

A proper integral for the electric field at the surface of a conducting sphere

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In a recent paper, I have shown that the electric field at the surface of a charged conducting sphere evaluates to *half* the field discontinuity across the surface of the sphere. This exact result has been found by solving an *improper* integral, which has been criticized by Assad in a very recent paper in this journal [6]. In this note, I present a simpler approach that yields a *proper* integral for that field. This integral also evaluates to exactly *half* the field discontinuity, contrarily to Assad's objections.

Keywords: Electrostatic fields, Gauss's law, Conducting sphere.

On seeking for a deeper understanding of a scientific topic, we often formulate simple questions which, though not realizable experimentally, should be satisfactorily answered. The electric field *at the surface* of a charged conducting sphere in electrostatic equilibrium, mentioned in virtually all introductory physics and electromagnetism textbooks, is among such topics because it is impossible to make a perfectly-shaped sphere.¹ Despite this practical impossibility, the field at the surface of an ideal, perfect sphere with radius R and charge Q may, *a priori*, be determined mathematically. As the net charge on any bulk *conductor* must reside on its surface, it follows from Gauss's law that the magnitude of the field of a conducting sphere must leap from a null value inside the sphere to $\sigma/\epsilon_0 = kQ/R^2$, just outside it (see, e.g., Refs. [2–4]).² Here, $\sigma = Q/(4\pi R^2)$ is the areal charge density on the surface of the sphere and $k = 1/(4\pi\epsilon_0)$ is the Coulomb constant, ϵ_0 being the electric permittivity of free space. For points located *exactly at the surface*, however, the amount of charge surrounded by a spherical gaussian surface with radius R is *ambiguous*, so Gauss's law is *inconclusive*. In a previous work, by treating the charge distribution as a collection of thin charged rings I have already shown that the field at the surface evaluates to $\frac{1}{2}kQ/R^2 = \sigma/(2\epsilon_0)$, i.e. *half the field discontinuity* across the surface [5]. As this result has been found by solving an *improper* integral, it has been criticized in a very recent paper [6]. I present below a simpler approach that yields a *proper* integral for that field.

Theorem. For a conducting sphere with radius R and net charge Q , in electrostatic equilibrium, the electric field in any point P on its surface is given by

$$\mathbf{E} = \frac{1}{2} \frac{kQ}{R^2} \hat{\mathbf{r}} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{r}},$$

where $\sigma = Q/(4\pi R^2)$ and $\hat{\mathbf{r}}$ is the radial (outward) unit-vector at point P .

Proof. Without loss of generality, take $Q > 0$ and the z -axis along the line from the center, at $x = y = z = 0$, to point P , at $x = y = 0$ and $z = R$, as seen in Fig. 1, below. On treating the spherical charge distribution as a collection of horizontal circular rings with radius r , varying from 0 to R , and variable charge $dQ = \sigma dA = \sigma(2\pi r ds)$, where $ds = R d\theta$ is the width of each ring, θ being the polar angle (in spherical coordinates), we begin deriving the electric field $d\mathbf{E}$ created in P by the charged ring centered at point $(0, 0, z)$, with $-R \leq z < +R$, which lies at a distance $R - z$ from P , as indicated in Fig. 1. The azimuthal symmetry of the charge distribution guarantees that the field in P must point along the z -axis direction, so $d\mathbf{E} = dE_z \hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is the unit-vector for the z -axis and

$$dE_z = k \frac{(R - z)}{[r^2 + (R - z)^2]^{\frac{3}{2}}} dQ, \quad (1)$$

according to the well-known formula for the field of a uniformly-charged ring, as given e.g. in Eq. (22-16) of Ref. [4]. Note that this expression is valid for all $-R \leq z < +R$, being an indeterminate form of the kind '0/0' for $z = R$. A direct integration $E_z = \int dE_z$ would lead us to an *improper* integral, as done in Ref. [5], so, in view to change it into an equivalent *proper* integral, let us express dE_z as a function of θ only (apart from multiplicative

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¹Even the surface of a polished metal, which seems smooth for the naked-eye, reveals a number of roughnesses under a microscope, with its surface being at an average distance R to the center [1].

²According to Gauss's law, the electric field of a uniformly-charged spherical shell is the same as that of a *conducting sphere*, both inside and outside it.

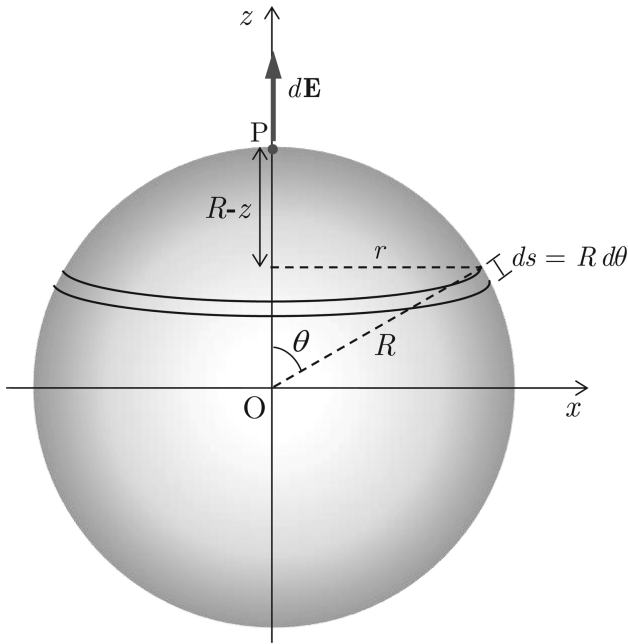


Figure 1: A conducting sphere with radius R and electric charge $Q > 0$ is cut into a large number of thin circular rings of radius $r = R \sin \theta$, each having a charge $dQ = \sigma(2\pi r ds)$. Each ring produces a vertical electric field $d\mathbf{E} = dE_z \mathbf{k}$ at point P.

constants). Since $z = R \cos \theta$ and $r = R \sin \theta$, one finds

$$\begin{aligned} dE_z &= 2\pi k \sigma \frac{R^2 \sin \theta R (1 - \cos \theta)}{\left[R^2 \sin^2 \theta + R^2 (1 - \cos \theta)^2 \right]^{\frac{3}{2}}} d\theta \\ &= 2\pi k \sigma \frac{\sin \theta (1 - \cos \theta)}{\left[\sin^2 \theta + (1 - \cos \theta)^2 \right]^{\frac{3}{2}}} d\theta \\ &= 2\pi k \sigma \frac{\sin \theta (1 - \cos \theta)}{(2 - 2 \cos \theta)^{\frac{3}{2}}} d\theta \\ &= \frac{\pi}{\sqrt{2}} k \sigma \frac{\sin \theta}{\sqrt{1 - \cos \theta}} d\theta. \end{aligned} \tag{2}$$

Though this last expression is still an indeterminate form of the kind ‘0/0’ for $\theta = 0$ (which corresponds to $z = R$),³ the trigonometric identities $\sin \theta = 2 \sin(\theta/2) \cos(\theta/2)$ and $\sin^2(\theta/2) = (1 - \cos \theta)/2$ allow us to simplify it to

$$\begin{aligned} dE_z &= \pi k \sigma \frac{\sin(\theta/2) \cos(\theta/2)}{\sqrt{\sin^2(\theta/2)}} d\theta \\ &= \pi k \sigma \cos\left(\frac{\theta}{2}\right) d\theta, \end{aligned} \tag{3}$$

which is a well-defined function for $\theta = 0$, being *bounded and continuous* there at that point. The integration over

³Note that $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sqrt{1 - \cos \theta}} = \sqrt{2}$, rather than ∞ , as would be the case for the Coulombian field from a *finite* point charge Q .

all rings then yields

$$E_z = \pi k \sigma \int_0^\pi \cos\left(\frac{\theta}{2}\right) d\theta, \tag{4}$$

which is a *proper integral*, since the integrand remains bounded and continuous throughout the integration interval, including its endpoints. On substituting $u = \theta/2$, one promptly finds

$$\begin{aligned} E_z &= 2\pi k \sigma \int_0^{\pi/2} \cos u du \\ &= \frac{\sigma}{2\epsilon_0}. \end{aligned} \tag{5}$$

□

We emphasize that the function at the right-hand side of Eq. (1) is undefined only at the point $z = R$, where it returns the indeterminate form ‘0/0’, remaining bounded and continuous for all other points in the domain $-R \leq z \leq +R$. Then, we have excluded that point from the domain, but we have properly included it there in Eq. (3), which has led to an elementary, *proper* integral in Eq. (4). Mathematically, this can be viewed as a regularization of the improper Riemann integral solved in Ref. [5], which is a valid procedure since *the value of the integral of a bounded function that is continuous in an interval of integration, except at a point belonging to this interval, does not change when we remove this point from the integration domain*, as follows from a well-known theorem by Cauchy (see, e.g., Theorem 8.1 and Sec. 8.4, as well as Ex. 8.4.6, of Ref. [7]).⁴ Physically speaking, this regularization is justified by noting that the charge is distributed *uniformly* on the surface of the sphere, with a constant density σ , so the charge $dQ = \sigma dA = 2\pi \sigma R^2 \sin \theta d\theta$ in each ring will tend to zero as $r \rightarrow 0$ (i.e., when $z \rightarrow R^-$ or $\theta \rightarrow 0$), which means that our point P, which encloses a null area, does not enclose a *finite* amount of charge (i.e., it cannot be treated as a finite point charge). Therefore, the expression for dE_z in Eq. (1) does not present a *non-integrable singularity* at $z = R$ (i.e., at $\theta = 0$), which shows that Assad’s argument below Eq. (15) of Ref. [6] is *incorrect*. In fact, even an older argument by Assad, as found at the end of Ref. [9], that the electric field is *undefined* at the surface of a conducting sphere *because there is electric charge there*, is also *incorrect*. The presence of an electric charge dQ at a point does not always create an infinite electric field there, otherwise the well-known result for the field *inside* a uniformly-charged sphere, namely $E(r) = (kQ/R^3)r$, valid for $0 \leq r \leq R$, as found, e.g., in Eq. (23-20) of Ref. [4], or Ex. 23.6(B) of Ref. [10], would not make sense!

The electric field is then *discontinuous* at the surface of a conducting sphere, leaping from 0 to $\sigma/(2\epsilon_0)$

⁴For a simple but rigorous proof of this property of Riemann integrals, see Ref. [8].

when we pass from points inside the sphere to any point *at* its surface, and then from this to σ/ϵ_0 for points just outside the sphere. Interestingly, the half-factor in our theorem also arises in more elaborate *microscopic* (quantum-mechanical) models, in a scale in which both the charge distribution and the electric field change continuously from inside to just outside a charged conductor [11]. According to these models, a thin transition slab with extension of a few atomic diameters is formed within which the field increases smoothly from nearly zero, inside the conductor, to its maximum value σ/ϵ_0 , attained at a point about 4 Å outside the conductor, as shown in Fig. I.5 of Ref. [11]. There in that figure, the field *at* the surface ($x = 0$) clearly equals $0.5 \sigma/\epsilon_0$.

In summary, though Gauss's law promptly reveals a leap discontinuity from 0 to σ/ϵ_0 in the electric field when we cross the surface of a charged conducting sphere, it does not provide a definite result for the field *at* this surface. In order to fill this gap, in this note I have shown, within the rigour of calculus, that this field evaluates to half that discontinuity. Our proof involves only basic integration rules, so it could well be adopted in introductory physics classes and textbooks.

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