Revista Brasileira de Ensino de Física, v. 31, n. 1, 1303 (2009)
www.sbfisica.org.br

The Saros cycle: obtaining eclipse periodicity from Newton's laws

(O ciclo do Saros: como obter a periodicidade dos eclipses a partir das leis de Newton)

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Recebido em 5/8/2008; Revisado em 19/9/2008; Aceito em 24/9/2008; Publicado em 30/4/2009

The Saros cycle has been known since antiquity and refers to the periodicity of eclipses. It is the least common multiple of three periods: the synodic, the draconic, and the anomalistic months. We show how to obtain these periods from Newton's laws with a precision greater than 0.02% using only the sidereal month and year as references.
Keywords: Saros cycles, eclipses, Newton's laws, restricted 3-body problem.

O ciclo do Saros é conhecido desde a antiguidade e se refere a periodicidade dos eclipses. Ele é o mínimo múltiplo comum de três períodos: os meses sinódico, dracônico e anomalístico. Mostramos neste trabalho como obter estes três períodos a partir das leis de Newton com uma precisão superior a 0.02% usando apenas os mês e ano siderais como referência.
Palavras-chave: ciclo de Saros, eclipses, leis de Newton, problema restrito de 3 corpos.

1. Introduction

The periodicity of solar and lunar eclipses is approximately 6585.3 days (18 years, 10 or 11 days, and 8 hours). This period, called the *Saros cycle*, has been known since at least the Babylons, who accurately determined it around 500 b.C., and it was probably known to the constructors of Stonehenge [1-4]. Detecting this regularity requires long-time observations and (oral or written) records. The tablets known as the "Babylonian Astronomical Diaries" record almost daily observations from the sky since the 8th century b.C. to the 1st century b.C. The Babylonians not only discovered the Saros period but also described precisely the lunar motion. It is still not clear how they derived their theory from the data [5]. Using the Babylonian theory, the Greeks were able to build a mechanical device, called the *Antikythera*, able to predict both solar and lunar eclipses. This device is supposed to be a mechanical realization of Hyparchos lunar theory and is considered to be the most complex known human-made mechanism in more than 1000 years [6]. Stonehenge, the prehistorical megalithic monument located in England, is supposed to be an astronomical observatory build in the second or third millennium b.C. Although its precise function is still under dispute, the most popular theory was advanced by the British archaeoastronomer Gerald Hawkins in 1963 [3]. According to Hawkins, Stonehenge was a "Neolithic computer" with dozens of alignments with the pre-historical sky. The monument's main purpose was to predict eclipses. Additional background on the history and the astronomy of lunar cycles [7-9] and solar eclipse [10, 11] can be found in the references.

The Saros period represents the least common multiple of three periods: the time between two full moons (the *synodic month*), the time between two passages of the Moon in the ascending node (the *draconic month*), and the time between two lunar apogees (the *anomalistic month*). That is, the relative geometry will be the same when the Moon is in the same point of its orbit (with respect to the Earth-Sun line), and the orbit is in the same plane and has the same form. Because the interval between two full moons is the same on the average as the interval between two new moons, the Saros cycle applies equally well to both lunar and solar eclipses. (However, these intervals are not always equal, depending on the position of the lunar perigee [12, 13]).

One Saros is approximately (to within 2 hours) equal to 223 synodic months, 242 draconic months, and 239 anomalistic months. After that time the Sun, the Earth, and the Moon return to approximately the same relative geometry. In other words, in one Saros period

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after a given eclipse, another very similar eclipse occurs, displaced by 120 degrees in longitude. This displacement is due to the remaining eight hours in the duration of the Saros cycle. The fact that successive eclipses are not equal implies that there is a maximum number of eclipses that generate a finite Saros series, starting from the first partial eclipse until the last partial eclipse related to this periodicity. Successive eclipses in the same Saros series are separated by one Saros cycle. For example, Saros series 131 began in 1427 and will last until 2707. This series consists of 70 eclipses or 69 cycles. The partial Solar eclipse of March 19, 2007 was the 20th partial eclipse of Saros series 149, whose first total eclipse will be in 2049 [14].

The Saros cycle is most frequently used in the prediction of eclipses [15]. However, the British astronomer E. Halley (who named this cycle), combined it with Newton's lunar theory to improve the accuracy of longitudinal calculations [16].

In this paper we show how to obtain the values of the synodic, draconic and anomalistic months and the Saros cycle using as inputs the sidereal month, that is, the time necessary for a full revolution of the Moon in the frame of the fixed stars, given by $T_0 = 27.321662$ days (1 day = 86400 s), and the terrestrial sidereal year $T_{\mathrm{y}} = 365.2555636$ days. The experimental values for the synodic, draconic and anomalistic months are given in Table 1. See also Table 2 for the astronomical constants refereed in the text.

Table 1 - Astronomical data and symbols relevant to the present work. All data here as measured on 1 January 2000. One day (d) means 86400 s.

Inclination of the ecliptic	β	5.145°
Terrestrial year (sidereal)	T_{y}	365.25636 d
Sidereal month	T_0	27.321662 d
Synodic month	T_{s}	29.530589 d
Draconic month	T_{d}	27.212221 d
Anomalistic month	T_{a}	27.554550 d

The synodic month is given exactly as a function of T_0 and T_{y} in section 2., and the other two months are expressed as a series in $T_0/T_{\mathrm{y}} \approx 0.0748$. The draconic month is obtained in section 3. following the techniques used in [17] for calculating the precession of the equinoxes. In section 4. we obtain the anomalistic month with tools developed for calculating the nonrelativistic contribution of the outer planets to Mercury's perihelion precession [18]. To the best of our knowledge, our procedure is the first direct calculation of the components of the Saros cycle using Newton's laws. This procedure gives a precision of more than 0.02%. The differences between the draconic and the sidereal months and between the anomalistic and the sidereal months are of 0.4% and 0.8% respectively. These differences are much smaller than the difference between the synodic month and the sidereal month, which is about 8%, because the synodic month differs from the sidereal month to first order in T_0/T_{y}, and the draconic and anomalistic months differ to second order.

We also give a partial solution to an important class of restricted three-body problems, $m_1 \gg m_2 \gg m_3$, where m_1 (Sun) is fixed, m_2 (Earth) revolves in a circular orbit around m_1, and m_3 (Moon) revolves around m_2 in a near-elliptic orbit (perturbed by m_1). We determine the periodicities of the relative geometry.

Because the Moon's eccentricity, $e_{\mathrm{m}} = 0.0549$, is 3.3 times larger than Earth's eccentricity, $e_{\mathrm{e}} = 0.0167$, we approximate the Earth's orbit by a circle. We first obtain the exact mean lunar period with respect to the moving Sun-Earth line (see section 2). Then we determine the precession of the lunar orbit using a first-order series expansion in the lunar eccentricity and a third-order expansion in the ratio of the average Moon-Earth distance to the average Sun-Earth distance. In both cases the mean effect of the highest order is zero (section 3). Finally, we determine the lunar apogee precession due to the Earth-Moon movement around the Sun (see section 4). Any interval that is commensurate with the synodic, draconic, and anomalistic months implies the repetition of the relative geometry. The Saros is the minimum of all these intervals.

The necessary background can be obtained in any book of Newtonian mechanics (see, for example [19]). Many aspects of Sun-Earth-Moon dynamics are similar to the treatment of the dynamics of artificial satellites, as perturbed by the Sun and the Moon [20].

Table 2 - Astronomical data and symbols used in the text. The values are not necessary and given only for completeness.

Sun (mass)	M_{s}	1.988435×10^{30} kg
Earth (mass)	M_{e}	5.9736×10^{24} kg
Moon (mass)	M_{m}	7.3477×10^{22} kg
Earth-Moon semi-major axis	R_{em}	3.84399×10^{8} m
Earth-Sun semi-major axis	R_{es}	1.4959787×10^{11} m
Constant of gravitation	G	6.67428×10^{-11} m^{3}kg^{-1}s^{-2}
Eccentricity of Moon's orbit	e_{m}	0.0549
Eccentricity of Earth's orbit	e_{e}	0.0167

2. The synodic month

The calculation of the synodic month is the easiest part. The synodic month is the average time between two full moons. So, we have to consider both the movement of the Moon around the Earth and the movement of the Earth around the Sun. This calculation follows from two simple proportions which can be understood with the help of Fig. 1.

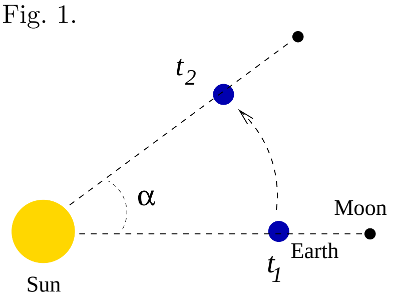

Figure 1 - In one synodic month, the Earth revolves around the Sun by an angle α, while the Moon revolves around the Earth by an angle $2\pi + \alpha$. The times t_1 and t_2 denote two successive full moons.

By comparing the sidereal terrestrial year T_y with the synodic month, we conclude that the angle α, the angular displacement of the Earth during one synodic month, is given by $\alpha = 2\pi T_s/T_y$. In one synodic month, the Moon revolves around the Earth by an angle of $2\pi + \alpha$, while in one sidereal month, it revolves around the Earth by exactly 2π. These two relations implies that $T_s = (2\pi + \alpha)T_0/(2\pi)$. We solve for α and T_s and find

$$T_s = \frac{T_0 T_y}{T_y - T_0} \approx 29.530589 \text{ days} \quad \text{(exact)}, \tag{1}$$

T_y and T_0 are given in Table 1.

3. The draconic month

The Moon's orbital plane is inclined with respect to the ecliptic by the angle $\beta = 5.145°$. The intersection of the two orbital planes (the Earth around the Sun and the Moon around the Earth) is called the *line of nodes*. Two successive passages of the Moon through the ascending node (the point where the Moon crosses the ecliptic moving to the Northern hemisphere) is called the *draconic month* T_d (see Fig. 2).

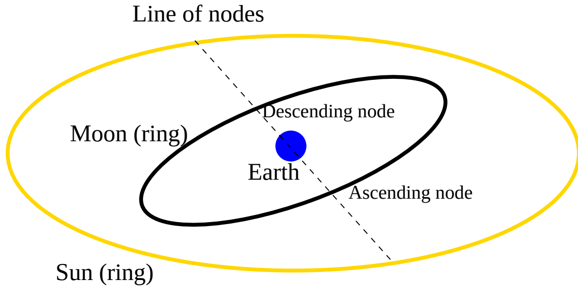

Figure 2 - We consider the Sun and the Moon to be in rings around the Earth in order to consider only their mean effect: the precession of the line of nodes.

The orbital path of the Moon precesses due to the influence of the Sun. We will estimate the average precession and the value of T_d. We consider the average effect caused by the Sun in one year. Then we replace the Sun by a ring of radius R_{es} and mass M_s. This method is called *averaging* in the field of dynamical systems. In this case, we replace a function by its average in order to obtain asymptotic expansions in periodic motions perturbed by secular terms. See Ref. [21] for a general introduction to the method and Ref. [17] for an application to celestial mechanics. We choose the origin to be at the center of the ring (the center of the Earth). The potential energy due to an element of mass of the Sun dM_s at a distance $r < R$ from the center of the ring is

$$dV(\mathbf{r}) = -G\frac{m\,dM_s}{\|\mathbf{R} - \mathbf{r}\|}, \tag{2}$$

where $\mathbf{R}$ and $\mathbf{r}$ are the position of the elementary mass dM_s and of the mass m, respectively. We choose a coordinate system such that the vector $\mathbf{r}$ is in the xz plane and the Solar ring is in the xy plane. The unit vectors are denoted by $\hat{x}$, $\hat{y}$, and $\hat{z}$. Then $\mathbf{r} = r\cos\phi\hat{x} + r\sin\phi\hat{z}$ and $\mathbf{R} = R\cos\theta\hat{x} + R\sin\theta\hat{y}$. We define the density of the Solar ring as $\lambda_s = M_s/(2\pi R_{es})$. Then

$$dV = -\frac{Gm\lambda_s R_{es} d\theta}{\sqrt{R_{es}^2 + r^2 - 2R_{es} r\cos\phi\cos\theta}} \approx -Gm\lambda_s\left[1 + \cos\phi\cos\theta\frac{r}{R_{es}} + \frac{1}{2}(3\cos^2\phi\cos^2\theta - 1)\left(\frac{r}{R_{es}}\right)^2 + \frac{1}{2}(5\cos^3\phi\cos^3\theta - 3\cos\phi\cos\theta)\left(\frac{r}{R_{es}}\right)^3\right]. \tag{3}$$

We integrate over θ from 0 to 2π and find

$$V(\mathbf{r}) = -\frac{GmM_s}{R_{es}}\left[1 + \frac{1}{2}\left(\frac{r}{R_{es}}\right)^2\left(\frac{3}{2}\cos^2\phi - 1\right)\right]. \tag{4}$$

We next consider the Moon as an elliptic ring of uniform density $\lambda_m = M_m/(2\pi R_{em})$ and choose a coordinate system with its origin in one of the foci (the position of the Earth), and with the same origin of the

xyz system, and x' coinciding with the line of nodes. The orbit of the Moon lies in the $x'y'$ plane. Then $\hat{x}' = \hat{x}$, $\hat{y}' = \cos\beta\hat{y} + \sin\beta\hat{z}$, and $\hat{z}' = -\sin\beta\hat{y} + \cos\beta\hat{z}$. Any point can be represented in spherical coordinates by $\mathbf{r} = r\sin\xi\cos\gamma\hat{x}' + r\sin\xi\sin\gamma\hat{y}' + r\cos\xi\hat{z}'$, where ξ is the co-latitude, and γ is the azimuth. (The lunar ring lies in the plane defined by $\xi = \pi/2$.) We have that

$$\sin\phi = \frac{\mathbf{r}\cdot\hat{z}}{r} = \frac{1}{r}(\sin\beta\mathbf{r}\cdot\hat{y}' + \cos\beta\mathbf{r}\cdot\hat{z}') = \sin\beta\sin\gamma\sin\xi + \cos\beta\cos\xi. \quad (5)$$

The potential energy of an element of mass $\mathrm{d}M_\mathrm{m}$ is given by

$$\mathrm{d}V(\mathbf{r}) = -\frac{G\mathrm{d}M_\mathrm{m}M_\mathrm{s}}{R_\mathrm{es}}\left[1 + \frac{r^2}{2R_\mathrm{es}^2}\left(\frac{3}{2}\left(1 - (\sin\beta\sin\gamma\sin\xi + \cos\beta\cos\xi)^2\right) - 1\right)\right]. \quad (6)$$

The corresponding force is given by $\mathrm{d}\mathbf{F} = -\nabla\mathrm{d}V$. The radial component $\mathrm{d}F_r$ cannot generate torque for obvious reasons. The torque associated with the polar component $\mathrm{d}F_\gamma$ is in the $\hat{z}'$ direction, and therefore cannot change the angular momentum nor the orbital plane. The only interesting component is given by

$$\mathrm{d}F_\xi = -\frac{1}{r\sin\frac{\pi}{2}}\frac{\partial}{\partial\xi}\mathrm{d}V(r, \xi = \frac{\pi}{2}, \gamma) = \frac{3G\mathrm{d}M_\mathrm{m}M_\mathrm{s}r}{4R_\mathrm{es}^3}\sin(2\beta)\sin\gamma. \quad (7)$$

In an elliptical orbit we have that

$$\mathbf{r} = \frac{R_\mathrm{em}(1 - e_\mathrm{m}^2)}{1 + e_\mathrm{m}\cos\gamma}(\cos\gamma\hat{x}' + \sin\gamma\hat{y}'). \quad (8)$$

The elementary torque in the $x'y'$ plane is given by

$$\mathrm{d}\mathbf{N}_{xy} = \mathbf{r}\times(\mathrm{d}F_\xi\hat{\xi}) = \frac{3G\lambda_\mathrm{m}M_\mathrm{s}r^3}{4R_\mathrm{es}^3}\sin(2\beta)\times \sin\gamma(-\sin\gamma\hat{x}' + \cos\gamma\hat{y}')\mathrm{d}\gamma \approx \frac{3G\lambda_\mathrm{m}M_\mathrm{s}R_\mathrm{em}^3}{4R_\mathrm{es}^3}\sin(2\beta)\times \sin\gamma(1 - 3e_\mathrm{m}\cos\gamma)(-\sin\gamma\hat{x}' + \cos\gamma\hat{y}')\mathrm{d}\gamma. \quad (9)$$

We integrate γ from 0 to 2π and find that

$$\mathbf{N}_{xy} = -\frac{3G\lambda_\mathrm{m}M_\mathrm{s}R_\mathrm{em}^3}{4R_\mathrm{es}^3}\pi\sin(2\beta)\hat{x} = -\frac{3GM_\mathrm{m}M_\mathrm{s}R_\mathrm{em}^2}{8R_\mathrm{es}^3}\sin(2\beta)\hat{x}. \quad (10)$$

The angular momentum $\mathbf{L}$ rotates in a circle of radius $L\sin\beta$. Thus the precession velocity is given by

$$\omega_\mathrm{d} = \frac{N_{xy}}{L\sin\beta} = -\frac{3GT_0M_\mathrm{s}}{8\pi R_\mathrm{es}^3}\cos\beta, \quad (11)$$

where $L = M_\mathrm{m}R_\mathrm{em}^2\omega$ and $\omega = 2\pi/T_0$.

To obtain the draconic month, we note that the ascending node goes backward, while the Moon goes forward. So in one draconic month T_d we have $\omega T_\mathrm{d} = 2\pi - |\omega_\mathrm{d}|T_\mathrm{d}$, that is,

$$T_\mathrm{d} = \frac{T_0}{1 + \frac{3GM_\mathrm{s}T_0^2\cos\beta}{16\pi^2R_\mathrm{es}^3}} = \frac{T_0}{1 + \frac{3T_0^2\cos\beta}{4T_y^2}} \approx 27.207946 \text{ days} \quad (\text{err} = 0.016\%), \quad (12)$$

where $(2\pi/T_\mathrm{y})^2 = GM_\mathrm{s}/R_\mathrm{es}^3$, and T_y, T_0 and β are given in Table 1.

4. The anomalistic month

In this section, we will calculate the *anomalistic month*, the time between two apogees. After such an interval, the distance between the Earth and the Moon is the same. The Moon's orbit is elliptic, with a small eccentricity, and therefore the precise time between two full Moons oscillates around the synodic month [22]. Strict periodicity occurs only when the Moon is in the same phase. Consequently the anomalistic month should be included in the calculation of the Saros cycle. This would not be necessary if the Moon's orbit were circular.

Our first step is to put a frame of reference on the center of the Sun, rotating with velocity $\omega = \sqrt{GM_\mathrm{s}/R_\mathrm{es}^3}$, such that the Earth-Sun line is fixed. Then, we put a second reference frame, at the center of the Earth, with its x axis pointing to the Sun. In this frame, the Moon is affected by two different centrifugal forces; the first one due to the rotation around the Sun (which will cancel, in average, the Sun's attracion on the Moon) and the second due to the fact that the reference frame rotates around itself. This second effect is exactly the perturbative effect causing the movement of the apogee.

We let μ be the Earth-Moon reduced mass and take $\rho \approx R_\mathrm{es} \pm R_\mathrm{em}$ to be the Sun-Moon distance and $\hat{\rho}$ the unity vector from the Sun to the Moon. Adding all radial forces over the Moon we find

$$\Phi(r) = -G\frac{M_\mathrm{e}\mu}{r^2} + \mu\omega^2 r + (\mathbf{F}_\mathrm{sun} - \mu\omega^2\rho\hat{\rho})\cdot\hat{r}, \quad (13)$$

where $\mathbf{F}_\mathrm{sun}$ is the gravitational force generated by the Sun (see Fig. 3).

We have that

$$\mu\ddot{R}_\mathrm{em} = -\frac{GM_\mathrm{e}\mu}{R_\mathrm{em}^2} + \mu\omega^2R_\mathrm{em} + GM_\mathrm{s}\mu\left(R_\mathrm{es}^{-2} - \rho^{-2}\right) \quad (14)$$

$$\approx \mu\omega^2R_\mathrm{em} - \frac{GM_\mathrm{e}\mu}{R_\mathrm{em}^2}\left(1 - \frac{2M_\mathrm{s}}{M_\mathrm{e}}\left(\frac{R_\mathrm{em}}{R_\mathrm{es}}\right)^3\right), \quad (15)$$

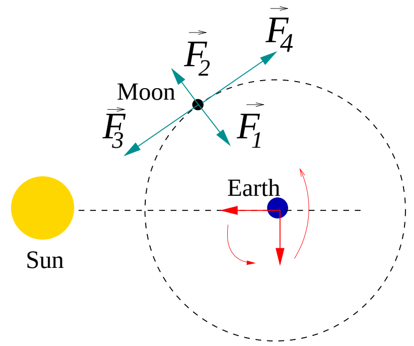

Figure 3 - The reference frame is centered on the Earth and rotates 2π radians per year around its origin and 2π radians per year around the center of the Sun. The forces over the Moon are given by the Earth's attraction $\mathbf{F}_1$, the centrifugal force due to the movement of the axis around itself $\mathbf{F}_2$, the Sun's attraction $\mathbf{F}_3$, and the centrifugal force due to the movement of the axis around the Sun $\mathbf{F}_4$. In the approximation used in the text, we consider $\mathbf{F}_3 + \mathbf{F}_4 \approx \mathbf{0}$.

with $\frac{2M_s}{M_e}(R_{em}/R_{es})^3 \approx 10^{-3}$. Therefore, the joint effect of the Sun's attraction and the centrifugal force due to the frame rotation around the Sun does not cause relative motion between the Earth and the Moon and will be henceforth omitted. Finally, we re-write Eq. (13) as

$$\Phi(r) = -G\frac{M_e\mu}{r^2} + G\frac{M_s\mu r}{R_{es}^3}\ . \tag{16}$$

The radial equation is given by $\Phi(r) = \mu(\ddot{r} - r\dot{\theta}^2)$, and the conserved angular momentum is $L = \mu r^2\dot{\theta}$, that is,

$$\Phi(r) = \mu\ddot{r} - \frac{L^2}{\mu r^3}. \tag{17}$$

For a circular orbit of radius r_0 we have $\Phi(r_0) = -L^2/(\mu r_0^3)$. We consider small perturbations of this radius, *i.e.*, we write $r = r_0 + \varepsilon r_1$, and conclude that the perturbative term obeys

$$\mu\ddot{r}_1 - \left[\frac{3\Phi(r_0)}{r_0} + \Phi'(r_0)\right] r_1 = 0. \tag{18}$$

Equation (18) implies (assuming that we have the correct sign in the bracket) an oscillation with period

$$T = \frac{2\pi\sqrt{\mu}}{\sqrt{-\left[\frac{3\Phi(r_0)}{r_0} + \Phi'(r_0)\right]}}. \tag{19}$$

If we assume constant angular velocity $\dot{\theta} = L/(\mu r_0^2) = \sqrt{-\Phi(r_0)/(\mu r_0)}$, during the interval T given by Eq. (19) and impose $r_0 = R_{em}$, the angular displacement between two successive apogees is given by

$$\Delta\Theta = T\dot{\theta} = \frac{2\pi}{\sqrt{3 + R_{em}\dfrac{\Phi'(R_{em})}{\Phi(R_{em})}}} \approx$$
$$2\pi\left(3 - \frac{2 + \dfrac{\omega^2 R_{em}^3}{GM_e}}{1 - \dfrac{\omega^2 R_{em}^3}{GM_e}}\right)^{-\frac{1}{2}} \approx 2\pi\left(1 + \frac{3M_s R_{em}^3}{2M_e R_{es}^3}\right). \tag{20}$$

We conclude that the anomalistic month is given by

$$T_a = T_0\left(1 + \frac{3M_s R_{em}^3}{2M_e R_{es}^3}\right) = T_0\left(1 + \frac{3T_0^2}{2T_y^2}\right)$$
$$\approx 27.550969\,\text{days} \qquad (\text{error} = 0.013\%), \tag{21}$$

where T_y and T_0 are given in Table 1.

From Eqs. (1), (12), and (21), we can calculate the Saros cycle, the least common multiple of T_s, T_d, and T_a.

5. Conclusions

As the introduction shows, the Saros cycle was known by many old societies. Unfortunately, it is almost absent from basic textbooks in both celestial dynamics and classical mechanics. Even the fact that eclipses are periodic seems not to be widely known, specially among students. The present work not only briefly introduces the history behind the empirical calculation of this period, but also shows that it is not difficult to obtain it with a high degree of accuracy using only Newtonian dynamics.

Furthermore, the Saros calculation seems to be a simple but non trivial way to explore different mathematical techniques like asymptotic expansions and homogenization. The fact that the final results depend only on the ratio between the sidereal month and year, and the inclination of the ecliptic, two non-dimensional numbers, comes as no surprise to anyone who had studied dimensional analysis and particularly the Buckingham Pi theorem. Unfortunately dimensional analysis, a simple and powerful technique, is generally not properly studied in basic courses.

Finally, this work can be used in classical or celestial mechanics intermediate courses as introduction to diverse mathematical techniques, and it can also be considered of cultural value.

Acknowledgments

This work was supported by FCT/Portugal. I also would like to thank M. Souza for a careful revision of the manuscript.

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