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Charged scalar field propagator under an external magnetic field: a connection between eigenfunction and proper time methods

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We present a comparison between the bosonic Feynman propagator in an external magnetic field calculated by Ritus' method with summation over Landau levels and its integral expression obtained via Schwinger's proper time method. Also, we have investigated the behavior boson system in a thermal reservoir under an increase in the external magnetic field strength.
Keywords: Scalar field, Landau levels, Closed expression.

1. Introduction

Magnetic fields of order $B \approx 10^{18}\ G$ created in ultra-relativistic heavy-ion collisions or present in neutron stars have a special influence on the phase transition of dense hadronic matter[1–4]. The presence of these magnetic fields on the physical systems brings a series of interesting phenomena to be analyzed, namely: magnetic catalysis (MC) [5–10], inverse magnetic catalysis (IMC) [11–17], and breaking-restoration of chiral symmetry [18, 19]. A mesonic system under an external magnetic background also was considered in [20–23]. In the finite size context, we have found that the magnetic field produces the direct and inverse catalysis effects in the Gross-Neveu model in the Refs. [24, 25] and [24, 26], respectively. In the Nambu–Jona-Lasinio model, the IMC phenomenon at finite size was found in [27] and the MC considering finite size of the system was found in [4, 28].

To include these magnetic effects in the theoretical model, we have to calculate the Feynman propagator. It is an important function to achieve results in quantum field theory (QFT). In particular, when we have to take into account magnetic effects coming from an external field, the Feynman propagator gets some difficulties to be calculated. Some very important papers in the literature are dedicated to exploring the external field effects in the Feynman propagator with different approaches [29–32]. In this paper, we propose investigating the relation between Landau levels obtained by Ritus' method (eigenfunctions method) and the method due to J. Schwinger (considering one extra dimension: the proper time S) in the calculation of the Feynman propagator. In other words, we shall investigate how to obtain one method from another, taking into account the zero spin quantum field.

V. Ritus developed the eigenfunction method in the 1970′s (see Ref. [31, 32]). This method allows writing the propagator under an external field as in the free case, i.e., in a diagonal form. A few years ago, Ritus' method was considered in low dimensions to describe the graphene [33] and in QFT scenario [34–37].

The main purpose of this manuscript is to get the expression written on the proper time representation to the zero spin field propagator under the constant external magnetic field present in Eq. (9) of Ref. [38]. For this, we will sum over all Landau levels that appear through the eigenfunctions method applied to the Klein-Gordon field.

The paper is organized as follows: In section 2, we discuss Ritus' method and apply it to find Green's function of the Klein-Gordon field under a constant external magnetic field in the z-direction. In section 3, we sum over Landau levels present in the Green's function and will obtain Schwinger's structure to the propagator, i.e., a closed expression on the proper time. In section 4, we compute the magnetic and thermal corrections on the mass parameter in a self-interaction zero spin system. Also, we will define dimensionless quantities for a better description of the system. Section 5 is reserved to investigate the phase structure of the model. In section 6, we conclude the paper and present some perspectives for further development.

We will use natural units $c = \hbar = k_B = 1$, metric tensor $\eta = diag(+,-,-,-)$ (unless we will be in the Euclidean space, explicitly written by bar "$-$").

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2. The Ritus' Method

Let us calculate the propagator of the bosonic field under an external magnetic field B, uniform, homogeneous, and along z-direction. In this case, the Klein-Gordon equation becomes

$$\left[-(iD)^2 + m_0^2\right] \Phi(u) = 0,$$

where $u \equiv u^\rho = (t, x, y, z)$, $D_\mu = \partial_\mu + ieA_\mu$ and we use the Landau gauge: $A^\mu = (0, -yB, 0, 0)$.

The propagator satisfies the equation

$$\left[-(iD)^2 + m_0^2\right] G(u, u', A) = -i\delta^4(u - u'). \quad (1)$$

Ritus' method is based on the existence of a complete set of eigenfunctions $E_p(u)$ of the operator $(iD)^2$. Furthermore, the method establishes the eigenvalue equation

$$(iD)^2 E_p = p^2 E_p. \quad (2)$$

If we are able to find a complete set of eigenfunctions E_p, we can write

$$G(u, u', A) = \int dp\, E_p(u)\, \mathcal{G}(p, A)\, E_p^*(u'). \quad (3)$$

Since,

$$\int dp\, E_p(u)\, E_p^*(u') = \delta^4(u - u'),$$

and take into account the Eqs. (1) and (3), we get

$$\mathcal{G}(p, A) = \lim_{\varepsilon \to 0} \frac{i}{p^2 - m_0^2 + i\varepsilon}, \quad (4)$$

and the problem of calculating the Feynman propagator of the bosonic field in an external magnetic field will be solved.

Let us find the analytical expression to $E_p(u)$ and p^2. From the gauge that we have chosen, we can show that the Klein-Gordon operator under the external magnetic field reads

$$(iD)^2 = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 + 2\, i\, \omega\, y\, \partial_x - \omega^2 y^2, \quad (5)$$

where $\omega \equiv |eB|$ is the cyclotron frequency.

Thus, to solve the eigenvalue equation, we make the ansätze [35–38]

$$E_p = const. \exp\left[-i\left(p_t t - \omega \xi x - p_z z\right)\right] Y(y). \quad (6)$$

Replacing Eq. (6) on Eq. (2) and taking into account Eq. (5), we obtain

$$\left[\partial_y^2 - \omega^2 (y + \xi)^2 + a\right] Y(y) = 0, \quad (7)$$

being $a \equiv p_t^2 - p_z^2 - p^2$. The Eq. (7) is the differential Hermite equation, which has finite solutions just to $a = \omega(2\ell + 1)$, being $\ell = 0, 1, 2, \cdots$. The normalized solution of Eq. (7) given in terms of Hermite polynomials, H_ℓ, is

$$\begin{aligned} E_p(u) = {} & \frac{1}{\sqrt{(2\pi)^3}} \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} \exp\left[-i\left(p_t t - \omega \xi x - p_z z\right)\right] \\ & \cdot \frac{1}{\sqrt{2^\ell \ell!}} \exp\left[-\omega (y + \xi)^2/2\right] H_\ell\left[\sqrt{\omega}\,(y + \xi)\right]. \end{aligned} \quad (8)$$

Using the orthogonality of Hermite polynomials and the delta function representation in terms of complex exponential, it is easy to show that

$$\begin{aligned} & \int d^4u\, E_p(u)\, E_{p'}^*(u) \\ & \quad = \delta_{\ell,\ell'}\, \delta(p_t - p_t')\, \delta(p_z - p_z')\, \delta\left[\omega(\xi - \xi')\right]. \end{aligned}$$

Therefore, the bosonic propagator of the zero spin field represented by the complete set of eigenfunctions E_p can be expressed as

$$G(u, u', A) = \sum_{\ell=0}^{\infty} \int dp_t\, dp_z\, \omega\, d\xi\, E_p(u)\, \mathcal{G}(p, A)\, E_p^*(u'), \quad (9)$$

where ℓ represents the Landau levels, $E_p(u)$ is given in the Eq. (8) and $\mathcal{G}(p, A)$ is given by Eq. (4) with $p^2 = p_t^2 - p_z^2 - \omega(2\ell + 1)$.

3. Sum Over Landau Levels

In this section, we will sum over Landau levels of Eq. (9). The explicit form of the propagator reads

$$\begin{aligned} G(u, u', A) = {} & \sum_{\ell=0}^{\infty} \left(\frac{\omega}{\pi}\right)^{1/2} \frac{1}{2^\ell \ell!} \int \frac{dp_t}{2\pi} \frac{dp_z}{2\pi} \frac{\omega d\xi}{2\pi} \\ & \cdot \exp\left\{-i\left[p_t(t - t') - p_z(z - z')\right]\right\} \\ & \cdot H_\ell\left[\sqrt{\omega}(y + \xi)\right] H_\ell\left[\sqrt{\omega}(y' + \xi)\right] \\ & \times \mathcal{G}(p, A) \exp\left\{-i\omega\left[-\xi(x - x')\right.\right. \\ & \left.\left. - \frac{i}{2}\left((y + \xi)^2 + (y' + \xi)^2\right)\right]\right\}. \end{aligned} \quad (10)$$

After some manipulations, we can write the term $[\cdots]$ in the last exponential of Eq. (10) as

$$\begin{aligned} [\cdots] = {} & -i\left[(\xi - \mathcal{A})^2 + \frac{1}{4}\left((x - x')^2 + (y - y')^2\right)\right. \\ & \left. + \frac{i}{2}(x - x')(y + y')\right], \end{aligned}$$

where $\mathcal{A} = \left[i(x - x') - (y + y')\right]/2$.

Thus, the Green's function of scalar field is given by

$$
\begin{aligned}
&G\left(u, u', A\right) \\
&= \exp\left[-i\omega(x-x')(y+y')/2\right] \sum_{\ell=0}^{\infty} \left(\frac{\omega}{\pi}\right)^{1/2} \\
&\times \frac{1}{2^{\ell}\ell!} \int \frac{dp_t}{2\pi}\frac{dp_z}{2\pi}\frac{\omega d\xi}{2\pi} \exp\left[-i\left(p_t(t-t') - p_z(z-z')\right)\right] \\
&\times \exp\left[-\frac{\omega}{4}(\mathbf{r}-\mathbf{r}')^{\mathbf{2}}_{\perp}\right] \exp\left[-\omega(\xi-\mathcal{A})^2\right] \\
&\times H_{\ell}\left[\sqrt{\omega}(y+\xi)\right] H_{\ell}\left[\sqrt{\omega}(y'+\xi)\right] \mathcal{G}(p, A). \qquad (11)
\end{aligned}
$$

Note that we factored the phase $\exp[-i\omega(x-x')(y+y')/2]$. This is the so-called Schwinger's phase, which arises in the proper time method by an appropriately chosen path, generally a straight line. Nevertheless, in the eigenfunction method, it is evident just from the reordering of the propagator.

Let us solve the integral in ξ separately. Making the change $\xi = \mathcal{A} + \Xi/\sqrt{\omega}$ and taking into account the well-known relation

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} d\Xi\, H_{\ell}(\Xi+y)\, H_{\ell'}(\Xi+y') \exp(-\Xi^2) \\
&\qquad = 2^{\ell'}\sqrt{\pi}\ \ell\,!(y')^{\ell'-\ell} L_{\ell}^{\ell'-\ell}(-2yy'),
\end{aligned}
$$

where $L_{\ell}^{\ell'-\ell}(-2yy')$ is the Laguerre polynomial associated, we obtain

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} d\xi \exp\left[-\omega(\xi-\mathcal{A})^2\right] \\
&\qquad \cdot H_{\ell}[\sqrt{\omega}(\xi+y)]\, H_{\ell}[\sqrt{\omega}(\xi+y')] \\
&\quad = 2^{\ell}\left(\frac{\pi}{\omega}\right)^{1/2} \ell!\ L_{\ell}\left[-2\omega(\mathcal{A}+y)(\mathcal{A}+y')\right]. \qquad (12)
\end{aligned}
$$

Noting that

$$
(\mathcal{A}+y)(\mathcal{A}+y') = -\frac{1}{4}\left[(x-x')^2 + (y-y')^2\right], \quad (13)
$$

which gives, after replacing Eq. (13) in Eq. (12) and the resulting equation in Eq. (11), the expression

$$
\begin{aligned}
&G\left(u, u', A\right) = \exp\left[-i\omega(x-x')(y+y')/2\right] \\
&\times \int \frac{dp_t}{2\pi}\frac{dp_z}{2\pi} \exp\left\{-i\left[p_t(t-t') - p_z(z-z')\right]\right\} \\
&\times \sum_{\ell=0}^{\infty} \frac{\omega}{2\pi} \exp\left[-\frac{\omega}{4}(\mathbf{r}-\mathbf{r}')^2_{\perp}\right] L_{\ell}[(\mathbf{r}-\mathbf{r}')^2_{\perp}\omega/2]\, \mathcal{G}(p, A). \\
&\qquad (14)
\end{aligned}
$$

Now we use the Eq. (ET II 13(4)a) of reference [39], namely

$$
\begin{aligned}
&\frac{(\alpha-\beta)^n}{\alpha^{n+1}} \exp\left(-\mathcal{Y}^2/2\alpha\right) L_n\left[\frac{\beta\mathcal{Y}^2}{2\alpha(\beta-\alpha)}\right] \\
&\quad = \int_0^{\infty} d\mathcal{X}\mathcal{X} \exp\left(-\alpha\mathcal{X}^2/2\right) L_n\left(\beta\mathcal{X}^2/2\right) J_0(\mathcal{X}\mathcal{Y}), \\
&\qquad (15)
\end{aligned}
$$

where $\mathcal{Y} > 0$, $\Re e(\alpha) > 0$ and J_0 is the Bessel function. Replacing Eq. (15) in Eq. (14) and labeling $\mathcal{X} = p_{\perp} = \sqrt{p_x^2 + p_y^2}$, $\mathcal{Y}^2 = (\mathbf{r}-\mathbf{r}')^2_{\perp}$ and $\alpha = \beta/2 = 2/\omega$, we obtain

$$
\begin{aligned}
&G\left(u, u', A\right) = \exp\left[-i\omega(x-x')(y+y')/2\right] \\
&\times \int \frac{dp_t}{2\pi}\frac{dp_z}{2\pi} \exp\left\{-i\left[p_t(t-t') - p_z(z-z')\right]\right\} \\
&\times \sum_{\ell=0}^{\infty} \frac{1}{\pi} \int_0^{\infty} dp_{\perp} p_{\perp} (-1)^{\ell} \exp\left[-p_{\perp}^2/\omega\right] L_{\ell}[2p_{\perp}^2/\omega] \\
&\times J_0\left(p_{\perp}\sqrt{(\mathbf{r}-\mathbf{r}')^2_{\perp}}\right) \mathcal{G}(p, A). \qquad (16)
\end{aligned}
$$

On the other hand, the Bessel function J_0 has the following representation,

$$
\begin{aligned}
&J_0\left(p_{\perp}\sqrt{(\mathbf{r}-\mathbf{r}')^2_{\perp}}\right) \\
&\qquad = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \exp\left[ip_{\perp}\sqrt{(\mathbf{r}-\mathbf{r}')^2_{\perp}}\cos\varphi\right]. \quad (17)
\end{aligned}
$$

After going to momenta cartesian coordinates and replace Eq. (17) in Eq. (16), we have

$$
\begin{aligned}
&G\left(u, u', A\right) = 2\exp\left[-i\omega(x-x')(y+y')/2\right] \\
&\times \int \frac{dp_t}{2\pi}\frac{dp_x}{2\pi}\frac{dp_y}{2\pi}\frac{dp_z}{2\pi} \exp\left[-ip\cdot(u-u')\right] \\
&\times \sum_{\ell=0}^{\infty} (-1)^{\ell} \exp\left[-(p_x^2+p_y^2)/\omega\right] L_{\ell}[2(p_x^2+p_y^2)/\omega] \\
&\times \lim_{\varepsilon\to 0} \frac{i}{p_t^2 - p_z^2 - \omega(2\ell+1) - m_0^2 + i\varepsilon}, \qquad (18)
\end{aligned}
$$

where we have used $\mathcal{G}(p, A)$ defined on Eq. (4).

Now, let us use Schwinger's idea and write the momenta space propagator $\mathcal{G}(p, A)$ in terms of an integral expression, namely

$$
\begin{aligned}
&\frac{i}{p_t^2 - p_z^2 - \omega(2\ell+1) - m_0^2 + i\varepsilon} \\
&\quad = \int_0^{\infty} dS \exp\left[iS\left(p_t^2 - p_z^2 - \omega(2\ell+1) - m_0^2 + i\varepsilon\right)\right], \\
&\qquad (19)
\end{aligned}
$$

being S the proper time variable. In terms of proper time, we have

$$
\begin{aligned}
&G\left(u, u', A\right) \\
&= 2\exp\left[-i\omega(x-x')(y+y')/2\right] \int \frac{dp_t}{2\pi}\frac{dp_x}{2\pi}\frac{dp_y}{2\pi}\frac{dp_z}{2\pi} \\
&\times \int_0^{\infty} dS \exp[iS(p_t^2 - p_z^2 - \omega + i\varepsilon)] \exp[-ip\cdot(u-u')] \\
&\times \exp\left[-(p_x^2+p_y^2)/\omega\right] \sum_{\ell=0}^{\infty} (-1)^{\ell} L_{\ell}[2(p_x^2+p_y^2)/\omega] \\
&\times \exp\left[-i2\omega\ell S\right]. \qquad (20)
\end{aligned}
$$

Finally, by using the expression

$$\sum_{\ell=0}^{\infty} z^{\ell} L_{\ell}^{\epsilon}(q) = \frac{1}{(1-z)^{1+\epsilon}} \exp\left[qz/(z-1)\right],$$

valid to $|z| < 1$, we get, for $\epsilon = 0$,

$$\sum_{\ell=0}^{\infty} \left[-\exp\left(-i2\omega S\right)\right]^{\ell} L_{\ell}[2(p_x^2+p_y^2)/\omega] = \frac{\exp(i\omega S)}{2\cos(\omega S)} \exp\left[\frac{(p_x^2+p_y^2)}{\omega}\left(\frac{2}{1+\exp(i2\omega S)}\right)\right]. \tag{21}$$

Replacing Eq. (21) in Eq. (20), we obtain the scalar field propagator under an external magnetic field

$$G(u,u',A) = \exp\left[-i\omega(x-x')(y+y')/2\right] \times \int \frac{d^4p}{(2\pi)^4} \exp\left[-ip\cdot(u-u')\right] \times \int_0^{\infty} dS \sec(\omega S) \exp\left[iS\left(p_t^2 - p_z^2 - m_0^2 + i\varepsilon - (p_x^2+p_y^2)\frac{\tan(\omega S)}{\omega i S}\right)\right]. \tag{22}$$

The Eq. (22), in the limit $\omega \to 0$, correctly gives the free propagator.

We can calculate the limit $u \to u'$ in the Eq. (22) and perform a kind of Wick rotation in the proper-time S (i.e., $S \to -i\bar{S}$ and $p_t \to i\bar{p}_t$) to get the Euclidean Green's function of the scalar field under the magnetic field in the z-direction:

$$G(u,u,A) = \int_0^{\infty} d\bar{S} \frac{1}{\cosh(\omega\bar{S})} \int \frac{d\bar{p}_t}{2\pi}\frac{dp_x}{2\pi}\frac{dp_y}{2\pi}\frac{dp_z}{2\pi} \times \exp\left\{-\bar{S}\left[\bar{p}_t^2 + p_z^2 + m_0^2 + (p_x^2+p_y^2)\frac{\tanh(\omega\bar{S})}{\omega\bar{S}}\right]\right\}. \tag{23}$$

The Eq. (23) was obtained in Ref. [38] (equations 8 and 9). It is the Feynman propagator taking into account all Landau levels for the zero spin field. This equation is important for including the corrections on the mass parameter of the scalar field, as we shall perform in the next section.

4. Corrections on Mass Parameter

4.1. Magnetic corrections

Let us consider a charged scalar particles gas system with quartic self-interaction under a magnetic constant field in z-direction. Initially, the Lagrangian density at zero temperature is defined on four-dimensional Euclidean space by

$$\mathcal{L} = |D_\mu\phi|^2 + m_0^2\,|\phi|^2 + \frac{\lambda_0}{4!}\,|\phi|^4.$$

Now the mass parameter m_0^2 will be corrected by the magnetic effects represented by ω. Then the mass parameter in one loop order is given by

$$m^2 = m_0^2 + \Sigma, \tag{24}$$

where the self-energy in one loop order is given by

$$\Sigma(\omega) \equiv \lambda_0\, G, \tag{25}$$

being G the scalar propagator under the constant external magnetic field, computed in Eq. (23).

Performing four Gaussian integrations on Eq. (23), we get the propagator at zero temperature and in the bulk form

$$G(u,u,A) = \frac{\omega}{16\pi^2}\int_0^{\infty} \frac{d\bar{S}}{\bar{S}} \frac{\exp(-\bar{S}\,m_0^2)}{\sinh(\omega\bar{S})}. \tag{26}$$

Below, we will analyze the changes in the mass parameter of the system when the magnetic field increases. For this, is appropriated defined the dimensionless quantities

$$\bar{s} = \bar{S}\,m_0^2; \quad \delta = \omega/m_0^2.$$

Therefore, Green's function written in terms of dimensionless quantities becomes

$$G(u,u,A) = \frac{m_0^2\,\delta}{16\pi^2}\int_0^{\infty} \frac{d\bar{s}}{\bar{s}} \frac{\exp(-\bar{s})}{\sinh(\delta\bar{s})}. \tag{27}$$

4.2. Thermal and magnetic corrections

The thermal effects are included by imaginary-time formalism [40–42]

$$\int \frac{d\bar{p}_t}{2\pi} f(\bar{p}_t,\vec{p}) \to \frac{1}{\beta}\sum_{n=-\infty}^{+\infty} f(\omega_n,\vec{p})$$

being

$$\bar{p}_t \to \omega_n \equiv \frac{\pi}{\beta}(2n) - i\mu, \quad n = 0, \pm1, \pm2, \cdots.$$

where $T = 1/\beta$ is the temperature of the system and μ its density (chemical potential).

The self-energy written on Eq. (25) taking into account thermal effects reads

$$\Sigma(\omega,T) = \frac{\lambda_0\,\omega}{16\,\pi^2}\int_0^{\infty} \frac{d\bar{S}}{\bar{S}} \frac{\exp\left(-\bar{S}\,m_0^2\right)}{\sinh(\omega\bar{S})} \cdot \theta_3\left[-i\frac{\mu\beta}{2}\,;\, \exp\left(-\beta^2/4\bar{S}\right)\right],$$

where we used the θ_3 definition function, namely [43, 44]

$$\theta_3(z;q) = \sum_{n=-\infty}^{+\infty} q^{n^2} \exp\left[(2n)\,iz\right].$$

Again, let us parametrized the model in terms of m_0:

$$\gamma = \mu/m_0; \quad t = T/m_0.$$

Therefore, the self-energy in terms of dimensionless parameters $(\delta, \gamma, t, \bar{s})$ is

$$\Sigma\,(\delta, t) = \frac{\lambda_0\, m_0^2}{16\,\pi^2} \int_0^\infty d\bar{s} \left[\frac{\delta}{\bar{s}\,\sinh(\delta\bar{s})}\right] \cdot\, \theta_3 \left[-i\frac{\gamma}{2t};\, \exp\left(-1/4t^2\bar{s}\right)\right] \exp\left(-\bar{s}\right). \quad (28)$$

The factor $[\delta/\bar{s}\,\sinh(\delta\bar{s})]$ presents the divergence of kind $1/\bar{s}^2$ for $\bar{s} \approx 0$. There is an analogous divergence in the Epstein-Huwrtiz zeta function approach found in Refs. [41, 42]. To contour this problem, let us consider just the finite part of self-energy in the proper time. Replacing Eq. (28) on Eq. (24), we obtain

$$\frac{M_{eff}^2}{m_0^2} = 1 + \frac{\lambda_0}{16\,\pi^2} \int_0^\infty d\bar{s} \left[\frac{\delta}{\bar{s}\,\sinh(\delta\bar{s})} - \frac{1}{\bar{s}^2}\right] \cdot\, \theta_3 \left[-i\frac{\gamma}{2t}\,;\, \exp\left(-1/4t^2\bar{s}\right)\right] \exp\left(-\bar{s}\right), \quad (29)$$

where we have defined the effective mass parameter as $M_{eff}^2 \equiv m^2 - \Sigma_\infty$ and the divergent part of the Eq. (28) is represented by Σ_∞.

In the following pictures, we shall investigate the phase structure of the system through Eq. (29). We have fixed $\lambda \equiv \lambda_0/(16\,\pi^2) = 0.095$.

5. Phase Structure of the System

The system will suffer phase transition in the points that define the effective mass parameter equal to zero. Therefore, we obtained the critical temperature of the system by transcendental expression $M_{eff}^2(t_c, \gamma, \delta) \equiv 0$.

From Fig. 1, we observe that the critical temperature of the system gets smaller values with the increase of the external magnetic field dimensionless δ for vanishing and non-vanishing chemical potential. However, the case where chemical potential dimensionless is finite shows a smaller critical temperature, considering the same values of the external magnetic field dimensionless. Still in Fig. 1, but focusing on the external magnetic field, we observe the inverse magnetic catalysis (IMC) phenomenon for both cases ($\gamma = 0$ and $\gamma \neq 0$), i.e., the external field δ driven the system for small critical temperatures t_c while δ increases. Recently, one of us (EBSC) and colleagues have found the IMC phenomena in a fermionic system, considering a magnetic dependence on the coupling constant of the Dirac field [27]. Nevertheless, this input was not necessary in the scalar field context, as treated here.

To corroborate the IMC phenomenon, the effective mass parameter is shown again in Fig. 2, but for several chemical potential values. We note that critical temperature values t_{1c} for $\delta_1 = 4.5$ is higher than t_{2c} values for $\delta_2 = 9$ (top and bottom panels, respectively). Now, for fixed δ, Fig. 2 shows that the effect on chemical potential over the system is lower critical temperature values while γ increases. Another characteristic of the phase transition bosonic covered here is the independence of the chemical potential suggested by effective mass parameter behavior for temperatures close to zero for any external magnetic field finite. To see that, just take a look at Fig. 2 (top or bottom panels) for dimensionless temperature in the limit $t \to 0$.

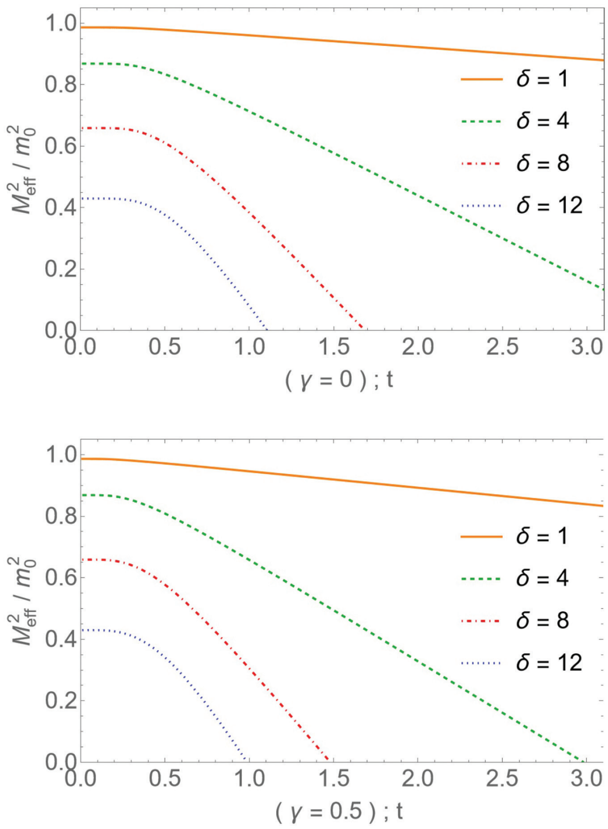
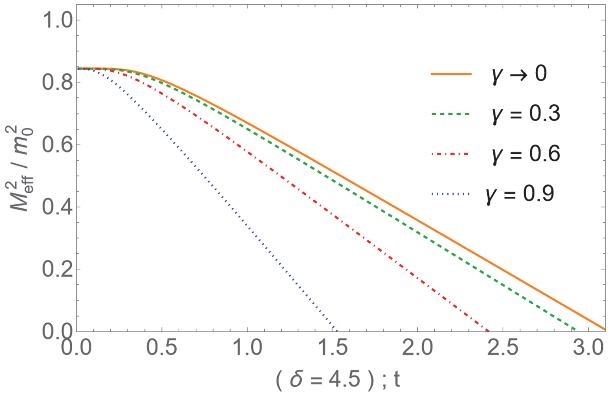


Figure 1: Effects arising from the finite temperature in the system for different values of dimensionless external magnetic field. In the upper panel, we have the dimensionless chemical potential set at zero. At the bottom, we have a finite dimensionless chemical potential.

6. Comments and Conclusions

Along this paper, we applied Ritus' method for calculating the Feynman propagator of the charged scalar field under a constant and homogeneous external magnetic field in the spatial z-direction. After we found the propagator, we did the summation over all Landau levels and got a closed expression for the propagator in terms of Schwinger's proper time. One of the main expressions calculated in this manuscript is the Eq. (23). It is easy to demonstrate that Eq. (23) describes correctly the charged propagator in the limit $\omega \to 0$ and we invited the reader to do it. Also, we have included thermal effects in the magnetic propagator. This was done by Matsubara formalism and from a mathematical point of view, the Jacobi theta function θ_3 was essential to compute all frequencies ω_n of the thermal scalar field. One of the



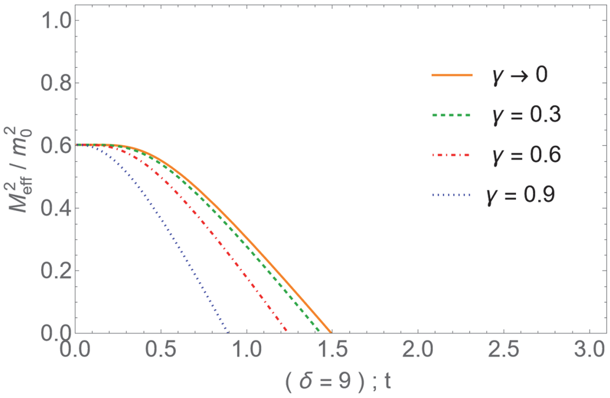


Figure 2: Effects arising from the finite temperature in the system for different values of dimensionless chemical potential. In the top panel, we have the dimensionless external magnetic field set at 4.5. At the bottom, we have $\delta = 9$.

findings here is the IMC effect. This result was found without the inclusion of magnetic dependencies on the coupling constant, as usual in the effective models.

We will continue studying relations between Ritus' and Schwinger's methods in other external field configurations, e.g., a magnetic field with decaying exponential.

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