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Analytic solution to the motion of mass-spring oscillator subjected to external force

(Solução analítica para o movimento de um oscilador massa-mola sujeito a uma força externa)

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The simple harmonic vibration, damping vibration and forced vibration of an oscillator attached to the massless spring are always discussed in general mechanics courses. In this article, we focus on the heavy-spring conditions. We first investigate the general situation where both viscous resistance and applied force are considered under the perspective of the renormalization group theory. Then we use analytic method to study the damped oscillation of an oscillator attached to the heavy spring, where renormalization method fails to work.
Keywords: mass-spring oscillator, damping vibration, forced oscillation.

A vibração harmônica simples, o amortecimento de vibrações e a vibração forçada de um oscilador ligado a uma moal sem massa são sempre discutidos em cursos gerais de mecânica. Neste artigo, vamos nos concentrar nas condições de mola com massa. Nós primeiro investigamos a situação geral em que tanto a resistência viscosa e a força aplicada são consideradas sob a perspectiva da teoria do grupo de renormalização. Então, nós usamos um método analítico para estudar a oscilação amortecida de um oscilador ligado a uma mola com massa, em que o método de renormalização não funciona.
Palavras-chave: oscilador massa-mola, vibração amortecida, oscilação forçada.

1. Introduction

An oscillation is a common but very important phenomenon in the physical world. If a physical quantity is displaced from the equilibrium a little, linear negative feedback may then lead to an oscillation. A familiar example is a simple harmonic oscillator. Also, damping vibrations and forced vibrations of an oscillator are normally focused [1]. The mass of the spring is neglected in models. However, the mass of the spring is unnecessarily neglected as to studying mass-spring system itself. In this article we try to solve the mass-spring system where the mass of the spring is not negligible.

In 1979, Weinstock studied the normal modes of the oscillator motion for the oscillator attached to a heavy spring by virtue of the Stieltjes integral [2]. In 1994, da Silva obtained the normal frequencies of elastic oscillations of a particle suspended on a spring of non-negligible mass again under the perspective of the renormalization group theory [3]. A continuous spring can be regarded as a chain of many small springs coupling an equal amount of small masses. Then mapping process is repeated by associating two consecutive small springs into a single one. At last, only the boundary effect matters. JM Nunes dealt with the problem only for the simplest situation, without the friction and applied force, thus the advantage of this method does not emerge in this case. In fact, we can not only find out the normal frequencies in the conservative system, but also obtain the specific equation of motion when external forces are acted on.

In the next sections, we first explore the most general condition, a forced vibration with viscous resistance, using the renormalization method. Then we deal with a special case analytically where the renormalization method fails to work. We investigate the orthogonality of the solutions of the PDEs in base set, and then obtain the motion of the damping oscillator attached to a heavy spring.

2. The forced mass-oscillator with damping

Hang an uniformly distributed spring with mass m vertically. The top side is fastened to stable fixture and

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the bottom side concatenate an object with mass M as oscillator. The free length of spring is L and the stiffness coefficient is k. To be discretized, the heavy spring is viewed as a series of small non-mass N equal springs and each small spring is coupled with a concentrated object with mass m/N. The natural length and elastic constant (labeled by s) of each small spring are L/N and $s = kN$ respectively. The damping on the oscillator M can be calculated as $-bv_M$ if the velocity of oscillator is v_M and damping coefficient is b. Besides, a time-dependent force $f(t)$ is applied on the oscillator.

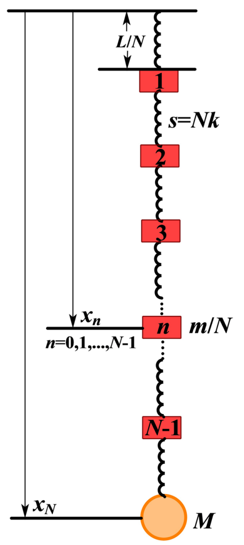


Figure 1 - A discretization model. The heavy spring with natural length L, mass m and stiffness coefficient k is divided into N small springs coupled with concentrated objects.

The positions of objects are denoted by $x_n (n = 0, 1, ..., N)$, then equations of motion can be set up as

$$x_0 = 0, \tag{1a}$$

$$\frac{m}{N}\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_n = s(x_{n-1} - 2x_n + x_{n+1}) + \frac{m}{N}g \quad (1 \le n \le N-1), \tag{1b}$$

$$M\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_N = -s(x_N - x_{N-1} - \frac{L}{N}) + Mg - b\frac{\mathrm{d}}{\mathrm{d}t}x_N + f(t). \tag{1c}$$

We can eliminate the constant terms derived from gravity in the Eq. (1) by changing the coordinates appropriately. So that we can use newly defined coordinates $(u_n = x_n - \frac{n}{N}L - \frac{ng}{s}[M + (2N-n-1)\frac{m}{2N}], n = 0, 1, ..., N)$ to describe the motion (da Silva proposed a way to approach these newly defined coordinates [3]). The equations with new coordinates are written as follows,

$$u_0 = 0, \tag{2a}$$

$$\frac{m}{N}\frac{\mathrm{d}^2}{\mathrm{d}t^2}u_n = s(u_{n-1} - 2u_n + u_{n+1}) \quad (1 \le n \le N-1), \tag{2b}$$

$$M\frac{\mathrm{d}^2}{\mathrm{d}t^2}u_N = -s(u_N - u_{N-1}) - b\frac{\mathrm{d}}{\mathrm{d}t}u_N + f(t). \tag{2c}$$

If we denote

$$\tilde{f} = \frac{1}{s}\mathcal{F}[f(t)] = \frac{1}{Nk}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t,$$

and

$$U_n = \mathcal{F}[u_n(t)] = \int_{-\infty}^{\infty} u_n(t)e^{-i\omega t}\,\mathrm{d}t \ (n = 0, 1, ..., N),$$

then the Eq. (2) can be Fourier transformed as

$$U_0 = 0, \tag{3a}$$

$$cU_n = U_{n-1} + U_{n+1} \quad (1 \le n \le N-1), \tag{3b}$$

$$CU_N = U_{N-1} + \tilde{f}, \tag{3c}$$

with $c = 2(1 - \frac{m\omega^2}{2N^2k})$ and $C = 1 + \frac{i\omega b - M\omega^2}{Nk}$.

By Eq. (3), we get $c^2U_{2n} = c(U_{2n-1} + U_{2n+1}) = U_{2n-2} + 2U_{2n} + U_{2n+2}$ and $cCU_N = cU_{N-1} + c\tilde{f} = U_{N-2} + U_N + c\tilde{f}$. We combine two small springs into a bigger small spring. That is, with this mapping process, the previous $(2n-1)^{\mathrm{th}}$ and $(2n)^{\mathrm{th}}$ springs now become the n^{th} bigger small spring. The new Fourier transformed position function of n^{th} spring is denoted by $\overline{U_n}$ and $\overline{U_n} = U_{2n}$. So we have

$$\overline{U_0} = 0, \tag{4a}$$

$$(c^2 - 2)\overline{U_n} = \overline{U_{n-1}} + \overline{U_{n+1}}, \tag{4b}$$

$$(cC - 1)\overline{U_{\frac{N}{2}}} = \overline{U_{\frac{N}{2}-1}} + c\tilde{f}. \tag{4c}$$

Comparing Eq. (4) and Eq. (3), the equations change regularly[2] after the process of combining two consecutive small springs into bigger spring. So we repeat the combination to renormalize. If we set $N = 2^p$, then after p^{th} repeat, Eq. (4c) finally becomes $C^{(p)}\overline{\overline{U_1}} = \overline{\overline{U_0}} + \tilde{f}\prod_{\kappa=0}^{p-1} c^{(\kappa)}$. $\overline{\overline{U_0}} = 0$ and the position function of the oscillator is

$$u_{\mathrm{oscillator}}(t) = \mathcal{F}^{-1}[\overline{\overline{U_1}}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\tilde{f}\prod_{\kappa=0}^{p-1} c^{(\kappa)}}{C^{(p)}} e^{i\omega t}\,\mathrm{d}\omega. \tag{5}$$

The next thing to do is to find out the iteration value $C^{(p)}$ and $\prod_{\kappa=0}^{p-1} c^{(\kappa)}$. Comparison between Eqs. (3)

[2] After each iteration, the coefficients of left terms in Eqs. (4b) and (4c) update to $c^{(l+1)} = c^{(l)2} - 2$ and $C^{(l+1)} = c^{(l)}C^{(l)} - 1$ from $c^{(l)}$ and $C^{(l)}$.

and (4) gives $c^{(1)} = c^2 - 2$ and $C^{(1)} = cC - 1$, so $2C^{(1)} - c^{(1)} = c(2C - c)$ and more generally,

$$2C^{(p)} - c^{(p)} = (2C - c)\prod_{\kappa=0}^{p-1} c^{(\kappa)}.$$

Introduce γ and let $c \equiv 2\cos\gamma$. $c^{(p)}$ and $\prod_{\kappa=0}^{p-1} c^{(\kappa)}$ are easily accessible with this variable substitution, *i.e.* $c^{(p)} = 2\cos 2^p\gamma$, $\prod_{\kappa=0}^{p-1} c^{(\kappa)} = \dfrac{\sin 2^p\gamma}{\sin\gamma}$. Finally,

$$C^{(p)} = \frac{1}{2}[\frac{\sin 2^p\gamma}{\sin\gamma}(2C - c) + 2\cos 2^p\gamma].$$

$c \equiv 2\cos\gamma = 2(1 - \frac{m\omega^2}{2N^2k})$. Notice that $N \gg 1$, so $\gamma \ll 1$ and $\gamma = \sin\gamma = \frac{\omega}{N}\sqrt{\frac{m}{k}}$. By Eq. (5),

$$\begin{aligned} u_{\text{oscillator}}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(\omega\sqrt{\frac{m}{k}})\tilde{f}}{\sin(2^p\gamma)(2C - c) + 2\sin\gamma\cos(2^p\gamma)} e^{i\omega t}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t}{i\omega b - M\omega^2 + \cot(\omega\sqrt{\frac{m}{k}})\omega\sqrt{km}} e^{i\omega t}\,\mathrm{d}\omega. \end{aligned} \tag{6}$$

Then $x_{\text{oscillator}}(t) = u_{\text{oscillator}}(t) + L + (M + \frac{m}{2})\frac{g}{k}$ (neglect $-\frac{mg}{2kN}$ because of $N \gg 1$) due to $u_N = x_N - L - \frac{Ng}{s}[M + (N-1)\frac{m}{2N}]$. It's the motion equation of the oscillator with consideration of mass of spring, external forces including damping, gravity and applied time-dependent force.[3] But one thing to note here is that $f(t)$ can not be 0 or the solution vanishes. We deal with this condition in following part.

In fact, if $g(t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{i\omega b - M\omega^2 + \cot(\omega\sqrt{\frac{m}{k}})\omega\sqrt{km}} e^{i\omega t}\,\mathrm{d}\omega (\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}\,\mathrm{d}\omega)$,[4] the motion equation of the oscillator can be given by the convolution of $f(t)$ and $g(t)$, *i.e.* $u_{\text{oscillator}}(t) = f(t) * g(t)$. From the following discussion, we will know that the frequencies which satisfy $i\omega b - M\omega^2 + \cot(\omega\sqrt{\frac{m}{k}})\omega\sqrt{km} = 0$, which make $G(\omega)$ divergent and lead to resonance, are exactly eigenvalues in no applied force condition (see Eq. (9)).

3. No applied force condition

Since we can no longer use renormalization method for no applied force condition, we then use mathematical physics equations to study this problem. Adopt appropriate coordinates as introduced before, and the problem can be analytically described as follows.[5]

$$\begin{cases} u_{tt} - \frac{kL^2}{m}u_{xx} = 0 \quad (t > 0, x \in [0, L)), \\ u(0,t) = 0, \\ u_x(L,t) = -\dfrac{M}{kL}u_{tt}(L,t) - \dfrac{b}{kL}u_t(L,t), \\ u|_{t=0} = \phi(x), \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \psi(x). \end{cases} \tag{7}$$

We consider using method of separation of variables to solve this equation and we take $u(x,t) = X(x)\exp(-i\mu L\sqrt{\frac{k}{m}}t)$ as the ansatz. The equations

[3]Notice that $\int_{-\infty}^{\infty} \frac{\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t}{i\omega b - M\omega^2 + \cot(\omega\sqrt{\frac{m}{k}})\omega\sqrt{km}} e^{i\omega t}\,\mathrm{d}\omega = \int_{\infty}^{-\infty} \frac{\int_{-\infty}^{\infty} f(t)e^{-i(-\omega)t}\,\mathrm{d}t}{i(-\omega)b - M(-\omega)^2 + \cot((-\omega)\sqrt{\frac{m}{k}})(-\omega)\sqrt{km}} e^{i(-\omega)t}\,\mathrm{d}(-\omega)$, so $\int_{-\infty}^{\infty} \frac{\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t}{i\omega b - M\omega^2 + \cot(\omega\sqrt{\frac{m}{k}})\omega\sqrt{km}} e^{i\omega t}\,\mathrm{d}\omega = \int_{-\infty}^{\infty} \frac{\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,\mathrm{d}t}{-i\omega b - M\omega^2 + \cot(\omega\sqrt{\frac{m}{k}})\omega\sqrt{km}} e^{-i\omega t}\,\mathrm{d}\omega$ and $u_{\text{oscillator}}(t)$ is pure real.

[4]By $\lim_{m\to 0}\cot\omega\sqrt{\frac{m}{k}} = \sqrt{\frac{k}{m}}/\omega$, we have $\lim_{m\to 0} G(\omega) = \frac{1}{i\omega b - M\omega^2 + k}$, which is consistent with the transfer function $G(s) = \frac{1}{Ms^2 + bs + k}$ for standard 2nd order mass/spring/damper system by Laplace transformation. [4] There is also another conclusion if we add one more expansion term in $\cot x$ ($\cot \simeq \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} + ...$ within radius of convergence), $\lim_{m\to 0}\cot\omega\sqrt{\frac{m}{k}} = \sqrt{\frac{k}{m}}/\omega - \frac{1}{3}\omega\sqrt{\frac{m}{k}}$, hence $\lim_{m\to 0} G(\omega) = \frac{1}{i\omega b - M\omega^2 + k - \frac{1}{3}m\omega^2}$. In no viscous case ($b = 0$), revised natural circular frequency for spring mass system is $\omega = \frac{k}{M + m/3}$ where spring's effective mass is $m/3$, [5] which can be obtained from $\lim_{m\to 0} G(\omega) = \frac{1}{-(M + m/3)\omega^2 + k}$.

[5]Take infinitesimal spring with length $\mathrm{d}x$, stiffness coefficient $\frac{kL}{\mathrm{d}x}$ and mass $\frac{\mathrm{d}x}{L}m$. The position deviation of infinitesimal spring from equilibrium is denoted by $\mathrm{d}u$. Then motion equation of the infinitesimal spring can be given by $\frac{\partial(\frac{kL}{\mathrm{d}x}\mathrm{d}u)}{\partial x}\mathrm{d}x = \frac{\mathrm{d}x}{L}m \cdot \frac{\mathrm{d}^2u}{\mathrm{d}t^2}$ according to Newton's second law.

above then become

$$\begin{cases} X''(x)+\mu^2 X(x)=0, & \text{(8a)}\\ X(0)=0, & \text{(8b)}\\ X'(L)=(i\mu\frac{b}{\sqrt{km}}+\mu^2\frac{ML}{m})X(L), & \text{(8c)}\\ X(x)=\phi(x), & \text{(8d)}\\ -i\mu L\sqrt{\frac{k}{m}}X(x)=\psi(x). & \text{(8e)} \end{cases}$$

By Eqs. (8a) and (8b), we substitute $X(x)=\sin\mu x$ into Eq. (8c) and get the eigenvalue equation

$$\cot\mu L = i\frac{b}{\sqrt{km}}+\mu\frac{ML}{m}. \tag{9}$$

We set that $X_p(x)$ and $X_q(x)$ are different solutions from the base set. To obtain expansion coefficients from initial conditions (8d) and (8e), we first derive the orthogonality relation between bases $X_p(x)$ and $X_q(x)$ within the boundary condition (8c). According to Eqs. (8a) and (8c), we have

$$X_p''(x)+\mu_p^2 X_p(x)=0, \tag{10a}$$

$$X_p'(L)=(i\mu_p\frac{b}{\sqrt{km}}+\mu_p^2\frac{ML}{m})X_p(L), \tag{10b}$$

$$X_q''(x)+\mu_q^2 X_q(x)=0, \tag{10c}$$

$$X_q'(L)=(i\mu_q\frac{b}{\sqrt{km}}+\mu_q^2\frac{ML}{m})X_q(L). \tag{10d}$$

Calculate Eq. (10b) $\times X_q(L)-$ Eq.(10d) $\times X_p(L)$ and we get

$$\begin{aligned} &\frac{ML}{m}(\mu_p^2-\mu_q^2)X_p(L)X_q(L)+i(\mu_p-\mu_q)\frac{b}{\sqrt{km}}X_p(L)X_q(L)\\ =&X_q(x)X_p'(x)\big|_0^L-\int_0^L X_p'(x)X_q'(x)\mathrm{d}x-X_p(x)X_q'(x)\big|_0^L+\int_0^L X_p'(x)X_q'(x)\mathrm{d}x\\ =&\int_0^L X_q(x)X_p''(x)\mathrm{d}x-\int_0^L X_p(x)X_q''(x)\mathrm{d}x\\ =&-\mu_p^2\int_0^L X_q(x)X_p(x)\mathrm{d}x+\mu_q^2\int_0^L X_p(x)X_q(x)\mathrm{d}x. \end{aligned} \tag{11}$$

From Eq.(11) we eventually reach the following equality under the boundary condition (8c),

$$\int_0^L X_q(x)X_p(x)\mathrm{d}x+\left[\frac{ML}{m}+i\frac{b}{(\mu_p+\mu_q)\sqrt{km}}\right]X_p(L)X_q(L)=0. \tag{12}$$

Eq. (12) shows that the solutions in the base set are generalized orthogonal [6]. The squared norms (denoted by N^2) of the eigenfunctions can be calculated as

$$\begin{aligned} N^2[X_p(x)]=&\int_0^L X_p^2(x)\mathrm{d}x+\left[\frac{ML}{m}+i\frac{b}{2\mu_p\sqrt{km}}\right]X_p^2(L)\\ &\overset{X_p(x)=\sin\mu_p x}{\underset{\text{Eq. (10b)}}{=\!=\!=\!=\!=}}\frac{L}{2}-\frac{1}{4\mu_p}\sin 2\mu_p L+\frac{\cos\mu_p L\sin\mu_p L}{2\mu_p}+\frac{ML}{2m}\sin^2\mu_p L\\ &=\frac{L}{2}+\frac{ML}{2m}\sin^2\mu_p L. \end{aligned} \tag{13}$$

So the solution to the Eq. (7) can be written as $u(x,t)=\sum_n A_n\sin\mu_n x\exp(-i\mu_n L\sqrt{\frac{k}{m}}t)$; the expansion coefficients A_n are determined by $\phi(x)$ and $\psi(x)$ collectively. We then expand $\phi(x)$ and $\psi(x)$ based on the orthogonality relation (12): $\phi(x)=\sum_n P_n\sin\mu_n x$, $\psi(x)=-i\sum_n\mu_n L\sqrt{\frac{k}{m}}Q_n\sin\mu_n x$ where

$$P_n=\frac{\int_0^L\phi(x)\sin\mu_n x\mathrm{d}x+\left[\frac{ML}{m}+i\frac{b}{2\mu_n\sqrt{km}}\right]\phi(L)\sin\mu_n L}{N^2[X_n(x)]}, \tag{14a}$$

$$Q_n=\frac{\int_0^L\frac{i}{\mu_n L}\sqrt{\frac{m}{k}}\psi(x)\sin\mu_n x\mathrm{d}x+\left[\frac{ML}{m}+i\frac{b}{2\mu_n\sqrt{km}}\right]\frac{i}{\mu_n L}\sqrt{\frac{m}{k}}\psi(L)\sin\mu_n L}{N^2[X_n(x)]}. \tag{14b}$$

Let $A_n = \alpha_n P_n + \beta_n Q_n$, then

$$\phi(x) = \sum_n (\alpha_n P_n + \beta_n Q_n) \sin \mu_n x, \tag{15a}$$

$$\sqrt{\frac{m}{k}}\psi(x) = -i \sum_n \mu_n L(\alpha_n P_n + \beta_n Q_n) \sin \mu_n x. \tag{15b}$$

Comparing Eq. (15a) + χ Eq. (15b) with $\phi(x) = \sum_n P_n \sin \mu_n x$, $\chi\sqrt{\frac{m}{k}}\psi(x) = -i\chi \sum_n \mu_n L Q_n \sin \mu_n x$, we get

$$P_n[(\alpha_n - 1) - i\chi\mu_n L\alpha_n] + Q_n[\beta_n - i\chi\mu_n L(\beta_n - 1)] = 0. \tag{16}$$

Hence, $\alpha_n = \frac{1}{1-i\chi\mu_n L}$ and $\beta_n = 1 - \alpha_n = \frac{-i\chi\mu_n L}{1-i\chi\mu_n L}$. Finally A_n can be given as

$$A_n = \frac{\int_0^L [\phi(x) + \chi\sqrt{\frac{m}{k}}\psi(x)] \sin \mu_n x \mathrm{d}x + [\frac{ML}{m} + i\frac{b}{2\mu_n\sqrt{km}}][\phi(L) + \chi\sqrt{\frac{m}{k}}\psi(L)] \sin \mu_n L}{N^2[X_n(x)](1 - i\chi\mu_n L)}, \tag{17}$$

where χ is determined by the initial conditions both $\phi(x)$ and $\psi(x)$.[6] For the simple but most common case, if $\psi(x) = 0$, *i.e.* $\frac{\partial u}{\partial t}\big|_{t=0} = 0$, we have $\chi = 0$ (hence $A_n = P_n$). According solutions satisfy Eqs. (7) when $t > 0$.

The eigenvalue equation reveals as $\cot \mu L = i\frac{b}{\sqrt{km}} + \mu\frac{ML}{m}$ (eigenvalue μ won't be 0). Notice that $-\overline{\mu}$ is also eigenvalue if μ is eigenvalue. The corresponding expansion coefficients have the relation $P(-\overline{\mu}) = -\overline{P(\mu)}$ and $Q(-\overline{\mu}) = -\overline{Q(\mu)}$ due to the Eq. (14). Considering $\sin(-\overline{\mu}x) = -\overline{\sin \mu x}$,

$$\sum_n P_n \sin \mu_n x = \sum_{\mathrm{Re}(\mu_n)>0} [P_n \sin \mu_n x + \overline{P_n} \sin \overline{\mu_n} x] = \sum_{\mathrm{Re}(\mu_n)>0} [P_n \sin \mu_n x + c.c.], \tag{18a}$$

$$-i\sum_n \mu_n L\sqrt{\frac{k}{m}} Q_n \sin \mu_n x = -i[\sum_{\mathrm{Re}(\mu_n)>0} [\mu_n L\sqrt{\frac{k}{m}} Q_n \sin \mu_n x] - c.c.]. \tag{18b}$$

Eq. (18) verifies that both $\phi(x)$ and $\psi(x)$ are pure real when ansatz

$$u(x,t) = \sum_n A_n \sin \mu_n x \exp(-i\mu_n L\sqrt{\frac{k}{m}}t),$$

is taken. Of course $u(x,t) = \sum_{\mathrm{Re}(\mu_n)>0} [A_n \sin \mu_n x \exp(-i\mu_n L\sqrt{\frac{k}{m}}t) + c.c.]$ is also pure real. We can obtain two types of independent eigen-vibration modes from this result. Let $\mu_n = \xi_n - i\zeta_n$[7], then

$$\begin{aligned}
&\sin \mu_n x \exp(-i\mu_n L\sqrt{\frac{k}{m}}t) = \\
&(\sin \xi_n x \cosh \zeta_n x - i \cos \xi_n x \sinh \zeta_n x)\times \\
&\exp(-\zeta_n L\sqrt{\frac{k}{m}}t)[\cos(\xi_n L\sqrt{\frac{k}{m}}t) - i \sin(\xi_n L\sqrt{\frac{k}{m}}t)] \\
&= [\sin \xi_n x \cosh \zeta_n x \cos(\xi_n L\sqrt{\frac{k}{m}}t) - \\
&\cos \xi_n x \sinh \zeta_n x \sin(\xi_n L\sqrt{\frac{k}{m}}t)] \exp(-\zeta_n L\sqrt{\frac{k}{m}}t) - \\
&i[\cos \xi_n x \sinh \zeta_n x \cos(\xi_n L\sqrt{\frac{k}{m}}t) + \\
&\sin \xi_n x \cosh \zeta_n x \sin(\xi_n L\sqrt{\frac{k}{m}}t)] \exp(-\zeta_n L\sqrt{\frac{k}{m}}t).
\end{aligned}$$

The two types of independent vibration modes are given

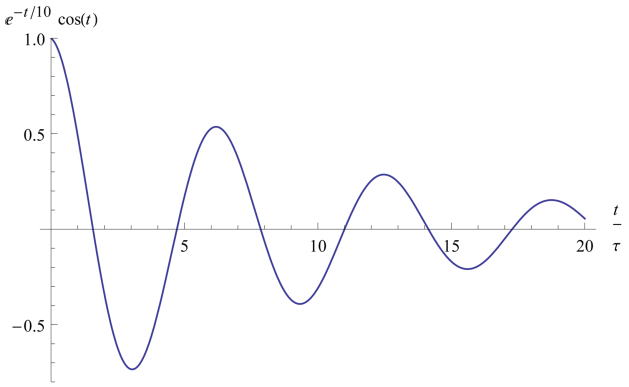
[6] If $\phi(x)$ and $\psi(x)$ are dependent, to be more precise, $P_n = Q_n$, χ can be any value. In this case, either Eq. (14a) or Eq. (14b) can be equally used to calculate the expansion coefficient A_n, and don't bother to introduce χ.

[7] It's hard to find all μs analytically from $\cot \mu L = i\frac{b}{\sqrt{km}} + \mu\frac{ML}{m}$ but we can use perturbation to investigate since imaginary part in RHS of eigenvalue equation $\frac{b}{\sqrt{km}}$ is feeble in generally underdamping conditions. The zero-order approximation, $\mu^{(0)}\tan\mu^{(0)}L = \frac{m}{ML}$ for zero friction situation ($b = 0$), is discussed in previous work [2]. Obviously, solutions are real numbers and come in pairs ($-\mu^{(0)}$ is also solution if $\mu^{(0)}$ satisfies zero-order eigenvalue equation). Perturbation calculation shows the first order modification is negative pure imaginary number and the paired zero-order eigenvalues $\pm\mu^{(0)}$ share the same first order modification. It coincides with the previous analysis that $-\overline{\mu}$ is eigenvalue if μ is eigenvalue.

by[8]

$$\text{mode 1:}[\sin\xi_n x\cosh\zeta_n x\cos(\xi_n L\sqrt{\frac{k}{m}}t) - \cos\xi_n x\sinh\zeta_n x\sin(\xi_n L\sqrt{\frac{k}{m}}t)]\exp(-\zeta_n L\sqrt{\frac{k}{m}}t),$$
$$\text{mode 2:}[\cos\xi_n x\sinh\zeta_n x\cos(\xi_n L\sqrt{\frac{k}{m}}t) + \sin\xi_n x\cosh\zeta_n x\sin(\xi_n L\sqrt{\frac{k}{m}}t)]\exp(-\zeta_n L\sqrt{\frac{k}{m}}t).$$

The oscillator ($x = L$) vibrates with damped amplitude (as is shown in Fig. 2) in both types of modes, which is reasonable.



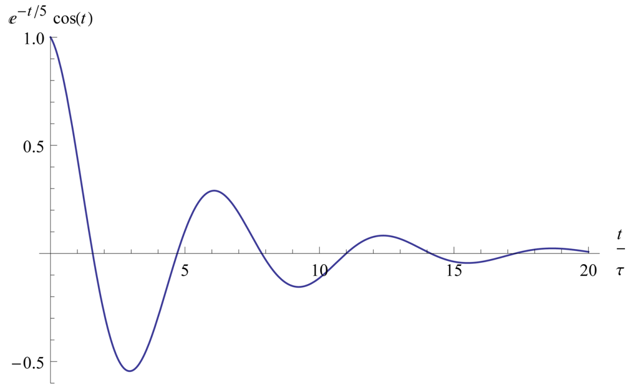


Figure 2 - Let t/τ be dimensionless with $\tau = \sqrt{\frac{m}{k}}/\xi_n L$, the eigen-vibration mode of the oscillator can be plotted under two cases with $\zeta_n/\xi_n = 0.1$ and $\zeta_n/\xi_n = 0.2$. The oscillator vibrates with damped amplitude.

For zero-friction case ($b = 0$), the solutions to eigenvalue equation $\cot\mu L = i\frac{b}{\sqrt{km}} + \mu\frac{ML}{m}$ are pure real, *i.e.* $\zeta = 0$. Then two types of vibration modes become[9]

$$\text{mode type 1:}\quad \sin\xi_n x\cos(\xi_n L\sqrt{\frac{k}{m}}t),$$
$$\text{mode type 2:}\quad \sin\xi_n x\sin(\xi_n L\sqrt{\frac{k}{m}}t).$$

Summarize the result: the solution to Eq. (7) is $u(x,t) = \sum_n A_n \sin\mu_n x\exp(-i\mu_n L\sqrt{\frac{k}{m}}t)$ where eigenvalue μ_n is given by Eq. (9). Expansion coefficient A_n and squared norms are given by Eqs. (17) and (13).

4. Conclusion

In this article, we detailedly studied the vibration of spring oscillator when the mass of the spring can't be neglected. Damped oscillation and forced vibration are especially focused. For general condition, oscillation with friction and applied force, renormalization method is employed to obtain the equation of the motion. Renormalization method shows superiority when applied force $f(t)$ exerts on the oscillator. We also investigate the damping vibration without applied force with theory of partial differential equations. For given boundary condition, the generalized orthogonality of base set is studied. We discussed the characters of the eigenvalue and the expansion coefficient and the discussions verified the validity of the solution.

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[8] To get a more detailed understanding of $\exp(-\zeta_n L\sqrt{\frac{k}{m}}t)$, we use $\cot z \simeq \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + ...$ again within the radius of convergence. The first-order approximation of $\cot(\xi_n - i\zeta_n)$ in eigenvalue equation gives that $\frac{\zeta_n}{\xi_n} = \frac{b/2M}{\sqrt{\frac{k}{m}}\xi_n L}$, so the dumping factor is $\exp(-bt/2M)$, which is the same as the damping factor of common damped oscillators. [1] The second-order approximation shows that the dumping factor is $\exp[-bt/2(M+m/3)]$.

[9] With $\tan\xi_n L = \frac{m}{M\xi_n L}$, the frequency $\xi_n L\sqrt{\frac{k}{m}}$ degrades if we neglect the mass of the spring, *i.e.* $\lim_{m\to 0}\xi_n L\sqrt{\frac{k}{m}} = \sqrt{\frac{k}{M}}$ for all ξ_n. Thus, many degrees of freedom are reduced in one body problem, $\omega = \sqrt{\frac{k}{M}}$.