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A problem regarding buoyancy of simple figures suitable for Problem-Based Learning

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A machine can make three-dimensional figures that can float on different types of liquids, using for each figure always the same quantity of mass. The machine had already made spheres when the engineers intended to design other figures that reached the same depth below the surface. They tested with cubes and realized that regardless of the liquid on which figures floated, spheres always reached a greater depth than cubes. Could their problem be solved using cones?

Keywords: Archimedes' principle, Problem-Based Learning.

1. Introduction

In elementary Fluid Mechanics, Archimedes' principle [1] is used for determining the equilibrium position of a floating body on a liquid. However, in Physics textbooks, it seems that Fluid Statics leads to simple problems when the geometry of the floating bodies is simple as well. The problem stated in the abstract appear to fall in the latter category, but as we will see later on, this is not the case. Despite the fact the statement of the problem is quite straightforward and easy to understand, it combines actually many topics, which at first glance are not evident. Moreover, the problem unfolds in different stages, so that is very suitable for Problem-Based Learning at undergraduate level [2]. Something that may puzzle students at the beginning is that no numeric data is available in the problem. Therefore, students are force to carry out the problem symbolically from the beginning to the end, which is a very good exercise for choosing an adequate nomenclature for the surprisingly large number of variables involved.

This paper is organized as follows. Section 2 is devoted to prove the assertion given in the problem about that spheres reach a greater depth than cubes. We will see that this problem is fortunately solved by using and aha! mathematical idea [3]. Section 3 develops the solution for a cone in terms of its half-aperture α as a function of the relative density ρ_r of the material of the cone with respect to the liquid on which floats. The final solution $\alpha\left(\rho_r\right)$ is a combination of the Physics involved in the problem with some mathematical developments used to neglect spurious solutions. Surprisingly, $\alpha\left(\rho_r\right)$ is a manifold solution, since we have two, one, or none solutions, depending on the range of ρ_r. In Section 4 a simple practice to be carried out by the students is proposed in order to verify the theoretical approach given above. Finally, we collect the conclusions in Section 5.

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2. Negative proof for the cube

In order to solve the problem, two assertions have to be considered: the figures float and each one contains the same mass of material.

Consider first that the figures float. If ρ denotes the density of the liquid, ρ' the density of the material of the figures, $\hat{V}$ the volume of the figure dipped into the liquid and V the volume of the figure, then, according to the Archimedes' principle, the buoyant force counteracts the weight of the figure, thus

$$\hat{V} = \rho_r V, \tag{1}$$

where $\rho_r = \rho'/\rho$ denotes the relative density of the figures with respect to the liquid. In the case of a cube of side a, consider $V_\square$ the total volume of it and $\hat{V}_\square$ the dipped volume, thereby

$$\begin{aligned} V_\square &= a^3, & (2)\\ \hat{V}_\square &= a^2\hat{h}_\square, & (3)\end{aligned}$$

where, according to Fig. 1, $\hat{h}_\square$ is the depth of the cube under the surface of the liquid. Applying (1) to (2) and (3), we obtain

$$\hat{h}_\square = \rho_r a. \tag{4}$$

We can perform the same calculation for a sphere of radius R_O (see Fig. 1), taking into account the formula of a spherical cap [4] of radius R_O and depth $\hat{h}_\mathrm{O}$, hence

$$\begin{aligned} V_\mathrm{O} &= \frac{4}{3}\pi R_\mathrm{O}^3,\\ \hat{V}_\mathrm{O} &= \frac{1}{3}\pi\,\hat{h}_\mathrm{O}^2\left(3R_\mathrm{O} - \hat{h}_\mathrm{O}\right).\end{aligned}$$

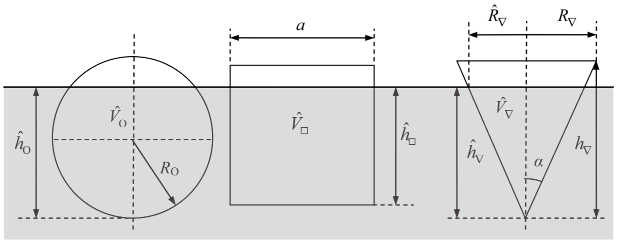

Figure 1: Nomenclature of floating figures.

Therefore, applying again (1) to (2) and (3), we arrive at

$$\hat{h}_\mathrm{O}^3 - 3R_\mathrm{O}\hat{h}_\mathrm{O}^2 + 4\rho_r R_\mathrm{O}^3 = 0. \tag{5}$$

Now, recall that all figures are made with the same quantity of mass, thus their volumes are also equal, $V_\square = V_\mathrm{O}$, so that

$$a^3 = \frac{4}{3}\pi R_\mathrm{O}^3 \quad \rightarrow \quad a = \beta R_\mathrm{O}, \tag{6}$$

where, for simplicity, we have defined the following numeric constant,

$$\beta = \sqrt[3]{\frac{4\pi}{3}} \approx 1.612. \tag{7}$$

Dividing (5) by $\hat{h}_\square^3$ and setting the dimensionless quantity

$$\xi = \frac{\hat{h}_\mathrm{O}}{\hat{h}_\square}, \tag{8}$$

we arrive at

$$\xi^3 - 3\xi^2\frac{R_\mathrm{O}}{\hat{h}_\square} + 4\rho_r\left(\frac{R_\mathrm{O}}{\hat{h}_\square}\right)^3 = 0. \tag{9}$$

To simplify (9), notice that from (4) and (6), we have

$$\hat{h}_\square = \rho_r\beta R_\mathrm{O} \quad \rightarrow \quad \frac{R_\mathrm{O}}{\hat{h}_\square} = \frac{1}{\rho_r\beta},$$

thus

$$\xi^3 - \frac{3\xi^2}{\rho_r\beta} + \frac{4}{\rho_r^2\beta^3} = 0. \tag{10}$$

According to the statement of the problem, the solutions of the cubic equation (10) must be greater than one, i.e. $\xi > 1$, in order to obtain $\hat{h}_\mathrm{O} > \hat{h}_\square$. Here we have two problems:

1. The general solution of a cubic equation is quite cumbersome.
2. We must prove that $\xi > 1$ for every possible value of ρ_r.

In order to surmount both difficulties, rewrite (7) as a quadratic equation for ρ_r, taking into account (6),

$$\xi\rho_r^2 - \frac{3}{\beta}\rho_r + \frac{3}{\pi\xi^2} = 0. \tag{11}$$

Notice that, in order to obtain a real value for ρ_r in (11), the discriminant must be greater than zero, thus

$$\frac{9}{\beta^2} - \frac{12}{\pi\xi} > 0 \quad \rightarrow \quad \xi > \frac{4\beta^2}{3\pi} \approx 1.214 > 1, \tag{12}$$

as we wanted to prove.

3. The cone solution

3.1. The basic solution

One the one hand, following the nomenclature given in Fig. 1, the volume of a cone and the part of it which is submerged is

$$V_\triangledown = \frac{\pi}{3}R_\triangledown^2 h_\triangledown, \tag{13}$$

$$\hat{V}_\triangledown = \frac{\pi}{3}\hat{R}_\triangledown^2 \hat{h}_\triangledown. \tag{14}$$

Notice that cones, unlike cubes and spheres, have two degrees of freedom in its shape, namely $R_\triangledown$ and $h_\triangledown$. Also, the geometry of cones provides the following equations:

$$\tan\alpha = \frac{R_\triangledown}{h_\triangledown} = \frac{\hat{R}_\triangledown}{\hat{h}_\triangledown}. \tag{15}$$

Therefore, according to (13), (14) and Archimedes' principle (1), we arrive at

$$\left(\frac{\hat{R}_\triangledown}{R_\triangledown}\right)^3 = \rho_r. \tag{16}$$

On the other hand, the equality of volumes between spheres and cones, $V_\mathrm{O} = V_\triangledown$, yields

$$\left(\frac{R_\mathrm{O}}{R_\triangledown}\right)^3 = \frac{1}{4\tan\alpha}, \tag{17}$$

which, taking into account (16), can be rewritten as

$$\left(\frac{R_\mathrm{O}}{\hat{R}_\triangledown}\right)^3 = \frac{1}{4\rho_r\tan\alpha}. \tag{18}$$

Notice also that according to (15) and (18), we have

$$\left(\frac{R_\mathrm{O}}{\hat{h}_\triangledown}\right)^3 = \left(\frac{R_\mathrm{O}/\hat{R}_\triangledown}{\hat{h}_\triangledown/\hat{R}_\triangledown}\right)^3 = \frac{\tan^2\alpha}{4\rho_r}. \tag{19}$$

Since we want the cone reaching the same depth under the liquid as the sphere, set the following dimensionless quantity equal to unity,

$$\eta = \frac{\hat{h}_\mathrm{O}}{\hat{h}_\triangledown} = 1.$$

Therefore, dividing (5) by $\hat{h}_\triangledown$ and taking into account (19), we arrive at

$$\eta^3 - 3\left(\frac{\tan^2\alpha}{4\rho_r}\right)^{1/3}\eta^2 + \tan^2\alpha = 0.$$

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Taking into account that $\eta = 1$, and solving for ρ_r, we get

$$\rho_r(\alpha) = \frac{27\tan^2\alpha}{4\left(1+\tan^2\alpha\right)^3}. \tag{20}$$

It is easy to prove that for the allowed angles of α in a cone, i.e. $0 < \alpha < \pi/2$,

$$0 < \rho_r(\alpha) < 1, \tag{21}$$

since

$$\rho_r'(\alpha) = \frac{27}{4}\cos^3\alpha \sin\alpha\,(3\cos 2\alpha - 1),$$

hence there are two minima at $\alpha_{\min} = 0, \pi/2$ and a maximum at $\alpha_{\max} = \cos^{-1}(1/3)/2$, for which $\rho_r(\alpha_{\min}) = 0$, and $\rho_r(\alpha_{\max}) = 1$. Physically speaking, since the cone floats, the relative density ρ_r must also be bounded as given in (21). Thereby, given ρ_r, we can design in principle the requested cone solving for α in (20). Fortunately, $\rho_r(\alpha)$ can be inverted as follows. First, perform the change of variables $z = \tan^2\alpha = \sec^2\alpha - 1$ and rewrite (20) as

$$\frac{4\rho_r}{27} = \frac{z}{(1+z)^3} = \left(\frac{1}{1+z}\right)^2 - \left(\frac{1}{1+z}\right)^3,$$

thus

$$u^3 - u^2 + \frac{4\rho_r}{27} = 0,$$

where $u = 1/(1+z) = \cos^2\alpha$. Recall now that a cubic equation [5]

$$u^3 + a\,u^2 + b\,u + c = 0,$$

admits three real solutions (trigonometric solution), $n = 0, \pm 1$,

$$\begin{aligned} u_n &= -\frac{a}{3} + 2\sqrt{P}\cos\left(\frac{2\pi n + \cos^{-1}\left(Q/\sqrt[3]{P}\right)}{3}\right), \\ P &= \frac{a^2-3b}{9}, \quad Q = \frac{ab}{6} - \frac{c}{2} - \frac{a^3}{27}, \end{aligned}$$

when $P > 0$, and the discriminant $D = P^3 - Q^2 > 0$. In our case, we have $P = 1/9 > 0$ and $Q = (1-2\rho_r)/27$, thus, according to (21), $D = 4\rho_r(1-\rho_r)/729 > 0$, and

$$u_n = \frac{1}{3} - \frac{2}{3}\cos\left(\frac{2\pi n + \cos^{-1}(2\rho_r - 1)}{3}\right). \tag{22}$$

Undoing the changes of variables, we arrive at

$$\alpha_n = \cos^{-1}\left(\sqrt{u_n}\right). \tag{23}$$

Mathematically speaking, for each ρ_r we have three possible angles α_n corresponding to $n = 0, \pm 1$. However, are all α_n physically allowable? To answer this question, note that from (21) it follows that $0 < \cos^{-1}(2\rho_r - 1) < \pi$, thus, taking $n = 0$, we have $1/2 < \cos\left(\frac{1}{3}\cos^{-1}(2\rho_r - 1)\right) < 1$. Therefore, $-1/3 < u_0 < 0$, and α_0 yields a complex number, being (22) reduced to

$$\alpha_\pm(\rho_r) = \cos^{-1}\left(\sqrt{\frac{1}{3} - \frac{2}{3}\cos\left(\frac{\cos^{-1}(2\rho_r - 1) \pm 2\pi}{3}\right)}\right). \tag{24}$$

Nonetheless, are always the $\alpha_\pm$ solutions still physically admissible? Not always. So far, we did not take into account the stability of the cone floating on the liquid. If the vertical position depicted in Fig. 1 for the cone is not stable, then the depth of the cone under the liquid will not be actually $\hat{h}_\triangledown$, but smaller.

3.2. Stability of the cone

The stability theory of floating bodies [6] asserts that if the *centre of gravity* of the body G is below the *metacenter* M (i.e. the *metacentric height* $\overline{GM} > 0$), then the floating body is stable, as shown in Fig. 2. In the case of floating figures with axial symmetry, the following criterion assures the stability

$$\left|\overline{GM}\right| > \left|\overline{GB}\right|, \tag{25}$$

where B is the *centre of buoyancy.*

3.2.1. Centre of gravity and buoyancy

According to the coordinate system XYZ shown in Fig. 3, the location of the centre of gravity G is on the Z-axis, due to the symmetry of the figure. Thereby, according to [7],

$$z_G = \frac{1}{M_\triangledown}\int_{V_\triangledown} z\, dm, \tag{26}$$

where the mass of the cone is given by

$$M_\triangledown = \rho' V_\triangledown. \tag{27}$$

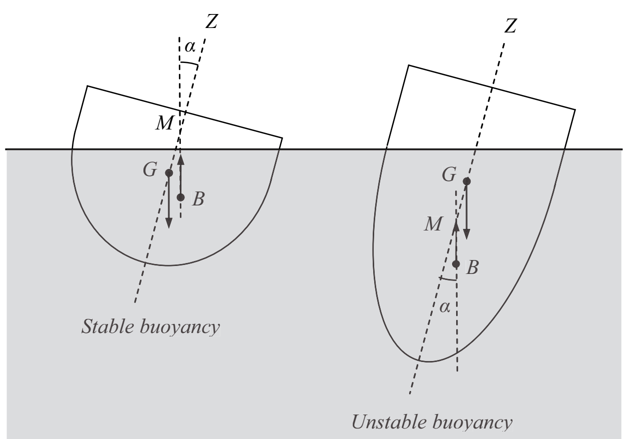

Figure 2: Metacentric height $\overline{GM}$ for stable and unstable buoyancy.

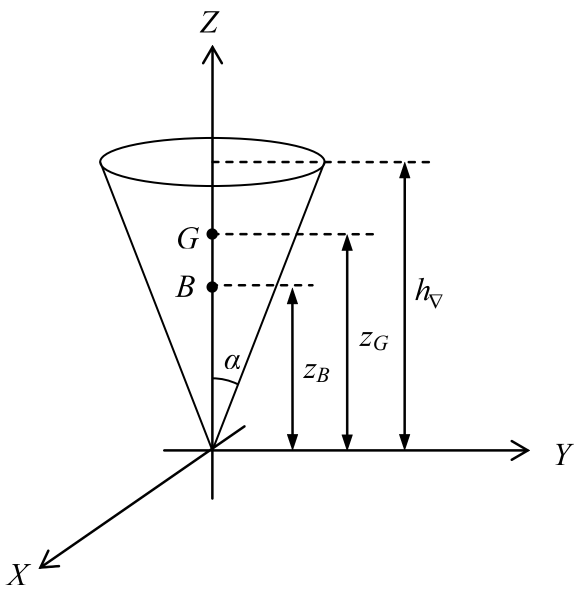

Figure 3: Centre of gravity G and centre of buoyancy B in a floating cone.

In order to calculate the volume integral given in (26), divide the volume of the cone as shown in Fig. 4, thus

$$dm = \rho' dV_\nabla = \rho' \pi \frac{R_\nabla^2}{h_\nabla^2} z^2 dz. \tag{28}$$

Substitute now (28) in (26), knowing that $\rho' = const.$, and take into account (13) and (27), to obtain

$$z_G = \pi \frac{\rho' R_\nabla^2}{M_\nabla h_\nabla^2} \int_0^{h_\nabla} z^3 \, dz = \frac{3}{4} h_\nabla. \tag{29}$$

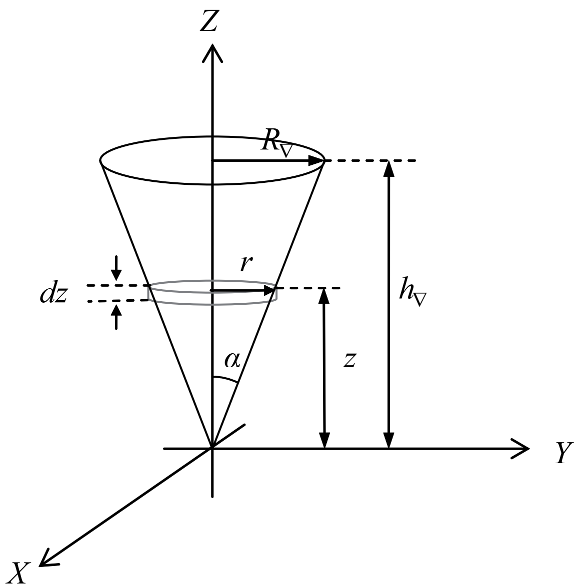

Figure 4: Integration parameters for the calculation of the centre of gravity and buoyancy.

The centre of buoyancy will be located at the centre of gravity of the volume of liquid displaced by the cone, $\hat{V}_\nabla$. Since $\hat{V}_\nabla$ is also cone-shaped, but of height $\hat{h}_\nabla$, similar to (29) the centre of buoyancy will be located at

$$z_B = \frac{3}{4} \hat{h}_\nabla. \tag{30}$$

According to (29) and (30), and knowing that the cone floats over the liquid, i.e. $h_\nabla > \hat{h}_\nabla$, we have

$$\left|\overline{GB}\right| = |z_G - z_B| = \frac{3}{4} \left(h_\nabla - \hat{h}_\nabla \right). \tag{31}$$

3.2.2. Metacentric height

When a symmetric body is tilted through a small angle around its symmetry axis at waterline level (see the MN line in Fig. 5), then the metacentric height is given by [6]

$$\left|\overline{GM}\right| = \frac{1}{\hat{V}_\nabla} \int_A x^2 dA. \tag{32}$$

where A is the waterline area. To calculate the surface integral given in (32), we can perform a polar change of coordinates, thus

$$\int_A x^2 dA = \int_0^{2\pi} \cos^2 \vartheta \, d\vartheta \int_0^{\hat{R}_\nabla} \varrho^3 \, d\varrho = \frac{\pi}{4} \hat{R}_\nabla^4. \tag{33}$$

Substituting (33) and (14) in (32), we arrive at

$$\left|\overline{GM}\right| = \frac{3}{4} \frac{\hat{R}_\nabla^2}{\hat{h}_\nabla}. \tag{34}$$

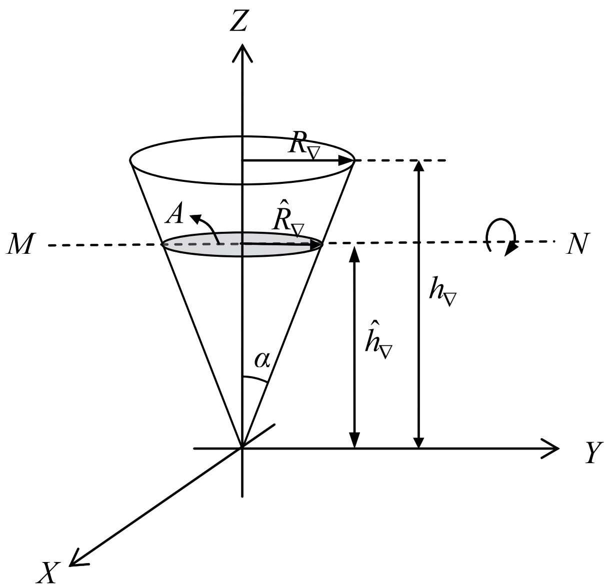

Figure 5: Stability of a floating cone tilted around the MN line.

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3.2.3. Stable solutions

Therefore, inserting the results (31) and (34) in the stability criterion given in (25), we have

$$\frac{\hat{R}_{\nabla}^{2}}{h_{\nabla}} > h_{\nabla} - \hat{h}_{\nabla}. \tag{35}$$

Setting now χ as the following dimensionless quantity,

$$\chi = \frac{\hat{h}_{\nabla}}{h_{\nabla}}, \tag{36}$$

and taking into account (15), we can rewrite (35) as

$$\tan^2 \alpha\ \chi^2 + \chi - 1 > 0. \tag{37}$$

However, note that, from (15) and (16), we have

$$\chi^3 = \rho_r, \tag{38}$$

hence, substituting (38) in (37) and solving for α, we have

$$\alpha > \alpha_s(\rho_r) = \tan^{-1}\left(\sqrt{\rho_r^{-2/3} - \rho_r^{-1/3}}\right), \tag{39}$$

where we have defined a limiting function $\alpha_s(\rho_r)$ for the stability of the cone.

Fig. 6 shows the plots for $\alpha_{\pm}(\rho_r)$ and $\alpha_s(\rho_r)$. Note that, according to (39), the solution is unstable under the graph of $\alpha_s(\rho_r)$. Numerically, we can compute easily the crossing points $\rho_r^{\pm}$ between $\alpha_{\pm}(\rho_r)$ and $\alpha_s(\rho_r)$ respectively, taking as initial bracketing (21) [8], obtaining

$$\begin{aligned} \rho_r^- &\approx 1.745 \times 10^{-3}, \\ \rho_r^+ &\approx 6.806 \times 10^{-1}. \end{aligned}$$

Therefore, the stable solutions for designing the floating cone are

$$\alpha(\rho_r) = \begin{cases} \alpha_-(\rho_r), & \rho_r^- < \rho_r < \rho_r^+ \\ \alpha_{\pm}(\rho_r), & \rho_r^+ < \rho_r < 1 \end{cases} \tag{40}$$

It is remarkable that the solution is unique in the region $\rho_r^- < \rho_r < \rho_r^+$. Outside this region, there is a small region ($0 < \rho_r < \rho_r^-$) wherein there is no solution, and surprisingly a region ($\rho_r^+ < \rho_r < 1$) wherein two solutions are found.

4. Experimental practice

As aforementioned in the Introduction, here we present a practice to be carried out by the students in the laboratory in order to design floating figures and thereby verify the theoretical predictions described above. We propose to use wax as the material for the figures and water as the liquid since both substances are cheap and easy to manage. The aim of the practice is to design a sphere and a cone of the same volume that reach the same depth under the surface.

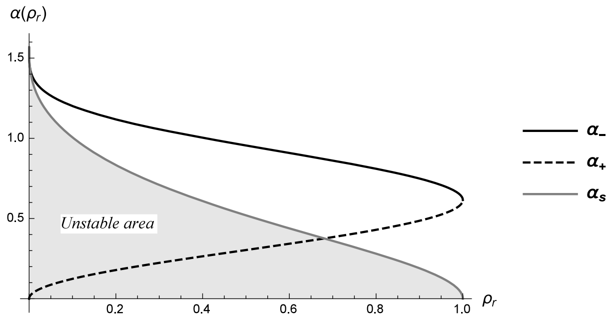

Figure 6: Allowable angles α for a stable floating cone.

First, we propose to the students to measure the density of the wax and the water they are going to use, and to obtain from these measurements the relative density. Here we provide an example.

$$\begin{aligned} \rho' &= 930 \text{ kg m}^{-3}, \\ \rho &= 988 \text{ kg m}^{-3}, \end{aligned}$$

thus the relative density is

$$\rho_r = \frac{\rho'}{\rho} = 0.932. \tag{41}$$

It is advisable to measure the density of the wax, once it is melted, because it might change significantly.

Second, for designing a sphere of a given radius, for instance

$$R_{\mathrm{O}} = 2.0 \times 10^{-2} \text{ m},$$

we can provide the students with different sizes of spherical molds. (Spherical molds made of plastic can be found in commercial packs for certain type of toys.) It is important to advise the students that they probably need to wait some hours to cool down the melted wax in the mold, depending on the size of the figure.

Third, we ask the students for computing the half-aperture of the cone, inserting (41) in (40),

$$\alpha(\rho_r) = \begin{cases} 0.725 \text{ rad} \\ 0.509 \text{ rad} \end{cases}$$

Notice that the selection of water and wax as working materials have led to a double solution in $\alpha(\rho_r)$, so that we can ask the students for designing the two cones and verify if both of them reach the same depth under water as the sphere. In order to design a cone with a given half-aperture, we can use the fact that the planar development of the cone is a circular sector.

Therefore, according Fig. 7, we need to know the generatrix length of the cone g_{∇} and the angle γ. Since $\ell_{\nabla} = g_{\nabla}\gamma = 2\pi R_{\nabla}$ and $\sin\alpha = R_{\nabla}/g_{\nabla}$, it follows that

$$\gamma = 2\pi \sin\alpha = \begin{cases} 4.17 \text{ rad} \\ 3.06 \text{ rad} \end{cases}$$

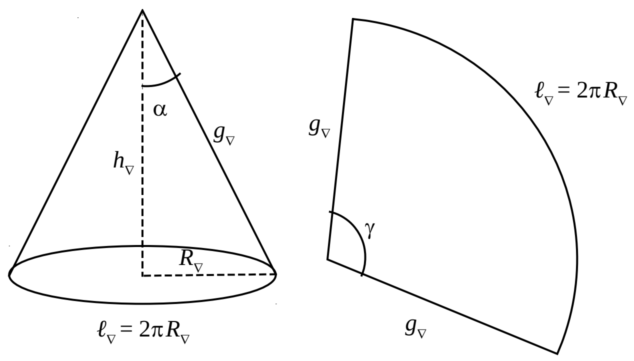

Figure 7: Stability of a floating cone tilted around the MN line.

Also, from (17) we have

$$R_\nabla = (4\tan\alpha)^{1/3}\, R_\mathrm{O},$$

hence, taking into account again that $\sin\alpha = R_\nabla/g_\nabla$, we arrive at

$$g_\nabla = 2R_\mathrm{O}\left(\csc 2\alpha \csc\alpha\right)^{1/3} = \begin{cases} 4.60\times 10^{-2}\ \mathrm{m} \\ 5.36\times 10^{-2}\ \mathrm{m} \end{cases}$$

Given g_∇ and γ is easy to make a mold with cardboard and refill it with melted wax.

Finally, we have to put our figures on the water of a glass container and see what happens. From my personal experience in the laboratory with the students, I can say that it is not easy for them to get enough accuracy in the cone and the sphere design to observe with the naked eye the same depth under the water. Nevertheless, this practice, as a final result of the previous theoretical work, has been very successful in terms of Problem-Based Learning.

5. Conclusions

Considering floating figures of the same material and mass, it has been shown that spheres always reach a greater depth beneath the surface than cubes, regardless the relative density of the material with respect to the liquid on which both float. The proof rests on an aha! idea, which consists in rewriting the cubic equation (10) as a quadratic one (11), and then using the discriminant of the latter.

The solution for the half-aperture of a cone reaching the same depth below the surface than a sphere has been also studied. The half-aperture has been given as a function of the relative density. To reach this solution, the analytical inversion of (20) turned out to be essential in order to discard complex spurious solutions, as well as unstable solutions. Eqns. (24) and (40) collect the allowable half-aperture angles for cones. Surprisingly, there is a region for which two solutions are possible.

Finally, we have proposed a practice to be carried out by students of undergraduate level in order to verify the theoretical predictions given above. We provide the guideline of the practice, which is quite easy and cheap to implement in the laboratory.

Acknowledgements

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