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Motion under a time-dependent driving force plus a linear velocity-dependent friction: From the unilateral Fourier transform to the Green function

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The problem of a particle subject to a time-dependent driving force plus a linear velocity-dependent friction can be addressed by utilizing the unilateral Fourier transform, despite the presence of derivatives of odd and even order in the differential equation. This technique yields a system of algebraic equations that combine the Fourier sine and cosine transforms. While this method is useless for solving the homogeneous equation, it can be effectively used to obtain an integral representation of the particular solution. Remarkably, this integral representation of the particular solution is expressed in terms of the Green function. This type of exactly solvable problem is relevant for students who are studying mathematical methods applied in the fields of physics and engineering at the undergraduate level, as it can serve as a useful illustration of how unilateral Fourier transforms can be employed to solve problems and to develop an understanding of Green functions, even in introductory calculus courses.

Keywords: Velocity-dependent friction, Unilateral Fourier transform, Green function.

1. Introduction

Integral transforms are powerful mathematical operations that facilitate the conversion of a function defined on one variable into a corresponding function defined on another variable. These transforms find wide-ranging applications in various fields. They are used to evaluate definite integrals, convert complex partial differential equations into simpler ordinary differential equations, transform ordinary differential equations into more manageable differential or algebraic equations, and play a crucial role in tackling theoretical aspects of applied problems.

The unilateral Fourier transform, also known as the one-sided Fourier transform, is a frequently employed mathematical tool for solving problems involving absolutely integrable functions over a semi-infinite interval, and has found extensive applications in causal signal processing and communication theory (see, e.g. [1–3]). However, it is important to be attentive to the appropriate homogeneous boundary (initial) conditions at the origin when using this method. The Fourier sine or cosine transforms should be used depending on whether the Dirichlet or Neumann boundary condition is satisfied at the origin. It is unfortunate that some authors overlook these important boundary conditions [4–13] (see [14] for a criticism), and there can also be issues with the interrelation between sine and cosine transforms in their derivative properties. Regarding this fact, Butkov mentions "The above result then indicates that the Fourier cosine and sine transforms are convenient under certain special conditions, such as in the absence of the derivatives of odd or even order in the differential equations ..." [15]. Notwithstanding these issues, the unilateral Fourier transform has been shown to be a useful tool for solving bound-state solution problems in non-relativistic quantum mechanics (see, e.g. [16–19]), and even for the homogeneous differential equation of the classical harmonic oscillator (in the sense of the Dirac delta distributions) [20]. To be accurate, all the aforementioned problems were solved under the convenient "special conditions" alluded by Butkov.

In this work, we venture to utilize the unilateral Fourier transform to solve a first-order ordinary differential equation that entails a combination of derivatives of both odd and even orders, and we obtain the Green function in the process.

The Green-function method is well-documented in textbooks (see, e.g. [15, 21–25]) and didactic papers (see, e.g. [26–31]), and has been shown to be intimately related to the integrating factor for first-order differential equations [32].

The problem we approach here is of practical interest to describe the motion of a particle subject to a time-dependent driving force plus a possible linear velocity-dependent friction ($dv(t)/dt + \gamma v(t) = f(t)$, $\gamma \geq 0$). While there are simpler methods available for solving this problem, incorporating unilateral Fourier transforms early on can provide several advantages. By employing this approach, even individuals with a

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basic understanding of first-order ordinary differential equations can gain valuable insights. This method can be particularly useful as an educational tool in science and engineering courses, introducing novice students not only to the unilateral Fourier transform of an exactly solvable problem but also to the concept of Green functions.

2. Concise Overview of the Unilateral Fourier Transform

To start, we will provide a concise overview of the unilateral Fourier transform and some of its essential characteristics. The direct Fourier sine transform of $y_s(x)$ and the direct Fourier cosine transform of $y_c(x)$ are represented by $\mathcal{F}_s\{y_s(x)\}$ and $\mathcal{F}_c\{y_c(x)\}$, respectively. These linear transforms are defined by the integrals (see, e.g. [15, 21–25])

$$\begin{aligned} \mathcal{F}_s\{y_s(x)\} &= Y_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty dx\, y_s(x) \sin kx, \\ \mathcal{F}_c\{y_c(x)\} &= Y_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty dx\, y_c(x) \cos kx. \end{aligned} \tag{1}$$

The original functions $y_s(x)$ and $y_c(x)$ can be reconstructed using the inverse unilateral Fourier transforms $\mathcal{F}_s^{-1}\{Y_s(k)\}$ and $\mathcal{F}_c^{-1}\{Y_c(k)\}$ expressed as

$$\begin{aligned} y_s(x) &= \mathcal{F}_s^{-1}\{Y_s(k)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty dk\, Y_s(k) \sin kx, \\ y_c(x) &= \mathcal{F}_c^{-1}\{Y_c(k)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty dk\, Y_c(k) \cos kx. \end{aligned} \tag{2}$$

The unilateral Fourier transform can be derived using the real form of the Fourier integral theorem, which applies to functions defined on the entire axis. Once the unilateral Fourier transform and its inverse are established, the behaviour of the functions on the other side of the axis becomes irrelevant. However, it is of the greatest importance to recognize that $y_s(x)$ and $y_c(x)$ are functions of different natures. The function $y_s(x)$ obtained by means of $Y_s(k)$ must meet the homogeneous Dirichlet boundary condition at the origin, whereas the function $y_c(x)$ obtained by means of $Y_c(k)$ must satisfy the homogeneous Neumann boundary condition at the origin:

$$y_s(x)\big|_{x=0} = \left.\frac{dy_c(x)}{dx}\right|_{x=0} = 0. \tag{3}$$

When dealing with the problem of satisfying boundary conditions at infinity, it is important to recognize that functions susceptible to the unilateral Fourier transform extend beyond those that approach zero as x becomes large. The key is to consider the Dirac delta symbol $\delta(x)$ defined by its assigned properties (see, e.g. [15, 21–25]):

$$\begin{gathered} \delta(x) = 0 \text{ for } x \neq 0, \\ \textstyle\int_{-\infty}^{+\infty} dx\, F(x)\delta(x - x_0) = F(x_0), \end{gathered} \tag{4}$$

for any function $F(x)$ continuous at x_0. By substituting (2) into (1), we find:

$$\begin{aligned} \delta(k - k') &= \frac{2}{\pi} \int_0^\infty dx\, \sin kx \sin k'x \\ &= \frac{2}{\pi} \int_0^\infty dx\, \cos kx \cos k'x. \end{aligned} \tag{5}$$

This last result implies that not only absolutely integrable functions are suitable for the unilateral Fourier transform. This point of view, with the functions $\sin \alpha x$ and $\cos \alpha x$, was used in [20] for solving the homogeneous differential equation of the classical harmonic oscillator.

To solve differential equations efficiently, it is vital to understand the method of expressing the unilateral Fourier transforms of the derivatives of $y_s(x)$ and $y_c(x)$ in terms of $y_s(x)$ and $y_c(x)$ themselves, while taking into account the relevant boundary conditions at the origin that are automatically incorporated in the approach. More specifically, when dealing with first-order derivatives, the following connections arise through the process of partial integration, where boundary terms vanish at infinity (if this do not hold, the functions can treated as Dirac delta distributions):

$$\mathcal{F}_s\left\{\frac{dy_c(x)}{dx}\right\} = -k\mathcal{F}_c\{y_c(x)\} \tag{6}$$

with $dy_c(x)/dx|_{x=0} = 0$,

$$\mathcal{F}_s\left\{\frac{dy_s(x)}{dx}\right\} = -k\mathcal{F}_c\{y_s(x)\} \tag{7}$$

with $y_s(x)|_{x=0} = dy_s(x)/dx|_{x=0} = 0$,

$$\mathcal{F}_c\left\{\frac{dy_s(x)}{dx}\right\} = +k\mathcal{F}_s\{y_s(x)\} \tag{8}$$

with $y_s(x)|_{x=0} = d^2y_s(x)/dx^2\big|_{x=0} = 0$, and

$$\mathcal{F}_c\left\{\frac{dy_c(x)}{dx}\right\} = +k\mathcal{F}_s\{y_c(x)\} \tag{9}$$

with $y_c(x)|_{x=0} = dy_c(x)/dx|_{x=0} = d^2y_c(x)/dx^2|_{x=0} = 0$.

The set (6)–(9) shows the aforementioned interrelation between sine and cosine transforms in unilateral transforms of the derivatives of odd order. Clearly, the functions $y_s(x)$ in (7) and $y_c(x)$ in (9) are also subjected to the Fourier sine and cosine transforms, respectively.

3. The Unilateral Fourier Transform Applied to a First-order Equation

Consider the first-order non-homogeneous ordinary differential equation

$$\frac{dv(t)}{dt} + \gamma v(t) = f(t), \quad \gamma \geq 0, \tag{10}$$

defined on $[0,\infty)$. Let $v_s(t)$ and $v_c(t)$ be solutions of (10) differing only in their boundary conditions at the origin: $v_s(t)|_{t=0} = dv_c(t)/dt|_{t=0} = 0$. By performing the Fourier sine transform on both sides of (10) and utilizing (7), we obtain:

$$-k\mathcal{F}_c\{v_s(t)\} + \gamma\mathcal{F}_s\{v_s(t)\} = \mathcal{F}_s\{f_s(t)\}, \qquad (11)$$

with the boundary conditions

$$f_s(t)|_{t=0} = v_s(t)|_{t=0} = \left.\frac{dv_s(t)}{dt}\right|_{t=0} = 0. \qquad (12)$$

For the Fourier cosine transform, we apply (9) to obtain

$$+k\mathcal{F}_s\{v_c(t)\} + \gamma\mathcal{F}_c\{v_c(t)\} = \mathcal{F}_c\{f_c(t)\}, \qquad (13)$$

with

$$\left.\frac{df_c(t)}{dt}\right|_{t=0} = v_c(t)|_{t=0} = \left.\frac{dv_c(t)}{dt}\right|_{t=0} = \left.\frac{d^2v_c(t)}{dt^2}\right|_{t=0} = 0. \qquad (14)$$

The system of coupled algebraic equations and their corresponding boundary conditions, described by Eqs. (11)–(14), is not suitable for solving the homogeneous equation because the requirements $v(t)|_{t=0} = dv(t)/dt|_{t=0} = 0$ in (10) with $f(t) = 0$ impose that the derivatives of all orders of $v(t)$ vanish at $t = 0$, leading to $v(t) = 0$ for all t. Despite these limitations, we can still seek a particular solution when $f(t) \neq 0$.

For convenience, let us disregard the symbols $f_s(t)$ and $f_c(t)$ and instead recall that $f(t)|_{t=0} = df(t)/dt|_{t=0} = 0$. It is important to note that because $v_s(t)$ and $v_c(t)$ are solutions of the same differential equation $\mathcal{F}_s\{v_c(t)\} = \mathcal{F}_s\{v_s(t)\}$ due to $v_s(t)|_{t=0} = v_c(t)|_{t=0} = 0$, and $\mathcal{F}_c\{v_s(t)\} = \mathcal{F}_c\{v_c(t)\}$ due to $dv_s(t)/dt|_{t=0} = dv_c(t)/dt|_{t=0} = 0$. Obviously, the boundary conditions imposed on the system imply that $v_s(t) = v_c(t)$. In the subsequent developments, we will refrain from considering $v_s(t) = v_c(t)$ and instead focus on working out the details separately. The system of coupled algebraic equations can be solved for $\mathcal{F}_s\{v_s(t)\}$ and $\mathcal{F}_c\{v_c(t)\}$:

$$\mathcal{F}_s\{v_s(t)\} = \frac{\gamma\mathcal{F}_s\{f(t)\} + k\mathcal{F}_c\{f(t)\}}{\gamma^2 + k^2} \qquad (15)$$

and

$$\mathcal{F}_c\{v_c(t)\} = \frac{\gamma\mathcal{F}_c\{f(t)\} - k\mathcal{F}_s\{f(t)\}}{\gamma^2 + k^2}. \qquad (16)$$

We can now solve for $v_s(t)$ and $v_c(t)$ by inverting (15) and (16). The inverse Fourier transforms yield the following integral representations for the particular solution of (10):

$$v_s(t) = \int_0^\infty d\tau\, G_s(t,\tau)\, f(\tau) \qquad (17)$$

for (15), and

$$v_c(t) = \int_0^\infty d\tau\, G_c(t,\tau)\, f(\tau) \qquad (18)$$

for (16). Because we speculated that $v_s(t) = v_c(t)$, we would expect $G_s(t,\tau) = G_c(t,\tau)$. The two-variable functions $G_s(t,\tau)$ and $G_c(t,\tau)$ are called Green functions. They satisfy boundary conditions at $t = 0$ in accordance with those ones imposed on $v_s(t)$ and $v_c(t)$, and are expressed by

$$G_s(t,\tau) = \sqrt{\frac{2}{\pi}}\,\mathcal{F}_s\{G_s(k,\tau)\} \qquad (19)$$

and

$$G_c(t,\tau) = \sqrt{\frac{2}{\pi}}\,\mathcal{F}_c\{G_c(k,\tau)\}, \qquad (20)$$

where

$$G_s(k,\tau) = \frac{\gamma\sin k\tau + k\cos k\tau}{\gamma^2 + k^2} \qquad (21)$$

and

$$G_c(k,\tau) = \frac{\gamma\cos k\tau - k\sin k\tau}{\gamma^2 + k^2}. \qquad (22)$$

The Green functions can also be expressed as:

$$\begin{aligned} G_s(t,\tau) &= \int_0^\infty dk\, \left\{\frac{1}{\pi}\frac{\gamma}{\gamma^2+k^2}[\cos k(t-\tau) - \cos k(t+\tau)] \right. \\ &\quad \left. + \frac{1}{\pi}\frac{k}{\gamma^2+k^2}[\sin k(t-\tau) + \sin k(t+\tau)]\right\} \end{aligned} \qquad (23)$$

and

$$\begin{aligned} G_c(t,\tau) &= \int_0^\infty dk\, \left\{\frac{1}{\pi}\frac{\gamma}{\gamma^2+k^2}[\cos k(t-\tau) + \cos k(t+\tau)] \right. \\ &\quad \left. + \frac{1}{\pi}\frac{k}{\gamma^2+k^2}[\sin k(t-\tau) - \sin k(t+\tau)]\right\}. \end{aligned} \qquad (24)$$

The Green functions clearly reveal the boundary conditions

$$G_s(t,\tau)|_{t=0} = \left.\frac{\partial G_c(t,\tau)}{\partial t}\right|_{t=0} = 0. \qquad (25)$$

Furthermore, there are additional boundary conditions to be validated at a later stage:

$$\left.\frac{\partial G_s(t,\tau)}{\partial t}\right|_{t=0} = G_c(t,\tau)|_{t=0} = \left.\frac{\partial^2 G_c(t,\tau)}{\partial t^2}\right|_{t=0} = 0. \qquad (26)$$

A convenient approach to solve (23) and (24) involves utilizing a table of integrals. Alternatively, we can resort to contour integration techniques, employing the Cauchy integral formula or employing the method of residues.

When $\gamma = 0$, we will employ the definite integral denoted as 3.721(1) in Ref. [33], expressed as:

$$\int_0^\infty dx\,\frac{\sin ax}{x} = \frac{\pi}{2}\,\mathrm{sgn}(a), \qquad (27)$$

where the sign function $\mathrm{sgn}(x)$ is defined such that $\mathrm{sgn}(x)$ equals ± 1 for $x \gtrless 0$. It can be established that:

$$G(t,\tau) = G_s(t,\tau) = G_c(t,\tau) = \frac{1+\mathrm{sgn}(t-\tau)}{2}\,. \quad (28)$$

For $\gamma \neq 0$, we will utilize the definite integrals labelled as 3.723(2) and 3.723(3) in Ref. [33]. These integrals are given by:

$$\int_0^\infty dx\,\frac{\cos ax}{\beta^2+x^2} = \frac{\pi}{2\beta}\,e^{-a\beta}, \quad a \geq 0,\, \mathrm{Re}\,\beta > 0, \quad (29)$$

$$\int_0^\infty dx\,\frac{x\sin ax}{\beta^2+x^2} = \frac{\pi}{2}\,e^{-a\beta}, \quad a > 0,\, \mathrm{Re}\,\beta > 0. \quad (30)$$

By utilizing these integrals, we can express the Green functions as follows:

$$G(t,\tau) = G_s(t,\tau) = G_c(t,\tau) = \frac{1+\mathrm{sgn}(t-\tau)}{2}\,e^{-\gamma(t-\tau)}, \quad (31)$$

As claimed, we have $G_s(t,\tau) = G_c(t,\tau)$. More compactly, for $\gamma \geq 0$, we can write:

$$G(t,\tau) = \theta\,(t-\tau)\;e^{-\gamma(t-\tau)}. \quad (32)$$

Here, $\theta(x)$ denotes the unit step function ($\theta(x) = 1$ for $x > 0$, and $\theta(x) = 0$ for $x < 0$). It is important to emphasize that the Green function aligns with (26), exhibits a jump discontinuity at $t = \tau$, and is constructed from the solution of the homogeneous equation, i.e. $e^{-\gamma t}$. Lastly, we can express the particular solution of equation (10) as

$$v(t) = \int_0^t d\tau\,e^{-\gamma(t-\tau)}f(\tau), \quad (33)$$

where $f(t)$ has at least a zero of order two at $t = 0$.

4. Final Remarks

We have demonstrated that the unilateral Fourier transforms are ineffective in providing the general solution to the problem of a particle subject to a time-dependent driving force plus a linear velocity-dependent friction. However, they do furnish an integral representation for the particular solution in terms of the Green function. From (17) and (18), we can show that both $G_s(t,\tau)$ and $G_c(t,\tau)$ (which we denote by $G(t,\tau)$) satisfy the following equation:

$$\frac{\partial G(t,\tau)}{\partial t} + \gamma G(t,\tau) = \delta(t-\tau)\,. \quad (34)$$

Note that $G(t,\tau)$ satisfies a homogeneous equation for all t, except $t = \tau$, where it has a singular point. Therefore, on each side of the singular point, the Green function is expressed as a solution of the homogeneous equation. By integrating (34) from $\tau - |\varepsilon|$ to $\tau + |\varepsilon|$, we observe that the Green function has a jump discontinuity at $t = \tau$. This jump discontinuity is given by

$$[G(\tau+|\varepsilon|,\tau) - G(\tau-|\varepsilon|,\tau)] \underset{|\varepsilon|\to 0}{\to} 1. \quad (35)$$

Furthermore, the boundary condition $G(t,\tau)|_{t=0} = 0$ implies that $G(t,\tau) = 0$ for all $t < \tau$. This realization underscores that $G(t,\tau)$ characterizes a retarded Green function, elucidating the causal relationship between the delta perturbation at $t = \tau$ and its subsequent influence on $G(t,\tau)$ for $t > \tau$. The boundary condition imposed on $G(t,\tau)$ mirrors that imposed on $v(t)$. In the physical context of the present problem, Eq. (33) is suggestive: $v(t)|_{t=0} = 0$ (remember, this is a sine qua non condition for the existence of the Fourier sine transform) and any change in $v(t)$ in the future time t is influenced by $f(\tau)$ for times preceding t.

Another problem that deviates from the convenient "special conditions" alluded by Butkov is the forced damped harmonic oscillator. This more intricate task is left to the readers.

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References

[1] S.J. Orfanidis, *Introduction to Signal Processing* (Prentice-Hall, Upper Saddle River, 1995).
[2] A.V. Oppenheim, A.S. Willsky, S.H. Nawab and J.J. Ding, *Signals and systems* (Prentice Hall, Upper Saddle River, 1997), v. 2.
[3] B.P. Lathi and R.A. Green, *Linear systems and signals* (Oxford University Press, Oxford, 2005), v. 2.
[4] C. Rubio-Gonzalez and J.J. Mason, J. Appl. Mech. **66**, 485 (1999).
[5] C. Rubio-Gonzalez and J.J. Mason, Comput. Struct. **76**, 237 (2000).
[6] C. Rubio-Gonzalez and J.J. Mason, Int. J. Fracture **108**, 317 (2001).
[7] E. Lira-Vergara and C. Rubio-Gonzalez, Int. J. Fract. **135**, 285 (2005).
[8] C. Rubio-Gonzalez and E. Lira-Vergara, Int. J. Fract. **169**, 145 (2011).
[9] M. Nazar, M. Zulqarnain, M.S. Akrama and M. Asif, Commun. Nonlinear Sci. Numer. Simulat. **17**, 3219 (2012).
[10] N. Shahid, M. Rana and I. Siddique, Bound. Value Probl. **48**, 1 (2012).
[11] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications* (Chapman and Hall, London, 2015), 3 ed.
[12] J.C. Araújo and R.G. Márquez, Rev. Eletr. Paul. Mat. **11**, 136 (2017).
[13] J.C. Araújo and R.G. Márquez, TEMA (São Carlos) **20**, 95 (2019).
[14] A.S. Castro, Rev. Bras. Ens. Fis. **42**, e20200129 (2020).
[15] E. Butkov, *Mathematical Physics* (Addison-Wesley, Boston, 1968).
[16] A.S. Castro, Rev. Bras. Ens. Fis. **36**, 2307 (2014).

[17] P.H.F. Nogueira and A.S. Castro, Eur. J. Phys. **37**, 015402 (2016).
[18] D.W. Vieira and A.S. Castro, Rev. Bras. Ens. Fis. **42**, e20200145 (2020).
[19] D.W. Vieira and A.S. Castro, Rev. Bras. Ens. Fis. **44**, e20220038 (2022).
[20] A.S. Castro, Rev. Bras. Ens. Fis. **45**, e20220318 (2023).
[21] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Wiley, New York, 1937).
[22] P.M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).
[23] M.L. Boas, *Mathematical Methods in the Physical Sciences* (Wiley, New York, 1966), 2 ed.
[24] G.B. Arfken and H.J. Weber, *Mathematical Methods for Physicists* (Harcourt/Academic Press, San Diego, 1996), 5 ed.
[25] K.F Riley, M.P. Hobson, S.J. Bence,*Mathematical Methods for Physics and Engineering* (Cambridge University Press, Cambrige, 2006), 3 ed.
[26] J. Bellandi Filho, E.C. Oliveira and H.G. Pavão, Rev. Bras. Ens. Phys. **6**, 9 (1984).
[27] J. Bellandi Filho, R.J.M. Covolan, A.B. Pádua and J.T.S. Paes, Rev. Bras. Ens. Fis. **10**, 50 (1988).
[28] J. Bellandi Filho, R.J.M. Covolan, A.B. Pádua and J.T.S. Paes, Rev. Bras. Ens. Fis. **11**, 74 (1989).
[29] J. Bellandi Filho, R.J.M. Covolan, A.B. Pádua and J.T.S. Paes, Rev. Bras. Ens. Fis. **12**, 2 (1990).
[30] M.M. Odashima, B.G. Prado and E. Vernek, Rev. Bras. Ens. Fis. **39**, e1303 (2017).
[31] A.S. Castro, Rev. Bras. Ens. Fis. **43**, e20210068 (2021).
[32] A.S. Castro, Rev. Bras. Ens. Fis. **45**, e20230052 (2023).
[33] I.S. Gradshteyn and I.M. Ryzhik, in: *Table of Integrals, Series, and Products*, edited by A. Jeffrey and D. Zwillinger (Academic Press, New York, 2007), 7 ed.