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Articles

Possible scenarios transgressing the nondegeneracy theorem

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In contrast to the nondegeneracy theorem, we present various scenarios in one-dimensional quantum mechanics that demonstrate how the Wronskian of two bound-state eigenfunctions with the same energy eigenvalue can be zero without implying that the eigenfunctions are linearly dependent. It is shown that the nondegeneracy theorem fails only when the potential makes different bound-state solutions corresponding to the same energy vanish at the singular point or region of singularity.
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The one-dimensional time-independent Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x) \tag{1}$$

requires continuous eigenfunctions. As a second-order differential equation, it has two linearly independent solutions that correspond to the same energy eigenvalue. However, the occurrence of the two-fold degeneracy is rare in describing one-dimensional bound states. Indeed, the absence of degenerate one-dimensional bound states is ensured by the nondegeneracy theorem (see, e.g. §21 in [1], Theorem 15 in [2], and also Problem 2.42 in [3] with the proviso that "the potential does not consist of isolated pieces separated by regions where $V=\infty$"). Motivated primarily by justifying the presence of degenerate bound states for the one-dimensional hydrogen atom $V(x)=-e^2/|x|$ [4], Loudon revisited the nondegeneracy theorem and properly concluded that it is not necessarily valid for a potential with singular points [5]. Since then a lot of controversy surrounds that problem [6–20]. Recently, the possibility of double degeneracy for a particle in a box has also been explored [21].

The usual proof of the nondegeneracy theorem considers the Wronskian of two eigenfunctions ψ_1 and ψ_2 corresponding to the same energy eigenvalue:

$$\begin{aligned} W\left(\psi_1(x),\psi_2(x)\right) &= \psi_1(x)\psi_2'(x)-\psi_1'(x)\psi_2(x) \\ &= \text{constant}, \quad \text{for all } x. \end{aligned} \tag{2}$$

For bound states, because ψ_1 and ψ_2 vanish for large $|x|$, $W\left(\psi_1,\psi_2\right)=0$ so

$$\psi_1(x)\psi_2'(x)-\psi_1'(x)\psi_2(x)=0. \tag{3}$$

Some authors simply divide (3) by $\psi_1\psi_2$, disregarding the zeros of ψ_1 and ψ_2, to obtain

$$\frac{\psi_2'(x)}{\psi_2(x)}=\frac{\psi_1'(x)}{\psi_1(x)}. \tag{4}$$

Therefore,

$$\int^x d\zeta\,\frac{\psi_2'(\zeta)}{\psi_2(\zeta)}=\int^x d\zeta\,\frac{\psi_1'(\zeta)}{\psi_1(\zeta)}+\text{constant}. \tag{5}$$

It follows that

$$\psi_2(x)=C\psi_1(x),\ \forall x, \tag{6}$$

where C is an arbitrary constant.

Loudon has cast doubt on the validity of (4) by drawing attention to possible troubles in the regions or at the points where the eigenfunctions have zeros [5]. To further elaborate on this problem we examine the behaviour of the eigenfunctions in the vicinity of a possible singular point x_0, where we assume linear dependence on each side of the point:

$$\begin{aligned} \psi_2(x)&=C_>\psi_1(x), \quad \text{for} \quad x\geqslant x_0, \\ \psi_2(x)&=C_<\psi_1(x), \quad \text{for} \quad x\leqslant x_0. \end{aligned} \tag{7}$$

We segregate the problem into two classes of eigenfunctions based on the behaviour of $\psi_1(x_0)$.

- *Class I:* $\psi_1(x_0)\neq 0$.
  For this class, $\psi_2(x_0)$ is also not equal to zero. As a result, we have

  $$C_>=C_<=\frac{\psi_2(x_0)}{\psi_1(x_0)}\neq 0, \tag{8}$$

  which implies that ψ_1 and ψ_2 are linearly dependent functions. To better understand what happens with the logarithmic derivatives, we

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substitute (7) into (3) and find that $\psi_1'(x_0+) - \psi_1'(x_0-)$ is indeterminate. Here, $\psi_1(x_0\pm)$ indicates the limit of $\psi_1(x)$ as x approaches x_0 from $x \gtrless x_0$. By setting $\psi_1'(x_0+) = \psi_1'(x_0-)$ into (7), we obtain

$$\frac{\psi_2'(x_0+)}{\psi_2(x_0)} = \frac{\psi_2'(x_0-)}{\psi_2(x_0)} = \frac{\psi_1'(x_0)}{\psi_1(x_0)}, \tag{9}$$

which states that the logarithmic derivatives are continuous functions at x_0. This always occurs when the potential is regular at x_0. However, if $\psi_1'(x_0+) \neq \psi_1'(x_0-)$ then

$$\begin{aligned} \frac{\psi_2'(x_0+)}{\psi_2(x_0)} &= \frac{\psi_1'(x_0+)}{\psi_1(x_0)} \\ \frac{\psi_2'(x_0-)}{\psi_2(x_0)} &= \frac{\psi_1'(x_0-)}{\psi_1(x_0)}. \end{aligned} \tag{10}$$

Note that the logarithmic derivative in this last case can be integrated across x_0 as in (5), despite its jump discontinuity. This scenario typically occurs when the potential gives a dominant contribution proportional to $\delta(x - x_0)$ at x_0 (see, e.g. [3, 16, 22]).

- *Class II:* $\psi_1(x_0) = 0$.
  In this class, $\psi_2(x_0) = 0$ and there is no logical connection between $C_>$ and $C_<$ as in Class I. Furthermore, Eq. (3) does not establish a connection between the first derivatives of ψ_1 and ψ_2 at the right and at the left of x_0. The logarithmic derivative is also meaningless for this class. This situation, with ψ_2 for $x > x_0$ independent of ψ_2 for $x < x_0$, arises from an infinite potential at the right or at the left of x_0. Examples of this include the problem of an infinite double well and also in the problem of a finite double well as the barrier width tends to infinity (see, e.g. Problem 2.44 in [3], Complement $\mathrm{G_{IV}}$ in [22], Sec. 6.6 in [23], and also Sec. 8.3.9 in [24]). Two-fold degenerate spectra also appear for the potentials proportional to $x^2 + \alpha x^{-2}$ (see, e.g. [25]) and $|x|^{-1} + \alpha x^{-2}$ (see, e.g. [26]). For example, if we have

$$\psi_1(x) \simeq C(x - x_0)^s \quad (s > 0), \quad \text{for} \quad x \simeq x_0, \tag{11}$$

  then the two-fold degeneracy can appear if the potential $\alpha|x - x_0|^{-2}$ dominates at x_0 (with $s > 1$ for $\alpha > 0$, and $1/2 < s < 1$ for $\alpha < 0$) [26], a conclusion that differs from that one found in Ref. [5]. A two-fold degenerate spectrum is also seen for a Dirac delta potential embedded in a box in the strong coupling limit [16].

A necessary condition for linear independence of two functions ψ_1 and ψ_2 is that the Wronskian does not vanish. With $\psi_2(x) = C\psi_1(x)$ one obtains $W(\psi_1, \psi_2) = 0$ but the converse is not necessarily true. If $W(\psi_1, \psi_2) = 0$ and ψ_1 is infinitely differentiable at x_0 with $\psi_1(x)|_{x_0} \neq 0$, it is easy to conclude that ψ_2 is proportional to ψ_1. Differentiating (3) repeatedly yields

$$\left.\frac{d^n\psi_2(x)}{dx^n}\right|_{x_0} = C \left.\frac{d^n\psi_1(x)}{dx^n}\right|_{x_0}, \quad C = \text{constant}, \ n \in \mathbb{N}. \tag{12}$$

Thus,

$$\begin{aligned} \psi_2(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n\psi_2(x)}{dx^n}\right|_{x_0} (x - x_0)^n \\ &= C\sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n\psi_1(x)}{dx^n}\right|_{x_0} (x - x_0)^n = C\psi_1(x). \end{aligned} \tag{13}$$

Even if $\psi_1(x)|_{x_0} \neq 0$ and $\psi_1'(x_0+) \neq \psi_1'(x_0-)$, we find $\psi_2 = C\psi_1$, according to the theory of distributions. However, if $\psi_1(x)|_{x_0} = 0$, then the process of repeated differentiation can not establish a connection between the nth order derivatives of ψ_1 and ψ_2. This implies that we can not connect ψ_2 for $x > x_0$ and ψ_2 for $x < x_0$. In other words, we can not establish whether ψ_1 and ψ_2 are linearly dependent. Therefore, we can not ruled out any possibility of a two-fold degeneracy.

In conclusion, we presented in a straightforward way that the vanishing of the Wronskian of two bound-state eigenfunctions does not guarantee their linear dependence and that the zeros of the eigenfunctions may cause the nondegeneracy theorem to fail, as noted by Loudon [5]. Additionally, we presented fair scenarios that illustrate two-fold degeneracies for bound states in one-dimensional quantum mechanics. The nondegeneracy theorem fails only when the potential has the pathological feature that makes different bound-state solutions corresponding to the same energy vanish at the singular point or region. In this case, the lack of connection between the derivatives of those eigenfunctions on each side of the singular point leads us to agree with Andrews [8, 10, 14]: the singular point or region acts as an impenetrable barrier.

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