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Notes

A note on potential energy in non-relativistic frames

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We show that, when associated with conservative forces that depend on two points in space: the center of the force and the position of the particle under its action, the value of potential energy is the same, up to an additive constant, in all frames of reference connected through an extended Galilean transformations. It is also assumed that this conservative force's direction is parallel to the straight line connecting the particle to this force's center. For both inertial and non-inertial frames, the result shown in this note is valid.
Keywords: Classical Mechanics, potential energy, extended Galilean transformations, conservative forces, inertial and non-inertial frames, mechanical energy.

The author once said: "*The potential energy of a conservative force is the same in any frame*", while speaking with a colleague. The colleague asked a straightforward question: "*Why? Do you include non-inertial frames in your statement?*". The query becomes understandable when we take into account that a particle's kinetic energy depends on the frame in which its motion is measured.

When examining the literature, we realize that when conservative forces are discussed, the movement of particles under their influence is studied in inertial frames [1–3].

We investigate in this note the well-known conservative forces: the elastic Hooke's force [3, p. 226–231, p. 387–391], the gravitational force [3, p. 140–144] and the Coulomb force [3, p. 145–147], when an extended Galilean transformation [3, p. 175] links the kinematics of a specific particle viewed from different frames in order to expand the analysis of particle dynamics to any non-relativistic frame.

Let $\mathcal{S}$ be an inertial frame from which the motion of a particle with mass m is followed; for details, see Fig. 1a.

Here, we study the motion of a particle of mass m in the inertial frame $\mathcal{S}$, moving under the action of Hooke's force with a spring of constant k.

$\vec{\mathbf{F}}_s^{(\mathcal{S})}$ is the force that, at instant t, the spring applies to the particle that is fixed to its end point P. It depends on the positions of two points: the point Q, which is the end point of the spring fixed in the wall, and its other end point P, which is connected to the particle. This is an important realization: this force depends on two points.

We measure the positions of P and Q in the inertial frame $\mathcal{S}$ at any given time t. $\vec{\mathbf{r}}_P(t)$ is the vector position of point P in this frame with respect to the origin O of the coordinate axes xyz. In the same frame the point Q is localized with respect to the same origin O, at the same moment t, by the vector position $\vec{\mathbf{r}}_Q(t)$. See the vectors $\vec{\mathbf{r}}_P(t)$ and $\vec{\mathbf{r}}_Q(t)$ in Fig. 1a. The coordinate axes xyz are fixed in the inertial frame $\mathcal{S}$.

Any observer at rest in the inertial $\mathcal{S}$ frame measures the spring force $\vec{\mathbf{F}}_s^{(\mathcal{S})}$ acting on the particle. Its expression is equal to:

$$\vec{\mathbf{F}}_s^{(\mathcal{S})}(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q) = -k\left(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\right). \tag{1}$$

The force $\vec{\mathbf{F}}_s^{(\mathcal{S})}$ is only dependent on the particle's relative position (point P) with respect to the force's center (point Q), or $\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q$.

The dynamics of the particle in the inertial $\mathcal{S}$ frame is described by Newton's Second Law. If the conservative force $\vec{\mathbf{F}}_s^{(\mathcal{S})}$ is the only force acting on it, then

$$m\frac{d^2\vec{\mathbf{r}}_P(t)}{dt^2} = \vec{\mathbf{F}}_s^{(\mathcal{S})}(\vec{\mathbf{r}}_P(t) - \vec{\mathbf{r}}_Q(t)). \tag{2}$$

It is important to mention that the point Q does not have to be at rest in the inertial $\mathcal{S}$ frame.

We apply an extended Galilean transformation [3, p. 175] to the vector locations of P and Q to obtain the dynamics of the same particle followed by observers at rest in the $\mathcal{S}'$ frame,

$$\vec{\mathbf{r}}'_P(t) \;=\; \vec{\mathbf{r}}_P(t) + \vec{\mathbf{R}}_O^{(O')}(t), \tag{3a}$$

$$\vec{\mathbf{r}}'_Q(t) \;=\; \vec{\mathbf{r}}_Q(t) + \vec{\mathbf{R}}_O^{(O')}(t). \tag{3b}$$

The vector positions of the points P and Q, measured from the origin O' of the coordinate axes $x'y'x'$, are denoted by $\vec{\mathbf{r}}'_P(t)$ and $\vec{\mathbf{r}}'_Q(t)$, respectively. These coordinate axes are fixed in the $\mathcal{S}'$ frame. The position of the origin O of the axes xyz with respect to the origin O' is determined by the vector $\vec{\mathbf{R}}_O^{(O')}(t)$, at time t. In the general case, the vector $\vec{\mathbf{R}}_O^{(O')}(t)$ can describe an

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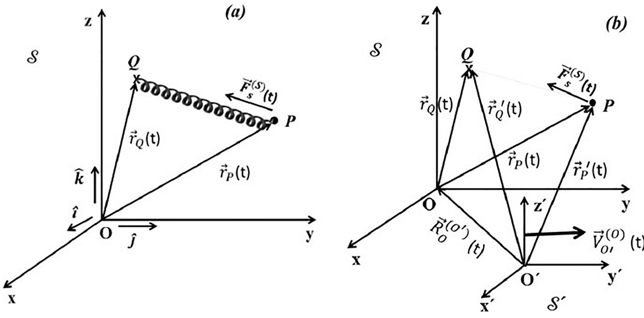

Figure 1: At position P, there is a particle attached to one end of a spring. Point Q is identified as the other end of the spring. a) From the inertial frame $\mathcal{S}$, these two positions are observed. In this frame, the particle is subject to the elastic force $\vec{\mathbf{F}}_s^{(\mathcal{S})}$. The force that the spring applies to the particle is centered at point Q. The coordinate axes xyz, which has the versors $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$, are fixed in the frame $\mathcal{S}$. Measured from the origin O of the axes xyz, the vector positions of the points P and Q are $\vec{\mathbf{r}}_P(t)$ and $\vec{\mathbf{r}}_Q(t)$, respectively. b) The velocity of the frame $\mathcal{S}'$ with respect to the inertial reference frame $\mathcal{S}$ is $\vec{\mathbf{V}}_{O'}^{(O)}(t)$. The coordinate axes $x'y'z'$ are fixed in the frame $\mathcal{S}'$. $\vec{\mathbf{r}}'_P(t)$ and $\vec{\mathbf{r}}'_Q(t)$ are respectively the position vectors of the points P and Q relative to the origin O' of the axes $x'y'z'$. $\vec{\mathbf{R}}_O^{(O')}$ is the vector that locates the origin O relative to the origin O'. The $\mathcal{S}'$ frame can be non-inertial or inertial. In both frames, $\mathcal{S}$ and $\mathcal{S}'$, all vectors in figures a and b are measured at time t.

accelerated motion between the two frames $\mathcal{S}$ and $\mathcal{S}'$. Fig. 1b shows the vectors $\vec{\mathbf{r}}'_P(t), \vec{\mathbf{r}}'_Q(t)$ and $\vec{\mathbf{R}}_O^{(O')}(t)$.

Through the use of the extended Galilean transformations (3a) and (3b), we obtain

$$\vec{\mathbf{r}}'_P(t) - \vec{\mathbf{r}}'_Q(t) = \vec{\mathbf{r}}_P(t) - \vec{\mathbf{r}}_Q(t), \tag{4}$$

valid for any $\mathcal{S}'$ frame. $\mathcal{S}'$ can be an inertial or a non-inertial frame.

The distance between the two locations P and Q and the relative vector position of the particle in point P to the center of the conservative force are both preserved at time t by the extended Galilean transformation in any $\mathcal{S}'$ frame, as demonstrated by the equality (4).

The dynamics (2) of the particle is rewritten in terms of the vector positions $\vec{\mathbf{r}}'_P(t)$ and $\vec{\mathbf{r}}'_Q(t)$, as measured by observers at rest in the $\mathcal{S}'$ frame, using the relations (3a) and (4),

$$m\,\frac{d^2\big(\vec{\mathbf{r}}'_P(t)\big)}{dt^2} = \vec{\mathbf{F}}_s^{(\mathcal{S})}\big(\vec{\mathbf{r}}'_P(t) - \vec{\mathbf{r}}'_Q(t)\big) - m\,\frac{d^2\big(\vec{\mathbf{R}}_O^{(O')}(t)\big)}{dt^2} \tag{5a}$$

$$\equiv \vec{\mathbf{F}}_s^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P(t) - \vec{\mathbf{r}}'_Q(t)\big) - m\,\frac{d^2\big(\vec{\mathbf{R}}_O^{(O')}(t)\big)}{dt^2}. \tag{5b}$$

$\vec{\mathbf{F}}_s^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P(t) - \vec{\mathbf{r}}'_Q(t)\big)$ is the force that acts on the particle of mass m, measured in the $\mathcal{S}'$ frame. When comparing the r.h.s. from equations (5a) and (5b), we obtain:

$$\begin{aligned}&\vec{\mathbf{F}}_s^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P(t) - \vec{\mathbf{r}}'_Q(t)\big)\\ &\quad= \vec{\mathbf{F}}_s^{(\mathcal{S})}\big(\vec{\mathbf{r}}'_P(t) - \vec{\mathbf{r}}'_Q(t)\big) = \vec{\mathbf{F}}_s^{(\mathcal{S})}\big(\vec{\mathbf{r}}_P(t) - \vec{\mathbf{r}}_Q(t)\big)\\ &\quad\Rightarrow \vec{\mathbf{F}}_s^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P(t) - \vec{\mathbf{r}}'_Q(t)\big) = \vec{\mathbf{F}}_s^{(\mathcal{S})}\big(\vec{\mathbf{r}}_P(t) - \vec{\mathbf{r}}_Q(t)\big).\end{aligned} \tag{6}$$

As can be seen from the result (6), the spring force acting on the particle corresponds to the same vector in any $\mathcal{S}'$ frame. The $\mathcal{S}'$ frame is an inertial or non-inertial frame.

In a given $\mathcal{S}'$ frame, a force is said to be conservative if its work to move a particle from its initial position $\vec{\mathbf{r}}'_P(t_1)$ to its final position $\vec{\mathbf{r}}'_P(t_2)$, where $t_2 > t_1$, depends only on these initial and final positions of the particle [2, p. 70–75], that is,

$$W_{\vec{\mathbf{F}}_s^{(\mathcal{S}')}}\big(\vec{\mathbf{r}}'_P(t_1) \to \vec{\mathbf{r}}'_P(t_2)\big) = \int_{\vec{\mathbf{r}}'_P(t_1)}^{\vec{\mathbf{r}}'_P(t_2)} \vec{\mathbf{F}}_s^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q\big)\cdot d\vec{\mathbf{r}}'_P \tag{7a}$$

$$= V^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P(t_1) - \vec{\mathbf{r}}'_Q(t_1)\big) - V^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P(t_2) - \vec{\mathbf{r}}'_Q(t_2)\big). \tag{7b}$$

The result (7b) is independent of the trajectory $\mathcal{C}$ that the particle takes (see Fig. 2) to move from its starting position $\vec{\mathbf{r}}'_P(t_1)$ to its final $\vec{\mathbf{r}}'_P(t_2)$. The particle's infinitesimal displacement, along the path $\mathcal{C}$, $d\vec{\mathbf{r}}'_P$, is depicted in Fig. 2.

In order for the result (7b) to hold true, the force $\vec{\mathbf{F}}_s^{(\mathcal{S}')}$ must satisfy the following requirement:

$$\vec{\mathbf{F}}_s^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q\big) = -\vec{\nabla}_{\vec{\mathbf{r}}'_P}\Big(V_s'^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q\big)\Big). \tag{8}$$

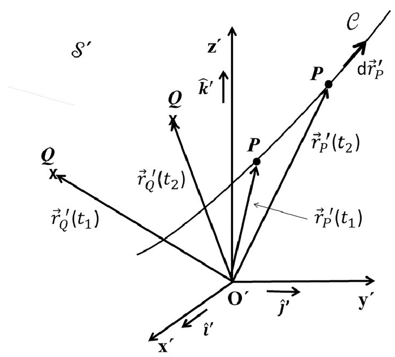

Figure 2: In $\mathcal{S}'$ frame, the particle moves along the trajectory $\mathcal{C}$. At the instants t_1 and t_2, we have the position vectors of the center of the conservative force (point Q) as $\vec{\mathbf{r}}'_Q(t_1)$ and $\vec{\mathbf{r}}'_Q(t_2)$, where $t_2 > t_1$. At distinct instants, the locations of the particle position vectors are: $\vec{\mathbf{r}}'_P(t_1)$ and $\vec{\mathbf{r}}'_P(t_2)$. At each instant t, the infinitesimal displacement of the particle along the trajectory $\mathcal{C}$ is represented by $d\vec{\mathbf{r}}'_P(t)$. The vector positions of the points P and Q are measured from the origin O' of the coordinate axes $x'y'z'$. These coordinate axes are fixed in the $\mathcal{S}'$ frame. $\hat{\imath}', \hat{\jmath}'$ and $\hat{k}'$ are versors of the axes x', y' and z', respectively. The $\mathcal{S}'$ frame can be an inertial or a non-inertial frame.

The function ${V_s'}^{(\mathcal{S}')}$ represents the potential energy of the particle as a result of the spring's action on it, as seen in the $\mathcal{S}'$ frame. $\vec{\nabla}_{\vec{\mathbf{r}}'_P}$ is the gradient of the function ${V_s'}^{(\mathcal{S}')}$ with respect to the vector $\vec{\mathbf{r}}'_P$[1]. The force $\vec{\mathbf{F}}_s^{(\mathcal{S}')}$ is conservative when the equality (8) holds.

It is simple to show that the potential energy of a particle acted on by a spring of constant k, with one end fixed at point Q and able of oscillating in three dimensions under the force (1), in the inertial $\mathcal{S}$ frame, is equal to:

$$\begin{aligned} &V_s^{(\mathcal{S})}\big(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\big) \\ &\quad = \frac{k}{2}\,|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q|^2 && \text{(9a)} \\ &\quad = \frac{k}{2}\,\left[(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2\right], && \text{(9b)} \\ &\quad = V_s^{(\mathcal{S})}\big(|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q|\big) && \text{(9c)} \end{aligned}$$

where

$$\begin{aligned} \vec{\mathbf{r}}_P(t) &= x_P(t)\,\hat{\imath} + y_P(t)\,\hat{\jmath} + z_P(t)\,\hat{k}, && \text{(10a)} \\ \vec{\mathbf{r}}_Q(t) &= x_Q(t)\,\hat{\imath} + y_Q(t)\,\hat{\jmath} + z_Q(t)\,\hat{k}. && \text{(10b)} \end{aligned}$$

$\hat{\imath}, \hat{\jmath}$, and $\hat{k}$ are the unitary vectors along the x, y and z axes, respectively. Fig. 1a shows these versors.

Equation (9a)/(9b) provides the potential energy $V_s^{(\mathcal{S})}$, which in the inertial $\mathcal{S}$ frame fulfills condition (8), that is,

$$\vec{\mathbf{F}}_s^{(\mathcal{S})}(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q)\underbrace{=}_{(9c)} -\vec{\nabla}_{\vec{\mathbf{r}}_P}\Big(V_s^{(\mathcal{S})}\big(|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q|\big)\Big). \quad \text{(10c)}$$

The results (6) and (4), and the expression (1) of the force $\vec{\mathbf{F}}_s^{(\mathcal{S})}$, permit us to write:

$$\vec{\mathbf{F}}_s^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q\big) = -k\,\big(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q\big). \quad \text{(11)}$$

The dependence of the force $\vec{\mathbf{F}}_s^{(\mathcal{S}')}$ on the measured relative position of the particle to the center of this force in the $\mathcal{S}'$ frame, $\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q$, is the same as the dependence of the conservative force $\vec{\mathbf{F}}_s^{(\mathcal{S})}(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q)$ on the measured relative position from the particle to the center of this same force in the inertial frame $\mathcal{S}$, $\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q$.

Due to the force $\vec{\mathbf{F}}_s^{(\mathcal{S}')}$, measured in any frame, having the same dependency on the particle's relative position to the force center, the particle's potential energy measured in the $\mathcal{S}'$ frame, ${V_s'}^{(\mathcal{S}')}\big(|\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q|\big)$, depends equally on the distance between the particle position P and the force center Q, which is measured in the $\mathcal{S}'$ frame. The expression of the function ${V_s'}^{(\mathcal{S}')}$ has to be similar to the result (9a), thus,

$$\begin{aligned} &{V_s'}^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q\big) \\ &\quad = \frac{k}{2}\left|\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q\right|^2 && \text{(12a)} \\ &\quad = \frac{k}{2}\,\left[\big(x'_P - x'_Q\big)^2 + \big(y'_P - y'_Q\big)^2 + \big(z'_P - z'_Q\big)^2\right] && \text{(12b)} \\ &\quad = {V_s'}^{(\mathcal{S}')}\big(|\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q|\big). && \text{(12c)} \end{aligned}$$

The components of the vectors $\vec{\mathbf{r}}'_P$ and $\vec{\mathbf{r}}'_Q$ in the coordinate axes $x'y'z'$, these axes being fixed in the $\mathcal{S}'$ frame, appear at r.h.s. of equation (12b),

$$\begin{aligned} \vec{\mathbf{r}}'_P(t) &= \quad x'_P(t)\,\hat{\imath}' + y'_P(t)\,\hat{\jmath}' + z'_P(t)\,\hat{k}', && \text{(13a)} \\ \vec{\mathbf{r}}'_Q(t) &= \quad x'_Q(t)\,\hat{\imath}' + y'_Q(t)\,\hat{\jmath}' + z'_Q(t)\,\hat{k}'. && \text{(13b)} \end{aligned}$$

$\hat{\imath}'$, $\hat{\jmath}'$ and $\hat{k}'$ are the versors of the axes x', y' and z', respectively. Fig. 2 shows the drawings of these unitary vectors.

As the mathematical dependence of the potential energy $V_s^{(\mathcal{S})}\big(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\big)$ of the components (x_P, y_P, z_P) and (x_Q, y_Q, z_Q), see eqs.(9a) and (9b), is the same as the function ${V_s'}^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q\big)$ in the components of (x'_P, y'_P, z'_P) and (x'_Q, y'_Q, z'_Q), see eqs.(12a) and (12b), therefore the following is true:

$$\vec{\mathbf{F}}_s^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q\big) = -\vec{\nabla}_{\vec{\mathbf{r}}'_P}\Big({V_s'}^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q\big)\Big), \quad \text{(14)}$$

Remember that ${V_s'}^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q\big)$ is the potential energy of the particle measured in the $\mathcal{S}'$ frame.

We remind the reader that the result (4) leads to equality (6). The result (4) is valid for any $\mathcal{S}'$ frame where the non-relativistic extended Galilean transformation, see eqs. (3a) and (3b), connects the information about the particle position and the center of the force to the ones measured in the inertial $\mathcal{S}$ frame. We follow the motion of the particle in a $\mathcal{S}'$ frame that can be inertial or non-inertial. The position of the points P and Q in this frame, from where the motion of the particle is observed, can vary over time.

The importance of equality (4) comes from the fact that the conservative force vector depends on the relative positions of two points in space: the position of the center of the force (point Q) and the position of the particle (point P) on which this force acts. The extended Galilean transformation (3a)/(3b) preserves the distance between any two points and the direction of the relative vector position that locates one of these points from the other.

Now, we address a subtle problem that arises when we write equation (8). It provides the necessary condition for the force $\vec{\mathbf{F}}_s^{(\mathcal{S})}$ to be a *conservative force*. We are

[1] We have:

$$\vec{\nabla}_{\vec{\mathbf{r}}'_P}\Big(G(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q)\Big) = \frac{\partial G}{\partial x'_P}\,\hat{\imath}' + \frac{\partial G}{\partial y'_P}\,\hat{\jmath}' + \frac{\partial G}{\partial z'_P}\,\hat{k}'.$$

$\hat{\imath}', \hat{\jmath}'$ and $\hat{k}'$ are the versors of the coordinate axes x', y' and z', fixed in the $\mathcal{S}'$ frame. G is any function of $\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q$. The variation of the vectors $\vec{\mathbf{r}}'_P$ and $\vec{\mathbf{r}}'_Q$ is independent of each other.

assuming here that the vectors $\vec{\mathbf{r}}'_P(t)$ and $\vec{\mathbf{r}}'_Q(t)$ have independent dynamics in any frame. In Nature, forces describe interactions between particles. When we have an interaction between two particles in which the mass of one of them is much, much, much greater than the mass of the other particle, the effect of the action of the lighter particle on the movement of the heavier particle is negligible. In this note, we suppose that the motion of the center of the force, point Q, which localizes the heavier particle's position in the interaction, is provided by an outside agent, that would be us!!! This hypothesis supports our assumption that when we calculate the partial derivative with respect to the components of the vector position $\vec{\mathbf{r}}'_P(t)$, the position vectors $\vec{\mathbf{r}}'_P(t)$ and $\vec{\mathbf{r}}'_Q(t)$ vary independently.

What is the relationship between the expressions of the particle's potential energy measured in the inertial frame $\mathcal{S}$, $V_s^{(\mathcal{S})}(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q)$ given by equation (9a)/(9b), and its potential energy measured in the frame $\mathcal{S}'$, ${V_s}'^{(\mathcal{S}')}(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q)$ given by equation (12a)/(12b)?

Beginning with equation (14), we obtain:

$$\vec{\mathbf{F}}_s^{(\mathcal{S}')}(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q)$$
$$= -\vec{\nabla}_{\vec{\mathbf{r}}'_P}\left({V_s}'^{(\mathcal{S}')}(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q)\right) \tag{15a}$$
$$\underbrace{=}_{(6)} \vec{\mathbf{F}}_s^{(\mathcal{S})}(\vec{\mathbf{r}}_P(t) - \vec{\mathbf{r}}_Q(t)) \tag{15b}$$
$$\underbrace{=}_{(10c)} -\vec{\nabla}_{\vec{\mathbf{r}}_P}\left(V_s^{(\mathcal{S})}(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q)\right) \tag{15c}$$
$$\underbrace{=}_{(4)} -\vec{\nabla}_{\vec{\mathbf{r}}'_P}\left(V_s^{(\mathcal{S})}(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q)\right) \tag{15d}$$
$$\Rightarrow\ \vec{\nabla}_{\vec{\mathbf{r}}'_P}\left({V_s}'^{(\mathcal{S}')}(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q)\right) = \vec{\nabla}_{\vec{\mathbf{r}}'_P}\left(V_s^{(\mathcal{S})}(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q)\right) \tag{15e}$$
$$\Rightarrow\ {V_s}'^{(\mathcal{S}')}(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q) = V_s^{(\mathcal{S})}(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q) + B. \tag{15f}$$

Any real number can be assigned to the constant B on the r.h.s. of equation (15f). Our choice of the value of the constant B corresponds to a particular choice of the zero value of potential energy in each physical problem discussed in the $\mathcal{S}'$ frame. The $\mathcal{S}'$ frame can be an inertial or a non-inertial frame.

By substituting the equality (4) on the r.h.s. of equation (15f), we arrive at:

$${V_s}'^{(\mathcal{S}')}(\vec{\mathbf{r}}'_P - \vec{\mathbf{r}}'_Q) = V_s^{(\mathcal{S})}(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q) + B. \tag{16}$$

According to the result (16), in the case of the conservative spring force acting on the particle, the potential energy in any reference frame $\mathcal{S}'$ has the same value, up to an additive constant.

We encourage the reader to proceed as before and consider the case of the conservative forces: the gravitational force [3, p. 140–144] and the Coulomb force [3, p. 145–147]

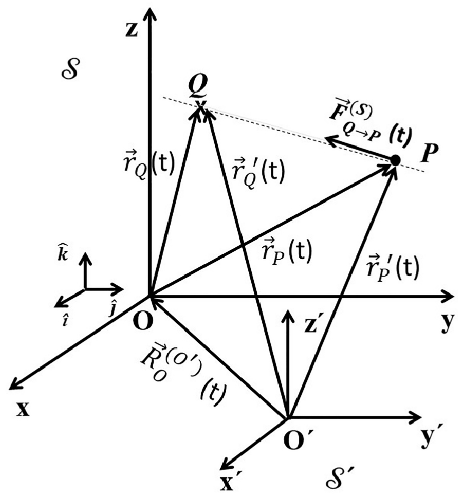

Figure 3: $\mathcal{S}$ is the inertial frame in which the coordinate axes xyz are fixed. In this frame, the conservative force $\vec{\mathbf{F}}^{(\mathcal{S})}_{Q\to P}$ is measured. Point Q is where the center of this conservative force is located. The particle localized at point P is subject to the force $\vec{\mathbf{F}}^{(\mathcal{S})}_{Q\to P}$. This force has the same direction as the line connecting the points P and Q. The vector positions of the points P and Q, measured from the origin O of the axes xyz, are denoted by $\vec{\mathbf{r}}_P(t)$ and $\vec{\mathbf{r}}_Q(t)$, respectively. A fixed coordinate axes $x'y'z'$ is located in the $\mathcal{S}'$ frame, which moves relative to the inertial $\mathcal{S}$ frame. The vector positions of the points P and Q, measured from the origin O' of the axes $x'y'z'$, are $\vec{\mathbf{r}}'_P(t)$ and $\vec{\mathbf{r}}'_Q(t)$, respectively. The location of the origin O with regard to the origin O' is given by the vector $\vec{\mathbf{R}}_O^{(O')}$. The versors of the axes x, y, and z, in the inertial $\mathcal{S}$ frame, are $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$, respectively. The $\mathcal{S}'$ frame can be an inertial or a non-inertial reference frame. All vectors in the figure are represented at time t.

Finally, we check whether the result (16), which is valid for the conservative spring force (1), can also be applied to the potential energy of conservative forces $\vec{\mathbf{F}}^{(\mathcal{S})}_{Q\to P}(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q)$ of the following type:

$$\vec{\mathbf{F}}^{(\mathcal{S})}_{Q\to P}(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q) = F(|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q|)\,(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q). \tag{17}$$

$\vec{\mathbf{F}}^{(\mathcal{S})}_{Q\to P}(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q)$ is the conservative force that acts on the particle, and it is measured in the inertial $\mathcal{S}$ frame, see Fig. 3. Since we want to describe physical forces, the F function on the r.h.s. of equation (17) is real and it depends only on the distance $|\vec{\mathbf{r}}_R - \vec{\mathbf{r}}_Q|$ at each instant t.

The vector position $\vec{\mathbf{r}}_Q(t)$ locates the center of the conservative force $\vec{\mathbf{F}}^{(\mathcal{S})}_{Q\to P}$ at any instant t in the inertial frame $\mathcal{S}$. The vector position $\vec{\mathbf{r}}_P(t)$ locates the particle in this frame at time t. The time evolution of the vector $\vec{\mathbf{r}}_Q(t)$ is an information provided in the physical scenario; therefore, it is independent of particle's movement, that is, it is independent of the vector $\vec{\mathbf{r}}_P(t)$.

The movement of the particle of mass m, observed from the inertial $\mathcal{S}$ frame, is given by Newton's Second Law, which describes the temporal evolution of the vector position $\vec{\mathbf{r}}_P(t)$,

$$m\,\frac{d^2\vec{\mathbf{r}}_P(t)}{dt^2} = \vec{\mathbf{F}}^{\,(\mathcal{S})}_{Q\to P}\big(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\big). \tag{18}$$

For the sake of simplicity, we assume that the only force acting on the particle is $\vec{\mathbf{F}}^{\,(\mathcal{S})}_{Q\to P}$, see equation (17).

For a conservative force of type (17), we show in Appendix A (Supplementary Material) that the potential energy $V^{\,(\mathcal{S})}$ associated with it is of type (equation (1): Supplementary Material),

$$V^{\,(\mathcal{S})}(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q) = V^{\,(\mathcal{S})}(\left|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\right|), \tag{19a}$$

and $\vec{\nabla}_{\vec{\mathbf{r}}_P}\big(V^{\,(\mathcal{S})}(\left|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\right|)\big)$ gives a vector of type (17), that is,

$$\vec{\nabla}_{\vec{\mathbf{r}}_P}\big(V^{\,(\mathcal{S})}(\left|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\right|)\big) = -F(\left|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\right|)\,(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q) \tag{19b}$$

$$= -\vec{\mathbf{F}}^{\,(\mathcal{S})}_{Q\to P}\big(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\big). \tag{19c}$$

Using the non-relativistic extended Galilean transformations (3a) and (3b), we may connect the vector positions measured in $\mathcal{S}'$ frame to the analogous vectors measured in the inertial $\mathcal{S}$ frame. For observers at rest in any $\mathcal{S}'$ frame, we use the extended Galilean transformation (3a) of the vector $\vec{\mathbf{r}}_P(t)$ to rewrite the equation (18) in terms of the vector $\vec{\mathbf{r}}\,'_P(t)$. $\vec{\mathbf{r}}\,'_P(t)$ measures the location of the particle (point P) by observers in the frame $\mathcal{S}'$. Equation (18) goes into:

$$m\,\frac{d^2\vec{\mathbf{r}}\,'_P(t)}{dt^2} = \vec{\mathbf{F}}^{\,(\mathcal{S})}_{Q\to P}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big) - m\,\frac{d^2\vec{\mathbf{R}}^{\,(O')}_O(t)}{dt^2}. \tag{20}$$

We rewrite the argument of the force $\vec{\mathbf{F}}^{\,(\mathcal{S})}_{Q\to P}$, on the r.h.s. of equation (20), using the equality (4). This relationship involving the relative position vector between the location of the particle (point P) and the center of the general conservative force of type (17) (point Q), $\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q = \vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q$, remains valid in any frame. $\vec{\mathbf{R}}^{\,(O')}_O(t)$ is the vector position of the origin O of the coordinate axes xyz, fixed in the inertial $\mathcal{S}$ frame, with respect to the origin O' of the axes $x'y'z'$, fixed in the $\mathcal{S}'$ frame.

The acceleration $\vec{\mathbf{A}}^{\,(O')}_O(t)$ of the inertial $\mathcal{S}$ frame relative to the $\mathcal{S}'$ frame is equal to:

$$\vec{\mathbf{A}}^{\,(O')}_O(t) \equiv \frac{d^2\vec{\mathbf{R}}^{\,(O')}_O(t)}{dt^2}. \tag{21}$$

When seen from the $\mathcal{S}'$ frame, the equation of motion of the particle with mass m takes the following form:

$$m\,\frac{d^2\vec{\mathbf{r}}\,'_P(t)}{dt^2} = \vec{\mathbf{F}}^{\,(\mathcal{S}')}_{Q\to P}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big) - m\,\vec{\mathbf{A}}^{\,(O')}_O(t), \tag{22}$$

where $\vec{\mathbf{F}}^{\,(\mathcal{S}')}_{Q\to P}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big)$ is the force that acts on the particle and is measured by observers at rest at the $\mathcal{S}'$ frame.

Comparing the first term on the r.h.s. of the equations (20) and (22), we obtain:

$$\vec{\mathbf{F}}^{\,(\mathcal{S}')}_{Q\to P}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big) = \vec{\mathbf{F}}^{\,(\mathcal{S})}_{Q\to P}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big) \tag{23a}$$

$$\underbrace{=}_{(4)} \vec{\mathbf{F}}^{\,(\mathcal{S})}_{Q\to P}\big(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\big). \tag{23b}$$

Equations (23a) and (23b) show that the conservative force of type (17) is the same in any $\mathcal{S}'$ frame. The latter frame has a non-relativistic movement in relation to the inertial $\mathcal{S}$ frame. The $\mathcal{S}'$ frame can be an inertial or a non-inertial frame.

The final query to be addressed is whether the force $\vec{\mathbf{F}}^{\,(\mathcal{S}')}_{Q\to P}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big)$, expressed as a vector measured in the $\mathcal{S}'$ frame,

$$\vec{\mathbf{F}}^{\,(\mathcal{S}')}_{Q\to P}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big) \underbrace{=}_{(23a)} \vec{\mathbf{F}}^{\,(\mathcal{S})}_{Q\to P}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big) \tag{24a}$$

$$\underbrace{=}_{(17)} F\big(\left|\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\right|\big)\,\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big), \tag{24b}$$

is a conservative force in all $\mathcal{S}'$ frames. If the last statement is correct there is a potential energy $V'^{\,(\mathcal{S}')}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big)$, that satisfies the relation (14), that is,

$$\vec{\mathbf{F}}^{\,(\mathcal{S}')}_{Q\to P}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big) = -\vec{\nabla}_{\vec{\mathbf{r}}\,'_P}\Big(V'^{\,(\mathcal{S}')}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big)\Big). \tag{25}$$

There is a function $V'^{\,(\mathcal{S}')}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big)$ that satisfies equality (25)?

Equations (23a) and (23b) allow us to obtain:

$$\vec{\mathbf{F}}^{\,(\mathcal{S}')}_{Q\to P}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big) = \vec{\mathbf{F}}^{\,(\mathcal{S})}_{Q\to P}\big(\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\big) \tag{26a}$$

$$\underbrace{=}_{(19c)} -\vec{\nabla}_{\vec{\mathbf{r}}_P}\Big(V^{\,(\mathcal{S})}\big(\left|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\right|\big)\Big). \tag{26b}$$

Due to the equality (4), we have

$$V^{\,(\mathcal{S})}\big(\left|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\right|\big) = V^{\,(\mathcal{S})}\big(\left|\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\right|\big). \tag{27}$$

The dependence of the function $V^{\,(\mathcal{S})}\big(\left|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\right|\big)$ on the components x_P, y_P and z_P of the vector $\vec{\mathbf{r}}_P$ in the axes xyz, see equation (10a), is the same as in the components x'_P, y'_P and z'_P of the vector $\vec{\mathbf{r}}\,'_P$ on the axes $x'y'z'$, see equation (13a), and consequently,

$$\vec{\nabla}_{\vec{\mathbf{r}}_P}\Big(V^{\,(\mathcal{S})}\big(\left|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q\right|\big)\Big) = \vec{\nabla}_{\vec{\mathbf{r}}\,'_P}\Big(V^{\,(\mathcal{S})}\big(\left|\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\right|\big)\Big). \tag{28}$$

The result (28) is used to rewrite equation (25),

$$\vec{\mathbf{F}}^{\,(\mathcal{S}')}_{Q\to P}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big) \underbrace{=}_{(25)} -\vec{\nabla}_{\vec{\mathbf{r}}\,'_P}\Big(V'^{\,(\mathcal{S}')}\big(\vec{\mathbf{r}}\,'_P - \vec{\mathbf{r}}\,'_Q\big)\Big) \tag{29a}$$

$$\underbrace{=}_{(26b)} -\vec{\nabla}_{\vec{\mathbf{r}}_P}\left(V^{(\mathcal{S})}\big(\vec{\mathbf{r}}_P-\vec{\mathbf{r}}_Q\big)\right)$$

$$\underbrace{=}_{(28)} -\vec{\nabla}_{\vec{\mathbf{r}}'_P}\left(V^{(\mathcal{S})}\big(|\vec{\mathbf{r}}'_P-\vec{\mathbf{r}}'_Q|\big)\right) \tag{29b}$$

$$\Rightarrow\ \vec{\nabla}_{\vec{\mathbf{r}}'_P}\Big[V'^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P-\vec{\mathbf{r}}'_Q\big)-V^{(\mathcal{S})}\big(|\vec{\mathbf{r}}'_P-\vec{\mathbf{r}}'_Q|\big)\Big]=0. \tag{29c}$$

Finally, we derive that from the preceding equation (29c) that:

$$V'^{(\mathcal{S}')}\big(\vec{\mathbf{r}}'_P-\vec{\mathbf{r}}'_Q\big)=V^{(\mathcal{S})}\big(|\vec{\mathbf{r}}'_P-\vec{\mathbf{r}}'_Q|\big)+B, \tag{30}$$

B being a real constant ($B\in\mathbb{R}$).

The result (30) brings us the conclusions: *i*) the force $\vec{\mathbf{F}}^{(\mathcal{S}')}_{Q\to P}\big(\vec{\mathbf{r}}'_P-\vec{\mathbf{r}}'_Q\big)$ of the type (17) is also a conservative force in any $\mathcal{S}'$ frame that connects to the inertial frame $\mathcal{S}$ through the non-relativistic extended Galilean transformations (3a) and (3b); *ii*) Up to an additive real constant B, the value of potential energy is the same in any $\mathcal{S}'$ frame. The choice of the value of the constant B leaves open the choice of zero potential energy in each physical situation in the $\mathcal{S}'$ frame from which the movement of the particle is observed. The $\mathcal{S}'$ frame can be an inertial or a non-inertial frame.

Conclusions

If the relative velocities between frames are much smaller than the velocity c of light, the conservative forces depend on two points: the force's center (point Q) and the particle's position (point P) that is being acted upon, and this force is also collinear with the particle's relative position to the force's center, (see equation (17)), the force's vector is the same in all frames. The frame may be non-inertial or inertial. The force is also conservative in all frames. Up to an additive constant, the potential energy's value remains the same over all these frames. We do, however, remind the reader that the value of the particle total mechanical energy varies with the frame. The total mechanical energy of the motion in the $\mathcal{S}'$ frame is not a constant if it is non-inertial. Conservative forces whose direction is not parallel to the relative location $\vec{\mathbf{r}}_P-\vec{\mathbf{r}}_Q$ are not included in the current result.

The present result is based on two properties of the extended Galilean transformations (3a)/(3b), that is, it preserves the distance between two points and their relative position vectors measured from one of these points.

Supplementary Material

The following online material is available for this article:
Appendix A

References

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