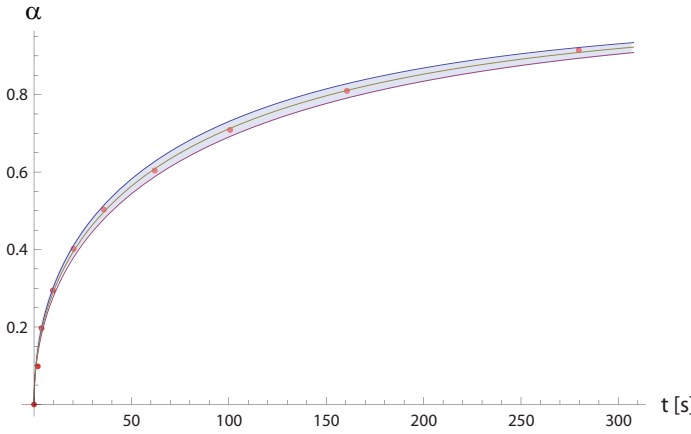
Figure 3 - Non-linear regression for experimental data given by [10].

Figure 4 shows the experimental data ($n = 16$) of water rising through a capillary tube of radius $r = 1.777 \times 10^{-4}$ m, inclined an angle $\beta = 45°$ [17], and the error band within the theoretical curves $\alpha\left(t, \kappa_{\text{fit}} + \Delta\kappa_{\text{fit}}\right)$ and $\alpha\left(t, \kappa_{\text{fit}} - \Delta\kappa_{\text{fit}}\right)$. The experimental maximum height obtained was $s_{\text{max}}^{\text{exp}} = 0.1217$ m. According to Eqs. (24) and (25)

$$\kappa_{\text{fit}} = 1.802 \times 10^{-1} \text{ s}^{-1},$$
$$\Delta\kappa_{\text{fit}} = 2.064 \times 10^{-2} \text{ s}^{-1}.$$

According to Eqs. (26) and (27), the experimental kinematic viscosity is

$$\nu_{\text{exp}} \pm \Delta\nu_{\text{exp}} = (1.247 \pm 0.143) \times 10^{-6} \text{ m}^2 \text{ s}^{-1},$$

which agrees with the value given by Ref. [18] within 10 °C and 20 °C ($\nu = 1.307 \times 10^{-6}$ m^2 s^{-1} for 10 °C and $\nu = 1.004 \times 10^{-6}$ m^2 s^{-1} for 20 °C). The adjusted coefficient of determination for the non-linear fitting is
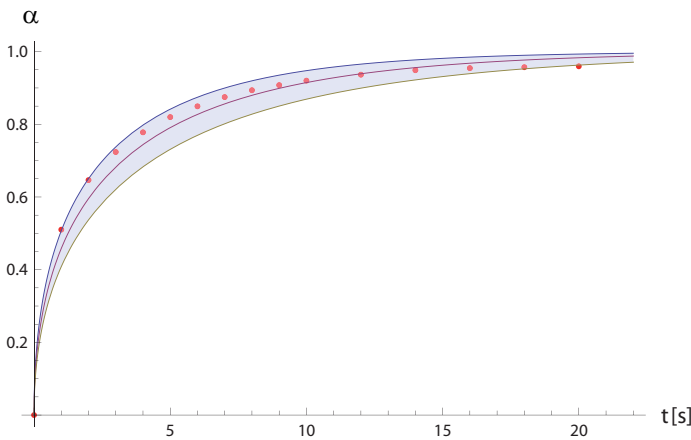
$$\bar{R} = 0.997866,$$

which indicates that the fitting is also good. The characteristic time for the liquid rise (19) is

$$t_0^{\text{exp}} = 5.63 \text{ s},$$

which is much more greater than the characteristic time for Poiseuille's law (22)

$$t_{\text{exp}}^* = 4.31 \times 10^{-3} \text{ s}.$$

Figure 5 shows the experimental data ($n = 22$) of silicone fluid rising through a vertical capillary tube of radius $r = 8.8 \times 10^{-5}$ m, inclined an angle $\beta = 57.7°$ [19], and the error band within the theoretical curves $\alpha\left(t, \kappa_{\text{fit}} + \Delta\kappa_{\text{fit}}\right)$ and $\alpha\left(t, \kappa_{\text{fit}} - \Delta\kappa_{\text{fit}}\right)$. The experimental maximum height obtained was $s_{\text{max}}^{\text{exp}} = 9.25$ cm. According to Eqs. (24) and (25)

$$\kappa_{\text{fit}} = 4.18 \times 10^{-2} \text{ s}^{-1},$$
$$\Delta\kappa_{\text{fit}} = 3.90 \times 10^{-3} \text{ s}^{-1}.$$

According to Eqs. (26) and (27), the experimental kinematic viscosity is

$$\nu_{\text{exp}} \pm \Delta\nu_{\text{exp}} = (1.31 \pm 0.12) \times 10^{-6} \text{ m}^2 \text{ s}^{-1}. \quad (28)$$

The value given by [19] is $\nu = 10^{-6}$ m^2 s^{-1} at 25 °C. This discrepancy could be explained because of the variation of viscosity with temperature. Let us assume that dynamic viscosity varies with temperature according to Andrade-Guzman equation [20, Eq. (4.15)]

$$\eta\left(T\right) = \eta_\infty \exp\left(\frac{T_0}{T}\right), \quad (29)$$

where T is the absolute temperature and η_∞ and T_0 are parameters which depend on the liquid considered. Knowing that at $T_1 = 25$ °C and at $T_2 = 0$ °C the density of the silicone fluid used in the experiment is $\rho\left(T_1\right) = 816$ kg m^{-3} and $\rho\left(T_2\right) = 840$ kg m^{-3}, and the kinematic viscosity is $\nu\left(T_1\right) = 10^{-6}$ m^2 s^{-1} and $\nu\left(T_2\right) = 1.68 \times 10^{-6}$ m^2 s^{-1} [21], then

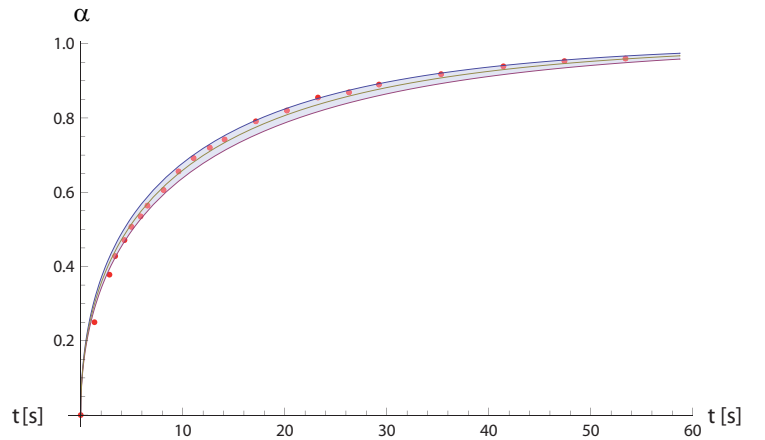Figure 4 - Non-linear regression for experimental data given by [17].

Figure 5 - Non-linear regression for experimental data given by [19].

$$\eta\left(T_1\right) = 8.16 \times 10^{-4} \text{ Pa s,} \tag{30}$$

$$\eta\left(T_2\right) = 1.41 \times 10^{-3} \text{ Pa s.} \tag{31}$$

Therefore, from Eqs. (30) and (31) we may calculate the parameters of Eq. (29)

$$T_0 = 1786 \text{ K,} \tag{32}$$

$$\eta_\infty = 2.04 \times 10^{-6} \text{ Pa s.} \tag{33}$$

Performing a linear interpolation of the density between T_1 and T_2 and taking into account Eq. (29) with the parameters found in Eq. (32) and Eq. (33), we may evaluate numerically the temperature at which the kinematic viscosity should be Eq. (28), obtaining a temperature for the experiment of $T_\text{exp} = (11.6 \pm 4.5)$ °C.

The adjusted coefficient of determination for the non-linear fitting is

$$\bar{R} = 0.997757,$$

which indicates that the fitting is good. The characteristic time for the liquid rise (19) is

$$t_0^\text{exp} = 23.9 \text{ s,}$$

which is much more greater than the characteristic time for Poiseuille's law (22)

$$t_\text{exp}^* = 1.02 \times 10^{-3} \text{ s.}$$

References

[1] M.G.C. Peiris and K. Tennakone, Am. J. Phys. **48**, 497 (1980).

[2] G.K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge Univ. Press, Cambridge, 1967).

[3] A.A. Elkarim, Am. J. Phys. **16**, 489 (1948).

[4] F.H. Hibberd, Am. J. Phys. **20**, 134 (1952).

[5] D.A. Barry, J.Y. Parlange, G.C. Sander and M. Sivaplan, Journal of Hydrology **142**, 29 (1993).

[6] N. Fries and M. Dreyer, Journal of Colloid and Interface Science **320**, 259 (2008).

[7] R. Lucas, Kolloid-Zeitschrift **23**, 15 (1918).

[8] E.W. Washburn, Physical Review **17**, 273 (1921).

[9] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Oxford, 1987).

[10] M.G.C. Peiris and K. Tenmakone, Am. J. Phys. **48**, 415 (1980).

[11] J. Jurin, Philosophical Transactions of the Royal Society **30**, 739 (1717-1719).

[12] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey and D.E. Knuth, Advances in Computational Mathematics **5**, 329 (1996).

[13] R. Fazio and A. Jannelli, Applied and Industrial Mathematics in Italy III **82**, 353 (2010).

[14] A. Siebold, M. Nardin, J. Schultz, A. Walliser and M. Oppliger, Colloids and Surfaces A **161**, 81 (2000).

[15] E.W. Weisstein, `http://mathworld.wolfram.com/LeastSquaresFitting.html`.

[16] S. Chatterjee, A. Hadi and B. Price, *Regression Analysis by Example* (Wiley, New York, 2000).

[17] W.E. Brittin, Appl. Phys. **17**, 37 (1946).

[18] D.R. Lide, *Handbook of Chemistry and Physics* (CRC Press, Florida, 2013), 94th ed.

[19] M. Stange, *Dynamik von Kapillarströmungen in Zylindrischen Rohren* (Cuvillier, Göttingen, 2004).

[20] D.S. Viswanath, T.K. Ghosh, D.H.L. Prasad, N.V.K. Dutt and K.Y. Rani, *Viscosity of Liquids. Theory, Estimation, Experiment, and Data* (Springer, Dordrecht, 2007).

[21] Silicone fluid technical data, `www.silicone.jp/e/catalog/pdf/kf96_e.pdf`.