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Notes

Moment of inertia through scaling and the parallel axis theorem

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A right triangular plate scaled by a factor two generates a bigger plate composed of four triangular plates the same size as the original. By using dimensional analysis we write the moment of inertia around the center of mass, for the original and the bigger plate, in terms of a common unknown parameter. Through the parallel axis theorem we relate the moments of inertia of both plates and finally solve a very simple equation to find out the unknown parameter. This procedure avoids to calculate integrals. The result is extended to a scalene triangular plate by recognizing it is composed of two right triangular plates. We also review the parallel axis theorem in an appendix.
Keywords: Moment of inertia, scaling, parallel-axis theorem.

1. Introduction

Efforts to introduce physics concepts with minimum calculus knowledge is something necessary in teaching careers others than engineering or science. It is something convenient even for science and engineering students with insufficient domain of calculus. In the mechanics course they are introduced to physics concepts as center of mass and moment of inertia which involve integrals, but integrals are introduced in the course of calculus, normally the same semester as those physics concepts. Here we propose an alternative to integral calculus in determining the moment of inertia of some plane figures, with help of the parallel axis theorem. Other alternative methods to avoid integrals are useful in cases of very symmetric bodies, and they also use the parallel axis theorem [1-2] or the perpendicular axis theorem and concepts of differential calculus [3].

A few years ago we proposed a method to find out the center of mass of a right triangular plate by using scaling and the parallel axis theorem instead of explicit integrals [4-5]. Now we propose a similar method in order to get the moment of inertia of a right triangular plate and apply the result to discuss the case of a scalene triangular plate. Also, at the end we include an appendix with a review of the parallel axis theorem [5-6]. So with help of the parallel axis theorem we have built a calculation path that begins with the intuitive knowledge of the center of mass of two equal masses, goes through the center of mass of triangular plates, and ends with their moments of inertia.

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2. The case of a bar

At first we review the moment of inertia of a homogeneous bar with length L and mass M, around an axis going through the center of mass and being perpendicular to the bar [1]. According to dimensional analysis we assume the moment of inertia I relates to length and mass by the model relation $I = k\,M\,L^2$, with k being unknown and dimensionless whose value we propose to find out through a procedure involving scaling and the parallel axis theorem. Next the model is applied to a homogeneous bar of length *2L* and mass *2M*, accordingly the moment of inertia of this bigger bar around a perpendicular axis going through its center of mass is $I_2 = k \cdot 2M \cdot (2L)^2 = 8k\,M\,L^2$. Note that the center of mass of this longer bar is a point located in the extremes of a couple of bars with length L. Then, other way to write the moment of inertia of the longer bar is by considering two bars with length L and the moment of inertia around an axis in their extremes. According to the parallel axis theorem [4], the moment of inertia for a bar of length L around an axis going through an extreme is $I_e = I + M\ (\mathrm{L}/2)^2$. Then, the moment of inertia of the longer bar, $I_2 = 2I_e$ can be written in terms of I as follows,

$$I_2 = 2I + 2M(L/2)^2. \tag{1}$$

By introducing the previous expressions for I and I_2 into Eq. (1) we get $k = \frac{1}{12}$, therefore the moment of inertia of a homogeneous bar is $I = \frac{1}{12} M\,L^2$.

3. A right triangular plate

The procedure we have just described in order to find the center of mass of a bar can be applied to other solids. Here we will describe the case of a homogeneous right triangular plate with equal cathetus. The plate and a convenient set of rectangular coordinates xy are shown in Fig. 1.

As before, we write the moment of inertia I around a perpendicular axis going through the center of mass of the triangular plate in Fig. 1, in terms of its mass M and length L of its cathetus. The proposed relationship is $I = k\,M\,L^2$, with k being unknown. As we know and will use forward, the center of mass of this plate is the point with coordinates $(L/3,\,L/3)$. Then we scale to a plate bigger by a factor 2, with cathetus $2L$, mass $4M$, and moment of inertia given by $I_2 = k \cdot 4M \cdot (2L)^2$ according to the proposed model. Inside the bigger plate we distinguish four smaller plates named A, B, C and D, all of them with cathetus L and mass M, as shown in Fig. 2.

Now we use the parallel axis theorem in order to write the moment of inertia I_2 around the center of mass of the bigger plate, in terms of moments of inertia corresponding to smaller plates A, B, C and D.

In Table 1 we synthesize data on the center of mass of this plates, in terms of coordinates $(x_i,\,y_i)$ defined in Fig. 2 and coordinates $(x'_i,\,y'_i)$ relatives to the center of mass of the bigger plate, with i being the index to identify each of the plates A, B, C and D. The last row in Table 1 contains the corresponding data for the bigger plate. The last column in Table 1 are distance between the center of mass of the bigger plate and the corresponding center of mass of each smaller plate. They are obtained from the relative coordinates$(x'_i,\,y'_i)$ and are the distances required in order to use the parallel axis theorem. As an example, we use data from Table 1 and the parallel axis theorem to identify that contribution of plate A to the moment of inertia I_2 of the bigger plate is $I + M\left(\sqrt{5}L/3\right)^2$.

Table 1: Coordinates and distances.

	Center of mass coordinates		
Plate	$(x_i,\,y_i)$	$\left(x'_i,\,y'_i\right)$	Distance
A	$(L/3,\,4L/3)$	$(-L/3,\,2L/3)$	$\sqrt{5}L\big/3$
B	$(L/3,\,2L/3)$	$(-L/3,\,0)$	$L/3$
C	$(2L/3,\,L/3)$	$(0,\,-L/3)$	$L/3$
D	$(4L/3,\,L/3)$	$(2L/3,\,-L/3)$	$\sqrt{5}L\big/3$
(A+B+C+D)	$(2L/3,\,2L/3)$	$(0,\,0)$	0

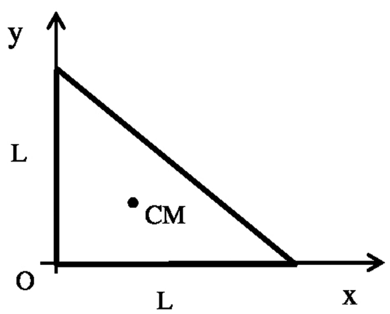

Figure 1: Right triangular plate

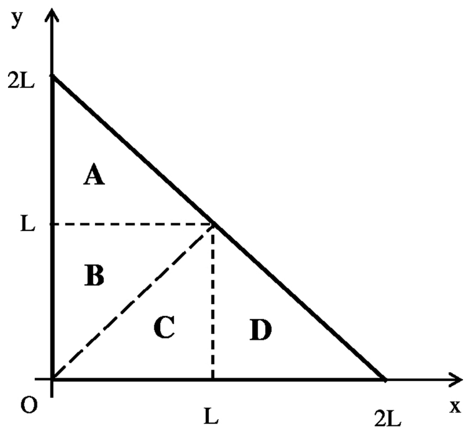

Figure 2: Bigger triangular plate

In considering contributions of all the plate components to the moment of inertia I_2 of the bigger plate, we have,

$$I_2 = 4I + 12M(L/3)^2. \tag{2}$$

By introducing previous expressions for I and I_2 into Eq. (2), we get a simple equation which when solved gives $k = \frac{1}{9}$. Therefore, the moment of inertia of the homogeneous triangular plate in Fig. 1, with mass M and cathetus L is $I = \frac{1}{9}M\,L^2$.

In order to practice the procedure, we suggest the reader to study a right triangular plate of cathetus L and H. With the model relation $I = M(\alpha\,L^2 + \beta\,H^2)$ it could be found $\alpha = \beta = \frac{1}{18}$, so the moment of inertia of a right triangular plate of cathetus L and H is $I = \dfrac{1}{18}M(\,L^2{+}H^2)$. This result and the parallel axis theorem provides an easy way to get the moment of inertia of a scalene triangular plate and other geometric figures.

4. A scalene triangular plate

Finally, we apply our discussion to get the moment of inertia of a scalene triangular plate of mass M with help of coordinate axis chosen as shown in Fig. 3. There we distinguish two right triangular plates with center of mass LC at $(-a/3,\,c/3)$ and RC at $(b/3,\,c/3)$ which compose the scalene triangular plate with center of mass CM at $((b-a)/3,\,c/3)$. In order to apply the parallel axis theorem we note that distances to CM are $b/3$ from LC and $a/3$ from RC. Because the plate is homogeneous, mass of component plates are proportional to their respective areas, so mass of component on the left side is $Ma/(a+b)$ and mass of component on the right side is $Mb/(a+b)$. By using our previous findings and in applying the parallel axis theorem we conclude that the moment of inertia of the scalene triangular plate in Fig. 3 is:

$$I_2 = \frac{1}{18}M(a^2 + ab + b^2 + c^2). \tag{3}$$

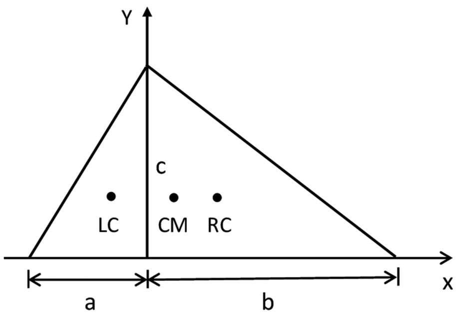

Figure 3: Scalene triangular plate

5. Conclusions

By using dimensional analysis we have assumed the moment of inertia of a right triangular plate of equal cathetus is a product of the mass M, the squared length L and an unknown dimensionless parameter k. Analogously, the moment of inertia of a bigger plate obtained by scaling the original by a factor two is written in terms of parameter k, length 2L and mass 4M. This is because we recognize the bigger plate as composed of four plates of the original mass and size. Then we use the parallel axis theorem to write the moment of inertia of the bigger plate in terms of the original plate. Two expressions for the moment of inertia corresponding to the bigger plate provide a simple equation to determine the unknown dimensionless parameter. So, the moment of inertia around the center of mass of a right triangular plate results $I = \frac{1}{9} M\, L^2$, in agreement with the well-known result obtained by using integral calculus. The method has been applied successfully to a right triangular plate with unequal cathetus and we have used this result in order to get the moment of inertia of a scalene triangular plate.

Supplementary material

The following online material is available for this article:

Appendix - A very simple proof of the parallel axis theorem

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