Revista Brasileira de Ensino de Física, v. 35, n. 4, 4305 (2013)
www.sbfisica.org.br

Experimental study of simple harmonic motion of a spring-mass system as a function of spring diameter

(Estudo experimental do movimento harmônico simples de um sistema massa-mola em função do diâmetro da mola)

C.A. Triana[1], F. Fajardo

Departamento de Física, Universidad Nacional de Colombia, Bogotá, Colombia
Recebido em 23/1/2013; Aceito em 14/3/2013; Publicado em 30/10/2013

The simple harmonic motion of a spring-mass system generally exhibits a behavior strongly influenced by the geometric parameters of the spring. In this paper, we study the oscillatory behavior of a spring-mass system, considering the influence of varying the average spring diameter Φ on the elastic constant k, the angular frequency ω, the damping factor γ, and the dynamics of the oscillations. It was found that the elastic constant k is proportional to Φ^{-3}, while the natural frequency ω_0 is proportional to $\Phi^{-3/2}$, and γ decreases as Φ increases. We also show the differences obtained in the value of the angular frequency ω when the springs are considered as ideal (massless), taking into account the effective mass of the spring, and considering the influence of the damping of the oscillations. This experiment provides students with the possibility of understanding the differences between theoretical models that include well-known corrections to determine the frequency of oscillations of a spring-mass system.
Keywords: spring-mass system, Hooke's law, elastic constant, simple harmonic motion, damping.

O movimento harmônico simples de um sistema massa-mola geralmente mostra um comportamento que é fortemente determinado pelos parâmetros geométricos da mola. Neste trabalho estuda-se o comportamento oscilatório de um sistema massa-mola considerando o efeito da variação do valor médio do diâmetro Φ da mola na constante elástica k, a frequência angular ω, o fator de amortecimento γ e a dinâmica das oscilações. Nós encontramos que a constante elástica k é proporcional a Φ^{-3}, enquanto a frequência ω_0 é proporcional a $\Phi^{-3/2}$, e que γ diminui quando Φ aumenta. Também se mostram as diferenças encontradas no valor da frequência angular ω quando as molas são consideradas ideais (sem massa), levando em conta a massa efetiva da mola, e levando em conta o amortecimento das oscilações. Esta experiência oferece aos estudantes a possibilidade de compreender as diferenças nos modelos teóricos que levam em conta correções que são bem conhecidas para determinar a frequência das oscilações de um sistema massa-mola.
Palavras-chave: sistema massa-mola, lei de Hooke, constante elástica, movimento harmônico simples, amortecimento.

1. Introduction

The study of the movement experienced by a mass suspended from the free end of a spring is a topic discussed in most introductory physics courses, from both theoretical and experimental outlooks. The physics of the spring-mass system oscillations have been widely studied in a great variety of texts, in which the relationship between the period and the oscillation frequency is shown in detail [1]. Similarly experiments allow observing the dependence of the oscillatory systems on the mechanical forces as established by Hooke's law [2]. Some studies included corrections in order to take into account the influence of the spring mass on the oscillations of the spring-mass system [3, 4]. Likewise, studies of the behavior of the oscillations of systems constructed of plastic and non-helical springs have been carried out [5, 6]. Previous studies have dealt with the influence of changing the natural length l_0 of the spring (for a fixed diameter) on the behavior of the elastic constant k, the angular frequency ω and the damping factor γ of the oscillations [7].

In this work we study the influence of varying the average spring diameter Φ, for a fixed length, on the behavior of the elastic constant k, the angular frequency ω, and the damping γ of the oscillatory motion, which are the principal variables that determine the simple harmonic motion of the spring-mass system. A point that should be emphasized is that this kind of experi-

[1]E-mail: ctrianae@unal.edu.co.

ment allows showing students that some variables that characterize a real physical system depend on the size of the system under consideration. It is important that students ask, for example, how the behavior of the oscillatory dynamics depends on the size of the spring used in the experimental arrangement. In other words, it is very important that students take into account that the change of size or geometry of one part of the physical system could have a large influence on the behavior of other physical variables that characterize its dynamic behavior. For the foregoing reasons, this experimental study is an excellent one for general physics courses. The paper is organized as follows: in section 2, a fast review of the theoretical concepts on which the work is based is performed; section 3 describes the device and the experimental procedure; in section 4, the discussion of results is addressed; and finally the conclusions of this work are presented.

2. Theoretical framework

Previous studies have shown that the longitudinal elastic constant k of a helical spring is determined by the diameter d of the wire, the average spring diameter Φ (which hereafter will be referred simply as the diameter Φ), the number of windings N, and the shear modulus G, which involves specific characteristics of the material of which the spring is manufactured [8]. In agreement with that result, it is possible to calculate the spring's elastic constant k through the ratio

$$k = \frac{Gd^4}{8\Phi^3 N}, \tag{1}$$

Eq. (1) allows one to determine the behavior of the elastic constant k as a function of the spring diameter Φ, and will be used to compare with our experimental results. Taking into account that we are studying the case of small oscillations, we have that the natural frequency ω_0 of oscillation of the spring-mass system is given by

$$\omega_0 = \sqrt{\frac{k}{m}}, \tag{2}$$

where m is the mass suspended from its free end. Equation (2) describes the behavior of an isolated system in which there are no dissipative forces and the spring has no mass (massless spring). In the case of a real physical system, one must take into account the equivalent mass $M/3$ of the spring [9, 10], then the angular frequency ω_e of the system oscillations taking into account this correction can be calculated by

$$\omega_e = \sqrt{\frac{k}{((M/3) + m)}}. \tag{3}$$

The principle of conservation of mechanical energy allows us to relate the angular frequency ω to the oscillation period T of the system, which in its fundamental mode [11] is given by $\omega = 2\pi/T$. The amplitude of the oscillations decreases as energy is dissipated due to the presence of non-conservative forces of friction in the medium in which the system is immersed. Therefore, the oscillatory behavior of a real physical system disappears after a certain time interval. Thus, a more realistic description of the behavior of the spring-mass oscillating system is determined by a damped oscillation, defined by an equation of motion of the form [12]

$$x = A_0 e^{-\frac{\gamma \cdot t}{2}} \cos(\omega t + \varphi), \tag{4}$$

where A_0 is the initial amplitude of the oscillations and φ is the initial phase. The damping factor γ responsible for the decrease in the amplitude of the oscillation as a function of time is defined by $\gamma = b/m$, b being the damping coefficient and m the mass suspended from the free end of the spring. Taking into consideration the damping, which makes the oscillation frequency of the system slightly smaller than the value defined by Eq. (2), the angular frequency is determined by

$$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}, \tag{5}$$

where $\omega_0^2 = k/m$ corresponds to the frequency in the absence of dissipative forces. Equation (5) suggests that the higher the damping factor γ, the greater will be the attenuation of the amplitude A of the oscillations of the spring-mass system. In this paper we will show the difference obtained in determining the frequency of oscillations according to theoretical models defined by Eqs. (2), (3) and (5) as a function of the spring diameter Φ.

3. Experimental procedure

Figure 1 shows a photograph of the experimental setup used to measure the k, ω and γ variables as a function of the spring diameter Φ. The experimental measurements were carried out using a series of eight springs of different diameters, made of a steel wire with a diameter of $d = 8.1 \times 10^{-4}$ m. All springs had a natural length of $l_0 = 10.1 \times 10^{-2}$ m and had no separation h between their coils (step $h = 0$). The number of coils for the springs was chosen to be $N = 124$. Table 1 summarizes the main characteristics of the springs used in this experiment. The cost of the springs was less than 10 dollars, so this is an experiment that requires low-cost materials. Using a tape measure, the measurement of the elongation Δx experienced by the spring was carried out when masses m were suspended at its free end (see Fig. 1a). Performing a graphical analysis of the applied force $F = mg$ as a function of the elongation Δx experienced by the spring, the elastic constant k was determined. The oscillation period T was recorded using a Vernier VPG-BTD photogate. To measure T, a mass $m = 0.182$ kg was suspended at the free end of the spring, and it was elongated by a length $\Delta x = 1.5 \times 10^{-2}$ m from the equilibrium position, and then it was allowed to oscillate freely. In order to obtain a more accurate value, the period T was obtained by means of averaging ten measurements. By replacing this value in the expression $\omega = 2\pi/T$, the value of the angular frequency ω was found. Using a digital force sensor Vernier Dual Force DFS-BTA, the force F of the spring-mass system was measured (see Fig. 1b). Adjusting the envelope of the oscillation by means of a decreasing exponential, the damping factor γ was determined.

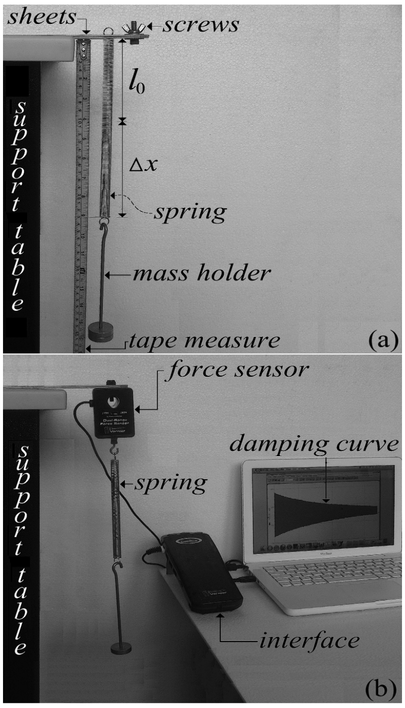

Figure 1 - (a) Photograph of the experimental setup used to measure the k and ω variables as a function of the diameter Φ of the spring. Note that the spring's upper coil is set between two plates in order to minimize external vibrations. (b) Experimental setup used to measure the force F in the spring-mass system with a Vernier sensor.

Table 1 - Diameters Φ and mass M of the springs[a,b] used to study the influence of Φ on the oscillations of a spring-mass system.

Spring	Diameter Φ ($\pm 0.01 \times 10^{-2} m$)	Mass M ($\pm 0.01 \times 10^{-3} kg$)
1	0.82	12.87
2	0.96	15.31
3	1.03	16.17
4	1.20	19.31
5	1.28	20.83
6	1.36	21.87
7	1.44	23.36
8	1.54	25.09

[a]All springs have a natural length $l_0 = 10.1 \times 10^{-2}$ m ($N = 124$ coils).
[b]The spring coils are in contact, without separation h between their coils (step $h = 0$).

4. Results and data analysis

Figure 2 shows the elongation Δx experienced by the springs when the masses that exerted the external forces F are suspended at their free end. We highlights that in all figures presented in this paper the error obtained in the calculation or in the measurement process are smaller that the point size. A close look shows that the initial value of the applied force is not the same for all springs, but increases as a function of the diameter Φ. This is because to the mass m suspended from the free end of the springs the fraction $M/3$ of the spring's mass M was added; springs having higher diameter Φ have greater mass M (see Table 1). This issue was done to graphically highlight the correction to the spring mass $M/3$, defined in Eq. (3), which becomes more noticeable as the spring diameter increases. The above procedure does not alter the slopes of the lines shown in Fig. 2, and allows seeing the effect of the correction in order to take into account the spring's mass. It can clearly be observed in Fig. 2 that the slopes of the straight lines that determine the elongation Δx produced by the applied force F increase as the spring diameter Φ decreases. This result implies that the spring's elastic constant, defined as $k = F/\Delta x$, is greatest for springs with smaller diameters Φ.

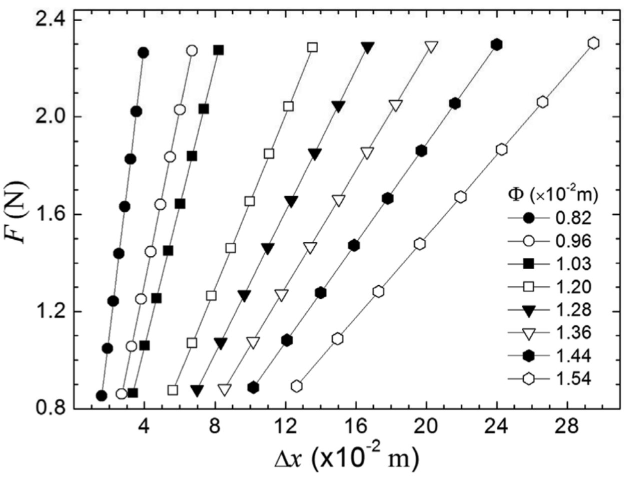

Figure 2 - Applied force F as a function of the spring elongation Δx taking as a parameter the spring diameter Φ.

Figure 3a shows the variation of the elastic constant k (full circles) as a function of the spring diameter Φ. In this figure, it can be seen that the value of elastic constant k decreases as Φ increases. The variation of the effective spring mass $M/3$ (empty circles) as a function of the spring diameter Φ is also shown in Fig. 3a. Note that the effective mass $M/3$ of the spring increases linearly with Φ, as expected because the increases in Φ rise linearly with the length of the wires that make up the spring, and then these linear increases in the length of the wires produce a linear increase in the effective mass $M/3$ of the springs. The linear fit between the spring's elastic constant k and Φ^{-3}, shown in Fig. 3b (which corresponds to a linear correlation coefficient of adjustment of 0.99), indicates that the results obtained fit perfectly with that established by Eq. (1).

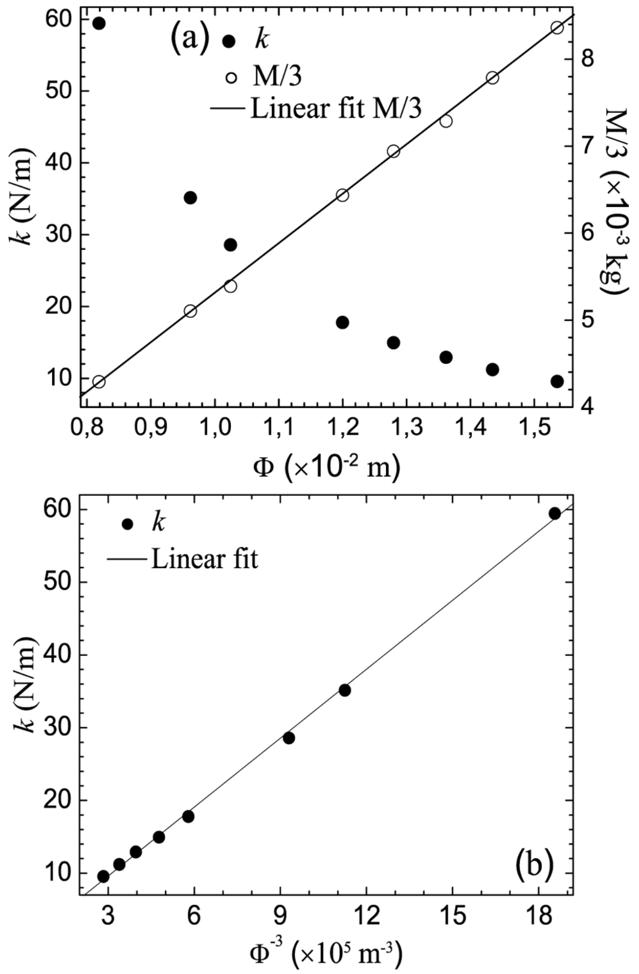

Figure 3 - Variation of the spring's elastic constant k (filled circles) and the effective spring mass $M/3$ (empty circles) as a function of the spring diameter Φ (a), and linear fit between the spring's elastic constant k and Φ^{-3} (b).

The functional relationship between the spring's elastic constant k and the spring diameter Φ is given by

$$k = \frac{3.16 \times 10^{-5}}{\Phi^3}, \tag{6}$$

where the quantity 3.16×10^{-5} corresponds to the constant value $Gd^4/8N$ defined in Eq. (1), which involves the diameter d of the wire with which the spring was manufactured, the number of coils N, and the shear modulus G. Taking into account that $3.16 \times 10^{-5} = Gd^4/8N$ and substituting the values of d and N, a value of $G = 72 \times 10^9$ N/m^2 is found for the shear modulus. This result is slightly lower than the shear modulus reported in the literature for steel [13], where $G = 79 \times 10^9$ N/m^2. We believe that this difference is due to the fact that the value of G found in our paper was obtained from a series of springs in which the diameter Φ varied, whereas the value of G reported in the literature is for a straight wire. In the springs' design, a relationship known as spring index C is defined, which is given by $C = \Phi/d$. Depending on this value, a series of second-order approximations in the calculation of the elastic constant is generated, as shown in previous theoretical studies with regard to the shear modulus G [14]. In our case, the spring indexes are $C = 10.1$ and $C = 18.9$ for the spring of lowest and highest diameter Φ respectively. These values of the spring index C suggest that as the diameter becomes smaller, the higher-order terms in the calculation of the elastic constant may have more influence on the oscillatory behavior of the system.

Figure 4a shows a comparison in the behavior of the angular frequencies ω, ω_0 and ω_e as a function of the spring diameter Φ. The ω value was calculated by replacing the period T in the expression $\omega = 2\pi/T$, while ω_0 was calculated by means of Eq. (2). The ω_e value was determined through Eq. (3), in which the correction to the spring mass $M/3$ was performed. In all cases the oscillations were induced by suspending a constant mass $m = 0.182$ kg and applying an initial amplitude of $A_0 = 1.5 \times 10^{-2}$ m. The results clearly show that the frequencies ω, ω_0 and ω_e are greatest for small values of the spring diameter Φ and decrease as Φ increases. This behavior is in complete agreement with the analysis performed in Fig. 3, since if the spring diameter Φ decreases, then the value of its elastic constant k will be greater. As established by Eqs. (2), (3) and (5) the frequencies ω, ω_0 and ω_e are highest for the greatest k, *i.e.* for springs with small diameter Φ.

Another important issue that can be seen in Fig. 4a is that the values of the frequencies are not equal, since $\omega_0 > \omega_e > \omega$. This behavior is due to the fact that the ω_0 value corresponds to the frequency determined in the absence of retarding forces, while the frequency ω_e includes the spring mass correction $M/3$. On the other hand, ω corresponds to a real physical system that experiences damped oscillations and compressive forces that tend to keep the spring coils together, an effect that is greater as the spring's diameter becomes smaller [15]. The fit shown in Fig. 4b shows the linear relationship between the angular frequency ω_0 and $\Phi^{-3/2}$, with a linear correlation coefficient of adjustment of 0.99. The functional relationship between the variables in Fig. 4b is given by

$$\omega_0 = \frac{0.013}{\Phi^{3/2}} = \frac{B}{\Phi^{3/2}}, \tag{7}$$

where by the substitution of Eq. (1) into Eq. (2) it is found that $B = \sqrt{Gd^4/8\,\mathrm{Nm}}$. Taking the value of the shear modulus $G = 72 \times 10^9$ N/m^2 previously calculated and replacing the values d, N and m in Eq. (7), it is obtained that $B = \sqrt{Gd^4/8\,\mathrm{Nm}} = 0.0132$, giving us a difference of 1.5% as compared with the result obtained in Eq. (7).

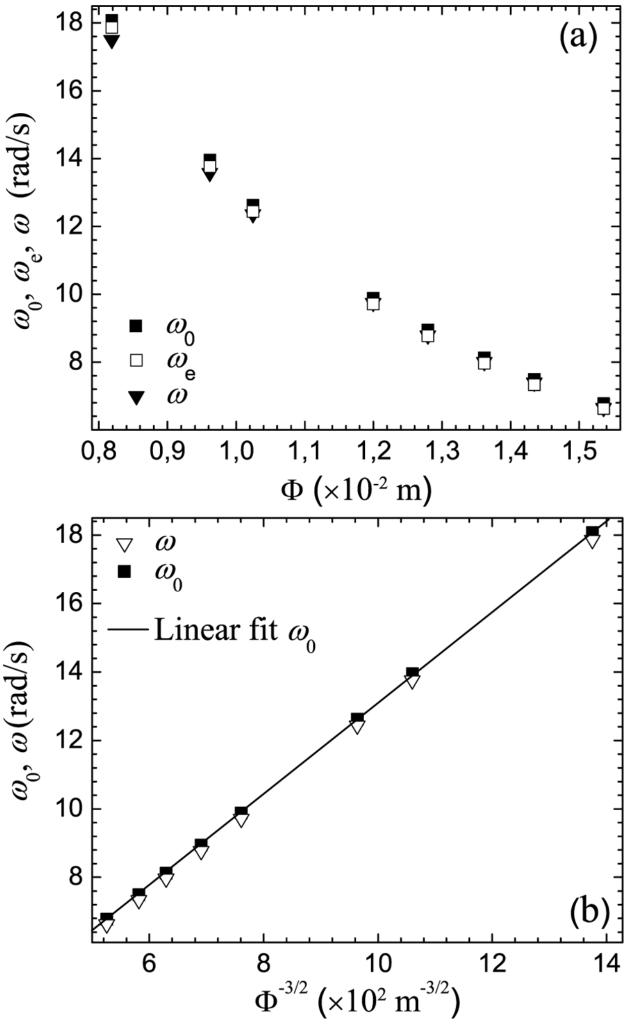

Figure 4 - Comparison of the behavior of the angular frequencies ω, ω_0 and ω_e as function of the spring diameter Φ (a). Note the linear fit between the natural frequency ω_0 and $\Phi^{-3/2}$ and an apparent linear relationship between the angular frequency ω and $\Phi^{-3/2}$ (empty triangles), which is not correct according to Eq. (5) (b).

In Fig. 4b, the angular frequency ω as function of the spring diameter $\Phi^{-3/2}$ is also shown, and it is possible to think that an apparent linear relationship between the variables ω and $\Phi^{-3/2}$ exists. However, we recall that ω is a value determined experimentally from measurements of period T of the oscillations, and therefore ω implicitly has the effect of damping. Consequently, in agreement with that established by Eq. (5), ω cannot be in a linear dependence with $\Phi^{-3/2}$.

The oscillatory dynamic of the spring-mass system addressed in this paper corresponds to a real physical system, where the amplitude of the oscillations decreases gradually and disappears after a certain interval of time. Consequently, there is a damped harmonic motion around an equilibrium position with decreasing amplitude, in which the action of non-conservative forces leads to the dissipation of energy into the medium in which the system is immersed. Figure 5a shows the damping experienced by the spring-mass system with the highest value of γ, *i.e.* for the spring with the smallest diameter Φ. Also, the envelope of the oscillatory curve is shown, which corresponds to the exponential factor defined in Eq. (4), from which the value γ_1 is obtained. For the springs with other diameters, only the envelope of the oscillatory curve is shown. In all cases, the experimental points, obtained by measuring the force F as a function of the time t, were fit according to the theoretical prediction given by Eq. (4), relating the values γ_i ($i = 1, ..., 8$) to the Φ values organized from smallest to largest diameter, according to the data summarized in Table 1. The results show that for the system built with the spring of the smallest diameter, which has the highest damping factor γ_1, the amplitude of the oscillations decreases rapidly in a time of $t \approx 300$ s, while the systems built with springs with lower damping factors γ_i require a much longer time for the oscillatory motion to vanish.

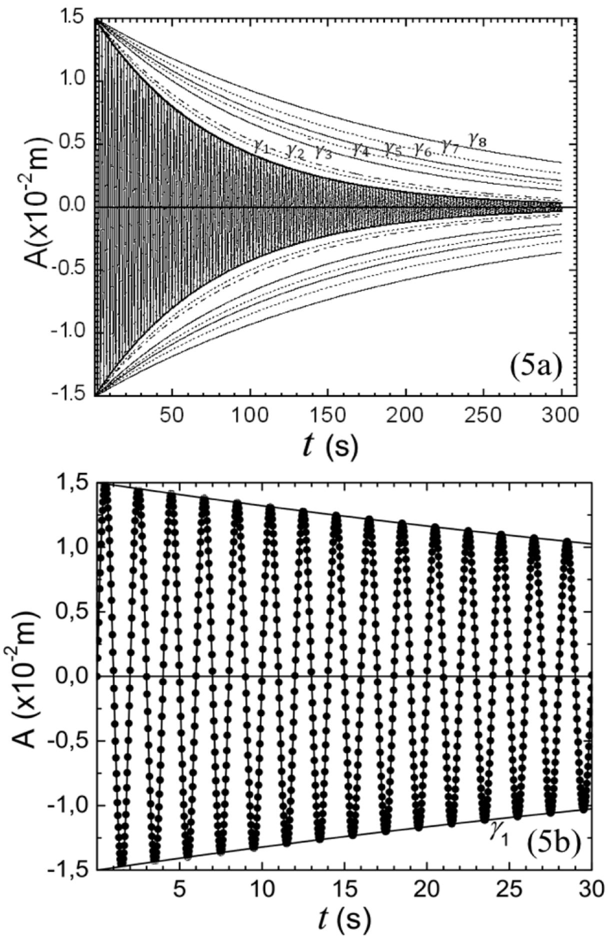

Figure 5 - Amplitude A of the oscillations as a function of time t for the spring with smallest diameter Φ and the highest damping factor γ_1. For the other springs, only the envelope of the oscillations is shown and the damping factor γ_i ($i = 1, ..., 8$) is indicated, which corresponds to the springs arranged from the smallest to largest diameter (a). Amplification of the amplitude A as a function of time t for the spring-mass system with damping factor γ_1 (smaller diameter Φ). The black dots in the oscillation correspond to the experimental values and the continuous curve to the theoretical fit (b).

Figure 5b shows an amplification of the oscillations shown in Fig. 5a for the spring with the smallest diameter Φ (*i.e.* the system with the largest γ). The black points in the oscillation correspond to the experimental values and the solid curve to the theoretical fit obtained by means of Eq. (4). An analysis of the results shows a correspondence of 99.9% between the experimental data fit with respect to the values obtained theoretically by using Eq. (4). It is found that the amplitude A of the oscillations (in meters) decreases in time according to the relationship

$$A = (1.5 \times 10^{-2})e^{-\frac{\gamma_i t}{2}}, \tag{8}$$

where the quantity $A_0 = 1.5 \times 10^{-2}$ m corresponds to the initial amplitude provided to the spring-mass system to induce the oscillations.

Figure 6 shows the variation of the damping factor γ as a function of the spring diameter Φ. Note that the damping factor γ given by Eq. (8) is greater for small values of the springs' diameter and decreases as Φ increases. This can be explained by taking into account that the springs with smaller diameter Φ have a larger elastic constant k, and therefore have higher damping factors γ. It is important to clarify that the spring constant is independent of the damping factor, because if, for example, we take the same spring-mass system and make the mass oscillate in two different media, two different values for the damping constant γ would be obtained, but the spring constant would be the same. The results obtained show that the spring's diameter considerably influences the behavior of the k, ω,and γ variables of the simple harmonic motion. Note that these variables decrease as the spring's diameter Φ increases, as shown in Figs. 3, 4 and 5, in accordance with predictions that can be made from Eqs. (1), (2), (3), (4) and (5). An issue to note in Fig. 6 is that at first glance there is an apparent linear relationship between the damping factor γ and the spring diameter Φ. However we believe that not is appropriate to make this generalization, because the obtained correlation coefficient of the linear adjustment is 0.86 which is far form the ideal value.

Figure 7a shows the behavior of the angular frequencies ω, ω_0, and ω_e as a function of the spring's elastic constant k. Observe that the frequencies increase as the k value increases (or likewise as Φ decreases), as is established by Eqs. (2), (3) and (5). Again we emphasize that the differences in the values of the frequencies ($\omega_0 > \omega_e > \omega$) are because ω_0 corresponds to the natural frequency, while ω_e includes the correction to the spring mass $M/3$ and ω corresponds to a damped oscillation under the effect of dissipative forces.

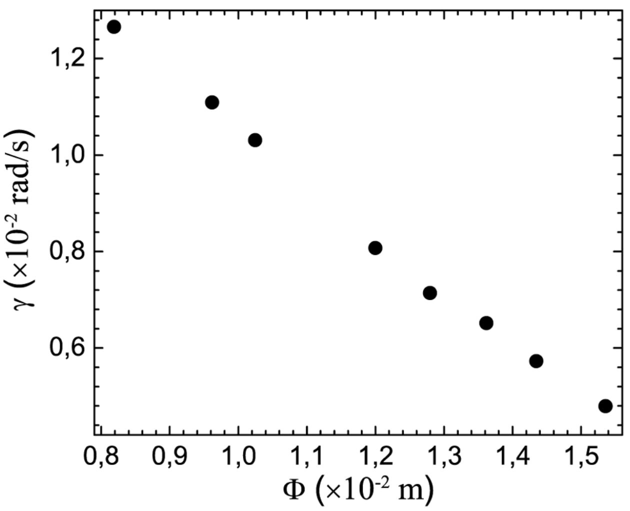

Figure 6 - Variation of the damping factor γ of the oscillations as a function of the spring's diameter Φ.

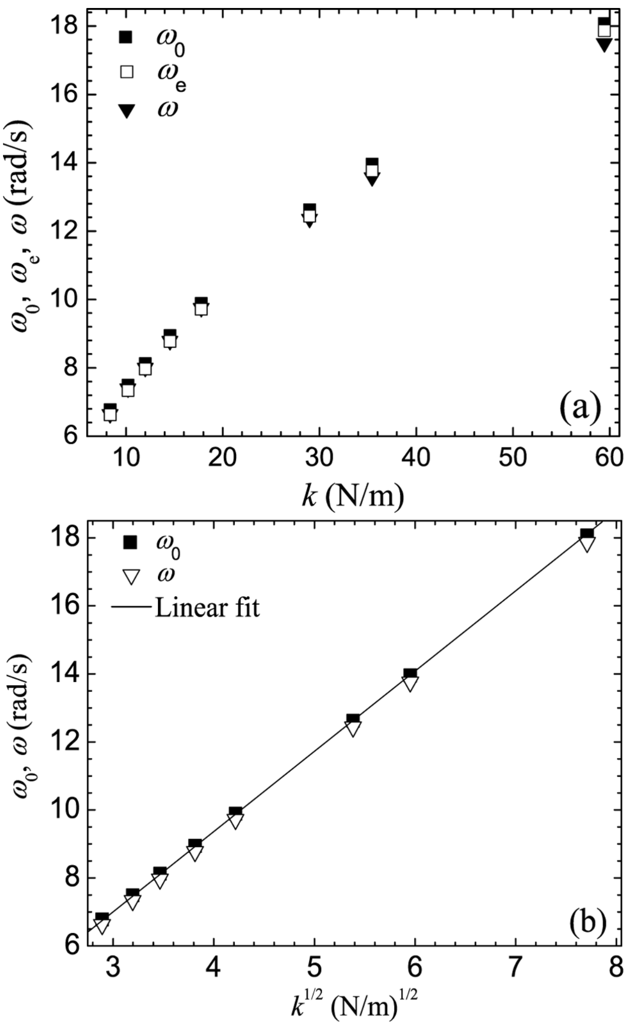

Figure 7 - Behavior of the angular frequencies ω, ω_0 and ω_e as a function of the spring's elastic constant k (a). The linear fit between the natural frequency ω_0 and $k^{1/2}$ (full squares) and an apparent linear relationship between the angular frequency ω and $\sqrt{k}$ (empty triangles), which is not correct, according to Eq. (5) (b).

The linear fit shown in Fig. 7b shows that the natural frequency ω_0 is directly proportional to $\sqrt{k}$ (full squares), as is established by Eq. (2). The linear correlation coefficient of adjustment is 0.99, and allows one to determine that the functional relationship between the variables is given by

$$\omega_0 = 2.34\sqrt{k}, \tag{9}$$

where the value $2.34 = 1/\sqrt{m}$. Replacing the mass $m = 0.182$ kg, suspended from the free end of the spring, we obtain a value of $1/\sqrt{m} = 2.344$, giving us a difference of 0.2% as compared with the result obtained in Eq (9). It is important to note, in Fig. 7b, that it gives the impression that there exists a linear relationship between the angular frequency ω and $\sqrt{k}$ (empty triangles); nevertheless this interpretation is not correct, since in agreement with Eq. (5) the angular frequency ω must involve the term $\gamma^2/4$, which is associated with the damping of the system.

Figure 8a shows the graphical representation of Eq. (5), *i.e.* the behavior of ω^2 as a function of $(k/m) - (\gamma^2/4)$, which is plotted taking into consideration the experimental values of ω, k and γ. Note that the functional relationship between the variables is linear, with slope 0.99 and intercept 0.9. These results are very close to those established by Eq. (5), where the slope and intercept are equal to 1 and 0, respectively. The small differences between experimental and theoretical values may be due to measurement uncertainties or to the influence of small effects that have not been taken into account in Eq. (5), such as the compression effects on the spring's coils [16]. It is important to note that the ω, k, and γ variables shown in Fig. 8a have all been determined experimentally, and show a linear correlation coefficient of adjustment between them of 0.99. Figure 8b shows that the value of the angular frequency ω increases as the damping factor γ increases. This behavior is due to the fact that the springs that have a higher damping factor have a higher elastic constant k, and therefore oscillate with higher angular frequency. This result can be corroborated taking into account that the values of the springs' elastic constant k correspond to powers of 10^1, while the values of the damping factor γ correspond to powers of 10^{-2}. This proves that in our experiment the quantity; $(k/m) >> (\gamma^2/4)$, and therefore Eq. (5) is always fulfilled taking positive values of the angular frequency ω.

5. Conclusions

The experimental study of simple harmonic motion of a spring-mass system shows that the principal physical variables that characterize the oscillations, such as k, ω, ω_0, ω_e, and γ, are strongly influenced by the spring's diameter Φ. The results obtained indicate that decreases in the spring's diameter Φ lead to increases in the elastic constant k, the angular frequency ω and the damping factor γ. The experiment is also very instructive for comparing the results of different models for finding the angular frequencies of the spring-mass oscillator, which involve the massless spring ideal approximation, a spring mass correction $M/3$, and the damping of the oscillations. This experiment, which uses low-cost materials, can be utilized so that through graphical analysis students can find a great variety of functional relationships between the variables that characterize the simple harmonic motion of a spring-mass system, making it an excellent practice or project for physics laboratory courses at the undergraduate level. One suggestion for carrying out this experiment is to use plastic springs such as are used in book binding, since they are available in a wide variety of diameters. Also, similar studies for other oscillating systems can be carried out.

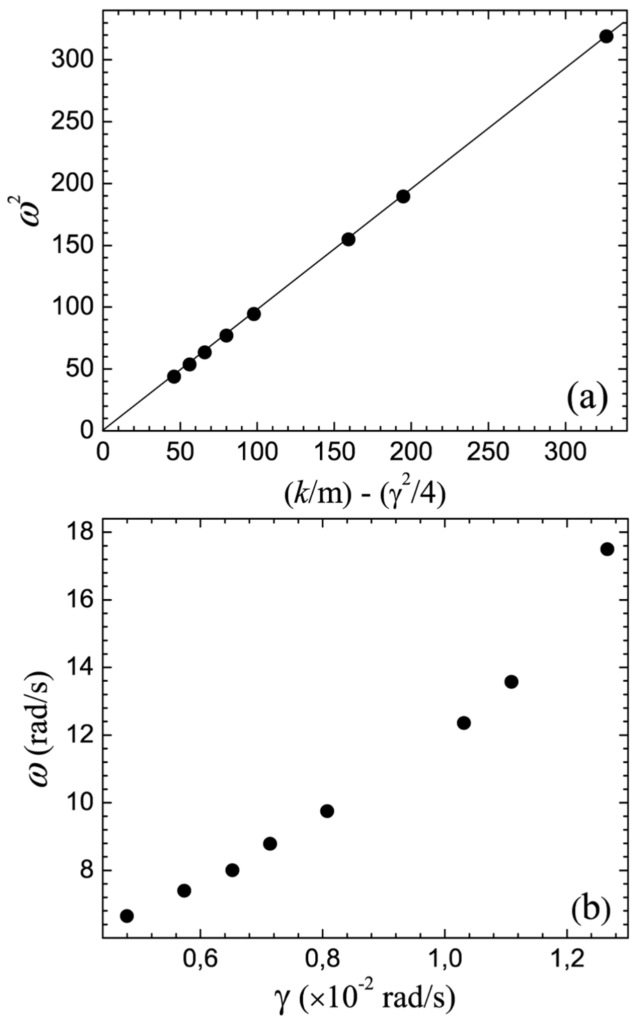

Figure 8 - Graphical representation of Eq. (5), behavior of ω^2 as a function of $(k/m) - (\gamma^2/4)$ (a). Angular frequency ω as a function of the damping factor γ (b).

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