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Articles



# Enhancing Learning of the Grad-Shafranov Equation through Scientific Literature: Part 1 of a Physics Education Series

A. Ojeda-González*[1], L. Nunes dos Santos[1,2], V. De La Luz[3], M. Cristaldo-Oliveira[1,5], A.N. Laurindo Sousa[4], A. Prestes[1], V. Klausner[1], S. Pilling[1]

[1]Universidade do Vale do Paraíba, Instituto de Física e Astronomia, São José dos Campos, SP, Brasil.
[2]Universidade Federal de Mato Grosso do Sul, Departamento de Matemática, Aquidauana, MS, Brasil.
[3]Universidad Nacional Autónoma de México, Unidad Morelia, Escuela Nacional de Estudios Superiores, Morelia, Michoacán, Mexico.
[4]Universidade Estadual do Maranhão, Centro de Estudos Superiores de Balsas, Departamento de Matemática, Balsas, MA, Brasil.
[5]Secretaria de Estado de Educação de Mato Grosso do Sul, Superintendência de Politicas Educacionais, Campo Grande, MS, Brasil.

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This article provides a comprehensive review of relevant studies in the fields of plasma physics, electromagnetism, and space physics. The aim is to demonstrate how the study of the scientific literature can be used to enhance problem-solving abilities and develop innovative solutions in physics. In this paper, we focus on the study of solutions of the specific Grad-Shafranov equation. Two of the new solutions proposed by Yoon and Lui (2005) are used as a basis for the development of a new solution. The new solution presented has singular points similar to the Yoon-Lui-2 solution, but with an inverted configuration, and also presents less rounded double islands compared to the Yoon-Lui-2 solution. Additionally, the new solution does not exhibit the formation of a current ring, a characteristic of the Yoon-Lui-1 solution, and varying its parameters may lead to higher plasma confinement efficiency. In summary, we illustrate how a thorough analysis of literature can serve as a powerful means for generating innovative approaches to resolving theoretical issues in physics.

**Keywords:** Grad-Shafranov equation, Magnetic flux-ropes, Plasma confinement, Singularity analysis.

## 1. Introduction

The role of the scientist is to observe the world around herself/himself and focus their attention on recurring phenomena, with the aim of discovering the universal laws responsible for the similarity observed in the results of experiments. After the initial observing phase, the scientist propose explanations, based on the existent knowledge, and test if these explanations indeed explain the problem and can predict future behaviors associated with the observed phenomena. This process is known as the scientific method and is fundamental to the understanding of various areas of knowledge, including physics.

In teaching physics, it is crucial to teach students how to use the results published in the scientific literature to foster the creation of new ideas and innovations, promoting the scientific method more broadly and effectively. As Isaac Newton wrote in a letter to Robert Hooke on February 5, 1676, based on a metaphor attributed to Bernard of Chartres, 'If I have seen further, it is by standing on the shoulders of giants.' Through evidence gathered from experiments and theoretical research, it is possible to provide students with a deeper understanding of physical concepts and how they relate to the world around them.

This article presents a literature review of relevant studies in the areas of plasma physics, electromagnetism, and space physics, with the aim of showing readers how it is possible to use the study of scientific literature to obtain new results and predict physical behaviors. In this specific case, we will focus on the article by Yoon and Lui (2005) [1], which presented new solutions to the specific Grad-Shafranov equation (also called the specific or simplified form of the Grad-Shafranov equation). This study is divided into two parts. In the first part, we will use two of these new solutions as the basis for developing an algebraic method to obtain a new solution. By following this specific example, readers can learn how to use scientific literature to solve practical problems and theoretical activities, providing a richer and more in-depth learning experience. The second part of this research will present another new solution, which will

*Correspondence email address: ojeda.gonzalez.a@gmail.com

be described in a separate article. This solution was also obtained from the solutions of Yoon and Lui (2005).

## 2. Theoretical Framework

To continue our reasoning, it is crucial to present the specific Grad-Shafranov equation and its corresponding analytical solution. Starting with the statement of Gauss's law for magnetism ($\nabla \cdot \vec{B} = 0$), which asserts that there are no observed magnetic "monopoles," and therefore, the magnetic field lines will always be closed curves [2]. A well-established concept in Electromagnetism is the definition of the magnetic vector potential $\vec{A'}$, which arises as a consequence of the mathematical identity that guarantees that the divergence of the curl of a vector field is zero [3]. It is possible to write the magnetic field as $\vec{B} = \vec{\nabla} \times \vec{A'}$ and use a vector identity in Ampere-Maxwell's law. According to [4], considering the invariant $y$-axis, the term $\nabla \cdot \vec{A}$ is zero, with $\vec{E}$ and $\vec{J}$ parallel to the $y$-axis, and therefore:

$$-\nabla^2 \vec{A'}(x,z) = \mu_0 \vec{J}(x,z). \tag{1}$$

The above vector equation is worked with only the $y$ component, therefore, from now on, we will consider $A'_y = A_y$. Thus, the resulting Poisson equation is given by:

$$\frac{\partial^2 A_y(x,z)}{\partial x^2} + \frac{\partial^2 A_y(x,z)}{\partial z^2} = -\mu_0 J_y(x,z). \tag{2}$$

Equation (2) represents the generalized *Ampère* equation, which holds great significance in the field of physics. It is worth noting that this equation can be converted into two new equations, one of which will be the primary focus of this study. These equations are:

*i)* The Grad-Shafranov equation (GS), applied when the current density is defined as a function of the first derivative of the magnetic vector potential and has no analytical solution [5, 6];

*ii)* A simplified form called the specific GS equation, applied when the current density is expressed as a function of the exponential of the magnetic vector potential, which has an analytical solution [1, 7–12].

The GS equation mentioned in item *(i)* is written in terms of Cartesian coordinates in the $x$-$z$ plane as follows:

$$\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial z^2} = -\mu_0 \frac{d}{dA_y}\left(p(A_y) + \frac{B_y^2(A_y)}{2\mu_0}\right), \tag{3}$$

where $A_y$ is the $y$-component of the magnetic vector potential, $p$ is the plasma kinetic pressure, and $B_y$ is the $y$-component of the magnetic field [9, 13, 14]. Equation (3) is a second-order partial differential equation that does not have an analytical solution but can be numerically solved as a Cauchy problem [14–16].

When explaining (3), it is important to emphasize that it is preferable to validate a proposed numerical solution by comparing it with an analytical solution. To achieve this, it is convenient to consider simplifications in the equation that allow the elimination of nonlinearity. In this way, a general analytical solution can be obtained that meets the initial conditions for the implementation of the numerical solution. This procedure will be explained, step by step, in following paragraphs, to provide an analytical solution for the equation (3).

The term on the right-hand side of (3) in the argument of the derivative defines the plasma transverse pressure ($P_t$) [13, 14], that is,

$$P_t(A_y(x,z)) = p(A_y(x,z)) + \frac{{B_y}^2(A_y(x,z))}{2\mu_0}. \tag{4}$$

An analytical solution of equation (3) is only possible for very specific cases of the expression $P_t$[1] [7, 17–19].

The equation (4) allow us to found a single expression to solve the equation (3) defined by

$$P_t = P_{t_0} e^{(-2\Psi)} \tag{5}$$

where

$$\Psi = -\frac{A_y}{LB_0} \tag{6}$$

is the normalized magnetic vector potential, where $B_0$ is the asymptotic magnetic field, $L$ represents the scale length, and

$$P_{t_0} = \frac{{B_0}^2}{2\mu_0} \tag{7}$$

is the transverse pressure when $A_y = 0$ [20]. Expressions (4-7) are substituted into (3) to obtain the specific GS equation, with the following expression:

$$\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Z^2} = e^{-2\Psi}, \tag{8}$$

considering new dimensionless variables: $\frac{x}{L} = X$ e $\frac{z}{L} = Z$ [1].

In our mathematical formulation adopted here, the component of the current density, $J_y$, is given by:

$$J_y(x,z) = \frac{B_0}{L\mu_0} e^{(-2\Psi)}. \tag{9}$$

The equation (3) was derived by [7] from Plasma Kinetic Theory by solving the set of Vlasov-Maxwell equations while considering a velocity distribution expression as a function of the Boltzmann factor of the Maxwell-Boltzmann statistics. For the detailed development of the entire physical-theoretical formulation using Kinetic Theory, refer to [21].

[1] Some of them applied to the tokamak, which is used to confine and heat plasma for nuclear fusion research.

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The mathematical expression given by (8) is a Poisson equation[2]. However, in the specific case where the nonhomogeneous term takes an exponential form, the equation is called a "two-dimensional Liouville equation", which in its original form is written as $\Phi_{xx}+\Phi_{yy}=ce^{d\Phi}$, with $c$ and $d$ being real constants [20, 22]. Note that equation (8) is the two-dimensional Laplacian of the normalized vector potential ($\Psi$) equals to the exponential of $\Psi$.

The previous equation, which was solved by [23], also appears in the literature as the "Liouville solution", but in this work, we prefer to call it the "Walker formula" or "Walker solution", thus using the notation already established in the area of Space Physics. Walker [23] proposed a general solution dependent on an analytic complex function called the generating function, $g(\zeta)$. The solution given by Walker is:

$$e^{-2\Psi(X,Z)}=\frac{4|g(\zeta)'|^2}{\left(1+|g(\zeta)|^2\right)^2}, \tag{10}$$

where $\zeta$ is a complex variable.

Walker's formula (10) allows us to propose new analytical solutions of (8). For example, the model proposed by [24] was the pioneer among a group of solutions that followed it [1, 7, 25–27]. The next section will present the context of a literature review that will allow us to propose a new solution.

## 3. Literature Review

This section is essential to show to the reader how science is supported by published results in the literature. As a case study, we will use the article by Yoon and Lui (2005), which presents nine solutions for (8) based on the Walker formula presented in equation (10).

To summarize, Table 1 lists the nine solutions presented in the work of Yoon and Lui (2005). The first column of the table presents the name of each solution, while the second column presents the corresponding generating function $g(\zeta)$. In the third column, we provide an exact solution for the GS equation by substituting the function $g(\zeta)$ into the Walker formula.

It is worth noting that solutions one to five are collectively referred to as the "Harris family". Solutions six to nine, on the other hand, are not part of the "Harris family", and have specific significance in this study. New solutions will be proposed based on solutions seven and eight, which will be henceforth referred to as Yoon-Lui-1 and Yoon-Lui-2, respectively. It is important to emphasize that all solutions were obtained from the Walker formula presented in Equation (10).

[2] The Poisson equation is a partial differential equation with broad applications in physics. Its general form is $\nabla^2\Phi=f$, where $\nabla$ represents the nabla operator, $\Phi$ and $f$ are general functions. For example, $\nabla^2 V=\frac{\rho}{\epsilon_0}$, where $V$ is the electric potential and $\rho$ is the charge density.

### 3.1. Magnetic singularities

Understanding magnetic singularities in magnetic fields poses a complex challenge in physics, as the unique physical conditions of these systems cannot be precisely reproduced in a laboratory setting. In many cases approaches are based in theoretical models of astrophysical observations that show extreme conditions as the initial conditions of the big-bang (derived of the CMB observations), inside of the event horizon of black holes, or magnetic reconnection in solar flares and in the interaction of Coronal Mass Ejections with planetary magnetic fields. Many solutions derived from the Walker formula exhibit such magnetic singularities [1]. Such promising results can be found in several modern theoretical physics models and in Albert Einstein's theory of general relativity, where singularities emerge in practical situations when solving Einstein's equations [30].

Furthermore, there are other areas of physics, such as cosmology and astrophysics, that also involve the study of singularities. For example, the Schwarzschild metric, which is a solution to Einstein's field equations, describes curvature singularities in black holes, and the Robertson-Walker metric describes the singularity at the isolated point in space that occurred at the moment of the Big Bang. Despite being challenging concepts to understand, singularities can provide essential information to fill gaps and solve important theoretical problems, such as the ultraviolet catastrophe, which resulted in failures of classical electromagnetic theory. Indeed, understanding singularities may be crucial to complete the puzzle of a theory and advancing our knowledge of fundamental physical phenomena [30].

The study of singularities in two-dimensional magnetohydrodynamic (MHD) fluid dynamics is investigated through direct numerical simulations [31]. Specifically, the formation of singularities in MHD-2D is influenced by the interaction between the magnetic field and the conducting fluid, which can lead to the formation of magnetic vortex structures and magnetic singularities [32]. These phenomena have applications in areas such as nanotechnology, where magnetic vortices have been used as data storage elements in random-access memory (RAM) and hard drives [33]. Magnetic singularities have also been explored as candidates for performing quantum computations and for developing new magnetic materials [34].

From now on, our focus will be on solutions derived from the Walker formula. Traditionally, the analysis of the singularities of equation (8) involved a thorough examination of the function $\Psi$, which could be a laborious and complicated process. This required checking the domain of the $\Psi$ function for mathematical inconsistencies that indicated the presence of singularities. However, a more straightforward approach was discovered by [35], who used the generating function $g(\zeta)$ to locate singularities. It was discovered that this function cannot be arbitrarily selected and must satisfy the following

**Table 1:** Analytical Solutions of the GS Equation according to the work of [1].

| Solution: | $g(\zeta)=$ | $\Psi=$ |
|---|---|---|
| 1- *Harris*[a] | $e^{\iota\zeta}$ | $\ln(\cosh Z)$ |
| 2- *Fadeev*[b] | $f_p + \sqrt{1+{f_p}^2}\, e^{\iota\zeta}$ | $\ln\left(f_p \cos X + \sqrt{1+{f_p}^2}\cosh Z\right)$ |
| 3- *Kan*[c] | $e^{\iota\zeta - \frac{\iota b}{\xi}}$ | $\ln \frac{\cosh[Z(1+b/R^2)]}{\sqrt{(1+b/R^2)^2 - 4bZ^2/R^4}}$ [g] |
| 4- *Manankova*[d] | $f_p + \sqrt{1+{f_p}^2}\, e^{\left(i\zeta - \frac{ib}{\zeta-a}\right)}$ | $\ln \frac{f_p \cos(X - \frac{b(X-a)}{R_0^2}) + \sqrt{1+{f_p}^2}\cosh(Z(1+\frac{b}{R_0^2}))}{\sqrt{\left(1+\frac{b}{R_0^2}\right)^2 - \frac{4bZ^2}{R_0^4}}}$ [h] |
| 5- H-F-K-M[e] | $f_p + \sqrt{1+{f_p}^2}\, e^{\left(i\zeta - \frac{ib}{(\zeta-a)^k}\right)}$ | $\ln \frac{f_p \cos(a + R_0\cos\theta - \frac{b\cos k\theta}{R_0^k}) + \sqrt{1+{f_p}^2}\cosh(R_0\sin\theta + \frac{b \sin k\theta}{R_0^k})}{\sqrt{\left(1+\frac{kb}{R_0^{k+1}}\right)^2 - \frac{4kb}{R_0^{k+1}}\sin^2\frac{(k+1)\theta}{2}}}$ [i] |
| 6- B-W [f] | $e^{[-\beta\rho(\zeta)]}$ [j] | $\ln \frac{\sqrt{r_+ r_- \cosh(\beta\xi)}}{\beta}$ [k] |
| 7- Yoon-Lui-1[e] | $\zeta^\nu$ | $\ln \frac{R(R^\nu + R^{-\nu})}{2\nu}$ |
| 8- Yoon-Lui-2[e] | $\zeta - \frac{a}{\zeta}$ | $\ln \frac{(R^2+a)^2 + R^2 - 4aX^2}{2[(R^2+a)^2 - 4aZ^2]^{1/2}}$ |
| 9- Yoon-Lui-3[e] | $\frac{\zeta}{(1-a^2\zeta^2)}$ | $\frac{1}{2}\ln(\frac{S(S+R^2)^2}{2T})$ [l] |

[a] [24]
[b] [25]
[c] [8]
[d] [26, 28, 29]
[e] [1, sections 3.5 (H-F-K-M), 3.7 (1), 3.8 (2), 3.9 (3)]
[f] [27]
[g] $R^2 = X^2 + Z^2$
[h] $R_0^2 = (X-a)^2 + Z^2$
[i] $\theta = \tan^{-1}\frac{Z}{X-a}$
[j] $\zeta = \alpha\cosh\rho, \quad \rho(\zeta) = \cosh^{-1}(\frac{\zeta}{\alpha}) = \ln\frac{\zeta \pm \sqrt{\zeta^2-\alpha^2}}{\alpha}$
[k] $r_\pm = \sqrt{(X\pm\alpha)^2 + Z^2}, \quad \xi = \ln(\tau + \sqrt{\tau^2-1})$ and $\tau = \frac{r_+ + r_-}{2\alpha}$
[l] $S = (1-a^2R^2)^2 + (2aZ)^2, \; T = (1-a^4R^4)^2 + (4a^2XZ)^2$

condition:

$$\nabla \ln |g'(\zeta)| = 0, \tag{11}$$

where $g(\zeta)$ is a complex function.

In other words, [35] stated that we can rewrite the equation (10) as follows:

$$\Psi = -\frac{1}{2}\ln\left(\frac{4\,|g'|^2}{(1+|g|^2)^2}\right). \tag{12}$$

Now, applying the nabla operator to equation (12) and performing some algebraic operations, we have:

$$\begin{aligned}\nabla\Psi &= -\nabla\ln|g'| + \nabla\ln\left(1+|g|^2\right)\\ &= -\nabla\ln|g'| + \frac{4\,|g'|^2}{(1+|g|^2)^2}.\end{aligned} \tag{13}$$

Note that equation (13) is equal to (10) if, and only if, $\nabla\ln|g'(\zeta)| = 0$. Therefore, if we really want to find the singularities, we must calculate $|g'(\zeta)| = 0$, which is the second condition of *Génot*.

The importance of equation (13) is that it allows us to determine, from $g'(\zeta)$, the singular points $(X, Z)$ of $\Psi(X, Z)$. In other words, the singularities can be obtained directly from $\Psi$, or from the zeros (roots) and poles of $g'(\zeta)$ [35, 36].

### 3.2. Yoon-Lui-1 solution

This solution was found by [1] using also the *Walker* formula, where the chosen generating function was:

$$g(\zeta) = \zeta^\nu, \tag{14}$$

where $\nu$ is an integer number. Its derivative will be:

$$g'(\zeta) = \nu\zeta^{\nu-1}, \tag{15}$$

Substituting (14) and (15) into (10) and performing the calculations, we obtain the following solution:

$$\Psi = \ln\frac{R(R^\nu + R^{-\nu})}{2\nu}, \tag{16}$$

where $R^2 = X^2 + Z^2$.

### 3.2.1. Calculating singular points

Let's start by using the derivative of $g(\zeta)$, presented in (15). Calculating the modulus of the first derivative, we have:

$$|g'(\zeta)| = \left(\nu\zeta^{\nu-1}\right)^{\frac{1}{2}} \cdot \left(\nu{\zeta^*}^{\nu-1}\right)^{\frac{1}{2}}. \tag{17}$$

Now, applying $\nabla \ln |g'(\zeta)| = 0$, we have:

$$\begin{aligned}
&\nabla \ln \left(\left(\nu\zeta^{\nu-1}\right)^{\frac{1}{2}} \cdot \left(\nu{\zeta^*}^{\nu-1}\right)^{\frac{1}{2}}\right)\\
&= \nabla \left[\ln \left(\nu\zeta^{\nu-1}\right)^{\frac{1}{2}} + \ln \left(\nu{\zeta^*}^{\nu-1}\right)^{\frac{1}{2}}\right]\\
&= 2\frac{\partial}{\partial\zeta}\left(\frac{\partial}{\partial\zeta^*}\left[\ln\left(\nu\zeta^{\nu-1}\right) + \ln\left(\nu{\zeta^*}^{\nu-1}\right)\right]\right)\\
&= 2\frac{\partial}{\partial\zeta}\left(\frac{\nu-1}{\zeta^*}\right) = 2\cdot 0 = 0.
\end{aligned} \tag{18}$$

With the first condition satisfied by (18), we proceed to calculate the singularities by substituting $\zeta = X + iZ$ into equation (17), as shown below:

$$|g'(\zeta)| = \nu\left((X+iZ)\cdot(X-iZ)\right)^{\frac{\nu-1}{2}} = 0. \tag{19}$$

Continuing with the algebraic manipulations, we have:

$$\nu(X^2+Z^2)^{\frac{\nu-1}{2}} = 0. \tag{20}$$

Assuming that $\nu \neq 0$[3], we have two situations: i) if $\nu < 1$, the point $(0,0)$ is an indeterminacy of the equation (19); ii) if $\nu > 1$, the point $(0,0)$ is a root. In both cases, the point $(0,0)$ is the only singularity in the Yoon-Lui-1 solution. In the case where $\nu = 1$, we have the expression $0^0$, which is considered a mathematical indeterminacy. However, when $\nu = 1$, the point $(0,0)$ is not an indeterminacy of the Yoon-Lui-1 solution, since $\Psi = \ln\frac{1}{2}$.

Figure 1 shows the current density plot of the Yoon-Lui-1 solution for the cases of $\nu = 1$ and $\nu = 4$. In the first case, there are no singularities, and in the second case, only the point $(0,0)$ is singular.

From a physical point of view, when applying this analytical result to the analysis of a magnetic flux rope with magnetic island configuration, it is recommended to use $\nu = 1$, since the current density is also maximum at the center of the tube. Additionally, the absence of singularities in the solution allows for it to be used to validate numerical solutions, as done in the work of [14] with the Fadeev solution.

### 3.3. Yoon-Lui-2 solution

This solution was found by [1], also using Walker's formula, where the chosen generating function was:

$$g(\zeta) = \zeta - \frac{a}{\zeta}. \tag{21}$$

[3] Note that $\nu = 0$ leads to an indeterminacy in the solution (16).

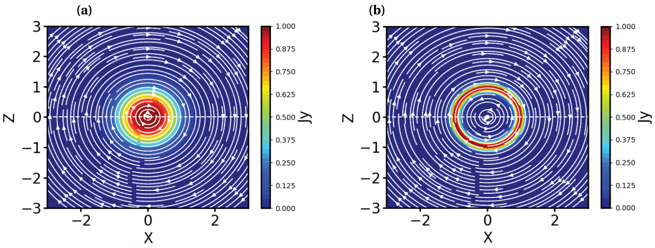


**Figure 1:** Current density graph of Yoon-Lui-1 solution given by equation (16), which presents only one magnetic island. The graphs in panel (a) were generated with a parameter of $\nu = 1$, while those in panel (b) were generated with a parameter of $\nu = 4$. In the case where $(\nu \neq 0, \nu < 1)$ or $(\nu > 1)$, the current density $J_y$ presents a magnetic singularity at $(0,0)$. In Figure (b), it is observed that the magnetic field has opposite directions around the singular point, forming a current ring where the magnetic field vanishes.

Substituting this into (10) and performing the calculations, we obtain the following solution:

$$\Psi = \ln \frac{\left(R^2+a\right)^2 + R^2 - 4aX^2}{2\left[\left(R^2+a\right)^2 - 4aZ^2\right]^{1/2}} \tag{22}$$

where $R^2 = X^2 + Z^2$.

### 3.3.1. Calculating singular points

Let's start by taking the derivative of the generating function, $g(\zeta) = \zeta - \frac{a}{\zeta}$:

$$|g'(\zeta)| = \left|\zeta - \frac{a}{\zeta}\right|' = \left|\frac{a}{\zeta^2} + 1\right|, \tag{23}$$

where using the definition of the modulus of the first derivative, we have:

$$|g'(\zeta)| = \left(\frac{a}{\zeta^2}+1\right)^{\frac{1}{2}} \cdot \left(\frac{a}{{\zeta^*}^2}+1\right)^{\frac{1}{2}}. \tag{24}$$

Now we will develop the equality $\nabla \ln |g'(\zeta)| = 0$ and after some algebraic manipulation, we have:

$$\begin{aligned}
&\nabla \ln \left(\left(\frac{a}{\zeta^2}+1\right)^{\frac{1}{2}} \cdot \left(\frac{a}{{\zeta^*}^2}+1\right)^{\frac{1}{2}}\right)\\
&= \frac{1}{2}\nabla\left[\ln\left(\frac{a}{\zeta^2}+1\right) + \ln\left(\frac{a}{{\zeta^*}^2}+1\right)\right]\\
&= 2\frac{\partial}{\partial\zeta}\left(\frac{\partial}{\partial\zeta^*}\left[\ln\left(\frac{a}{\zeta^2}+1\right) + \ln\left(\frac{a}{{\zeta^*}^2}+1\right)\right]\right)\\
&= 2\frac{\partial}{\partial\zeta}\left(\frac{-2a}{{\zeta^*}^3\left(\frac{a}{{\zeta^*}^2}+1\right)}\right) = 2\cdot 0 = 0.
\end{aligned} \tag{25}$$

Making some algebraic manipulations, we find:

$$\left(\frac{a^2 + 2a(X^2-Z^2) + (X^2+Z^2)^2}{(X^2+Z^2)^2}\right)^{\frac{1}{2}} = 0. \tag{26}$$

If both X and Z are non-zero, the quadratic equation will not have real roots. However, if one of them is fixed to zero, the roots can be found. When $a$ is positive, there are two distinct situations in (26): *i)* if $Z = 0$, the roots of the quadratic equation are obtained from $(Z^2-|a|)^2 = 0$, which has no real solutions; *ii)* if $X = 0$, the roots of the quadratic equation are obtained from $(Z^2 - |a|)^2 = 0$, which has two symmetrical roots given by $Z = \pm\sqrt{a}$.

Thus, we found the following singularities for the case $a > 0$: $(0, +\sqrt{a}), (0, -\sqrt{a})$. Note that for the case $a < 0$, when $X = 0$, the roots of (26) are obtained from $(Z^2 + |a|)^2 = 0$. In this case, there are no roots, while when $Z = 0$, the roots of (26) are obtained from $(X^2 - |a|)^2 = 0$. In this case, there are two roots $X = \pm\sqrt{a}$. Therefore, we conclude that we found the following singularities for the case $a < 0$: $(-\sqrt{a}, 0)$, $(+\sqrt{a}, 0)$.

Moreover, it is important to highlight that when $a = 0$, the term on the right-hand side of equation (26) becomes:

$$\left(\frac{(X^2+Z^2)^2}{(X^2+Z^2)^2}\right)^{\frac{1}{2}} = 1^{\frac{1}{2}} = 1. \quad (27)$$

Thus, the equation reduces to $1 = 0$, which is a contradiction. Therefore, there is no solution for equation (26) when $a = 0$, so there is no singular point in this case.

Next is showed the current density plot for the Yoon-Lui-2 solution in Figure 2. The plot for the Yoon-Lui-2 solution, given by equation (22) with $a = 2$, can be interpreted as follows: note that above the X-axis, there are two magnetic islands located at $X = \pm\sqrt{a}$ with finite current density entering the plane, which is characteristic of regions of plasma confined in magnetic fusion devices. At the neutral point $X$ between the magnetic islands, the current density is zero and the structure is stable. On the Z-axis, we can observe two magnetic singularities with zero current density at the center. This type of singularity is the same as that appeared in Kan's solution at $(0, \pm\sqrt{b})$ (see Table 1). This singularity is unwanted in analytical models used to generate initial conditions in MHD simulations or tests to improve the numerical solution of the Grad-Shafranov equation. However, the Yoon-Lui-2 model allows these singularities to be excluded in a more elegant way, simply by increasing the value of $a$, for example, $a = 6$. By increasing the value of $a$, the two singularities are moved away from the origin, but at the same time, the two magnetic islands are also moved away.

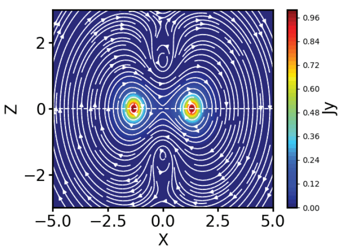


**Figure 2:** Density plot of the Yoon-Lui-2 solution given by equation (22), using $a = 2$. We note that there are two magnetic islands on the $X$ axis, located at $(-\sqrt{a}, 0)$ and $(\sqrt{a}, 0)$, and a neutral type-X point between them. On the $Z$ axis, we can observe two singular points and three neutral type-X points, one at the origin and the other two near the singularities $(0, -\sqrt{a})$ and $(0, \sqrt{a})$. If $a$ is equal to $-2$, the islands shift to the $Z$ axis and the singular points to the $X$ axis.

In summary, [1] proposed this model as an alternative to the model of [27]. The previous model had an equilibrium structure with a point $X$ characterized by a pair of parallel currents, whose density diverged at the two centers of the magnetic islands. In the alternative model, however, the current density at the center of the neighboring magnetic islands is finite. However, the alternative model has a drawback in that it has two magnetic singularities above the Z-axis (for $a > 0$), as shown in Figure 2.

## 4. Methodology

The Yoon-Lui-1 and Yoon-Lui-2 solutions, as analytical solutions of the specific Grad-Shafranov equation, describe a possible morphology of a magnetic field in confined plasma regions. Both solutions are expressed in terms of elementary functions on the Cartesian plane. However, there are significant differences between them.

The Yoon-Lui-1 solution is an exact analytical solution for the Grad-Shafranov equation that describes the structure of the magnetic field in a magnetic island-type configuration. This solution is characterized by a single singularity at $(0, 0)$, which is an indeterminacy if $\nu < 1$ and a root if $\nu > 1$, and has maximum current density at the center of the "magnetic island". The solution is particularly useful for validating numerical solutions and analyzing the stability of magnetic structures.

On the other hand, the Yoon-Lui-2 solution is an extension of the Yoon-Lui-1 solution and describes the structure of the magnetic field in a configuration with two magnetic islands. This solution has two magnetic singularities above the Z-axis, but these singularities can be avoided by increasing the value of $a$, which controls the separation of the magnetic islands. The current density is finite at the center of each magnetic island and zero at the neutral point between them.

In summary, both the Yoon-Lui-1 and Yoon-Lui-2 solutions are useful for describing the structure of the magnetic field in confined plasma regions and are expressed in terms of elementary functions. The Yoon-Lui-1 solution is simpler and only describes the structure of a magnetic island, but it is useful for validating numerical solutions and analyzing stability. The Yoon-Lui-2 solution describes the structure of two magnetic islands, but has two singularities above the Z-axis that can be avoided by increasing the value of $a$.

In this methodology section, a new solution of the specific Grad-Shafranov equation will be presented. This new solution is obtained by combining the generating functions of the Yoon-Lui-1 and Yoon-Lui-2 models, taking the quotient of them.

The new solution obtained from the combination of the generating functions of these models may present a unique combination of features, possibly leading to greater plasma confinement efficiency [37].

To obtain the new solution, specific mathematical methods, such as combining generating functions, are used. These methods will be described in detail in the results section to ensure reader understanding. The hypothesis of the new solution will be presented, but its validity will be evaluated later in the following section.

## 5. Results

The necessary steps to obtain the new solution are presented below. By taking the quotient of (14) and (21), we obtain the following generating function:

$$g(\zeta) = \frac{\zeta^{\nu}}{\frac{\zeta^2 - a}{\zeta}} = \frac{\zeta^{\nu+1}}{\zeta^2 - a}, \tag{28}$$

where $\nu$ and $a$ are constants.

The square modulus of $g(\zeta)$ after substituting $\zeta = X + iZ$ is as follows:

$$|g(\zeta)|^2 = \frac{(X^2+Z^2)^{\nu+1}}{a^2 - 2a\left(X^2 - Z^2\right) + \left(X^2+Z^2\right)^2}. \tag{29}$$

Adding one to both sides of the previous equation, we have:

$$1 + |g(\zeta)|^2 = \frac{(X^2+Z^2)^{\nu+1} + a^2 - 2a\left(X^2 - Z^2\right) + \left(X^2+Z^2\right)^2}{a^2 - 2a\left(X^2 - Z^2\right) + \left(X^2+Z^2\right)^2}. \tag{30}$$

Continuing with the reasoning, the first derivative of the generating function is:

$$g'(\zeta) = \frac{\zeta^{\nu}\left[(\nu-1)\zeta^2 - a(\nu+1)\right]}{\left(\zeta^2 - a\right)^2}. \tag{31}$$

The modulus of the derivative is:

$$|g'(\zeta)| = \sqrt{\frac{(X^2+Z^2)^{\nu}\left[a(\nu+1) - (\nu-1)(X-iZ)^2\right]}{\left(a^2 - 2a\left(X^2-Z^2\right) + \left(X^2+Z^2\right)^2\right)^2}} \cdot \sqrt{\left[a(\nu+1) - (\nu-1)(X+iZ)^2\right]}. \tag{32}$$

With some algebraic work, we eliminate the imaginary unit from the modulus of the generator function's derivative, as follows:

$$|g'(\zeta)| = \frac{\sqrt{(X^2+Z^2)^{\nu}}}{a^2 - 2a\left(X^2-Z^2\right) + \left(X^2+Z^2\right)^2} \cdot \sqrt{\left[(\nu-1)\left(X^2-Z^2\right) - a(\nu+1)\right]^2 + 4(\nu-1)^2X^2Z^2}. \tag{33}$$

Finally, substituting (30) and (33) into the Walker formula given by:

$$\Psi = \ln\left[\frac{1+|g(\zeta)|^2}{2|g'(\zeta)|}\right], \tag{34}$$

the result is as follows:

$$\Psi = \ln\left[\frac{\frac{(X^2+Z^2)^{\nu+1}+a^2-2a\left(X^2-Z^2\right)+\left(X^2+Z^2\right)^2}{a^2-2a(X^2-Z^2)+(X^2+Z^2)^2}}{2\frac{\sqrt{(X^2+Z^2)^{\nu}\left[[(\nu-1)(X^2-Z^2)-a(\nu+1)]^2+4(\nu-1)^2X^2Z^2\right]}}{a^2-2a(X^2-Z^2)+(X^2+Z^2)^2}}\right]. \tag{35}$$

Three parameters are introduced in order to simplify the form of (35), namely:

$$R^2 = X^2 + Z^2, \tag{36}$$

$$U^2 = 4X^2Z^2, \tag{37}$$

$$T^2 = X^2 - Z^2. \tag{38}$$

The final expression of Equation (35) is:

$$\Psi = \ln\left[\frac{R^{2(\nu+1)} + a^2 - 2aT^2 + R^4}{2\sqrt{R^{2\nu}\left[\left[(\nu-1)T^2 - a(\nu+1)\right]^2 + (\nu-1)^2U^2\right]}}\right]. \tag{39}$$

### 5.1. Calculating singular points

Calculating the derivative of the generating function, $g(\zeta) = \frac{\zeta^{\nu+1}}{\zeta^2-a}$, we have:

$$g'(\zeta) = \left(\frac{-a\zeta^{\nu} - a\nu\zeta^{\nu} + \nu\zeta^{\nu+2} - \zeta^{\nu+2}}{(\zeta^2-a)^2}\right), \tag{40}$$

from which, calculating the modulus of the first derivative, we have

$$|g'(\zeta)| = \left(\frac{-a\zeta^{\nu} - a\nu\zeta^{\nu} + \nu\zeta^{\nu+2} - \zeta^{\nu+2}}{(\zeta^2-a)^2}\right)^{\frac{1}{2}} \cdot \left(\frac{-a\zeta^{*\nu} - a\nu\zeta^{*\nu} + \nu\zeta^{*\nu+2} - \zeta^{*\nu+2}}{(\zeta^{*2}-a)^2}\right)^{\frac{1}{2}}. \tag{41}$$

Starting from the definition $\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} = \frac{4\partial^2}{\partial\zeta\partial\zeta^*}$ and doing some algebraic work, one can demonstrate that:

$$\nabla \ln|g'(\zeta)| = 0. \tag{42}$$

Based on equation (42), the first condition is satisfied. Next, we will calculate the singularities using the equation $|g'(\zeta) = 0|$. We substitute $\zeta = X + iZ$ in our equation (41), as follows:

$$|g'(\zeta)| = \sqrt{\left(\frac{-a\zeta^{\nu} - a\nu\zeta^{\nu} + \nu\zeta^{\nu+2} - \zeta^{\nu+2}}{(\zeta^2 - a)^2}\right)} \cdot \sqrt{\left(\frac{-a\zeta^{*\nu} - a\nu\zeta^{*\nu} + \nu\zeta^{*\nu+2} - \zeta^{*\nu+2}}{(\zeta^{*2} - a)^2}\right)} = 0 \quad (43)$$

Making some algebraic manipulations in equation (43), if $\zeta \neq \sqrt{a}$, we can write:

$$(-a(1+\nu)\zeta^{\nu} + (\nu-1)\zeta^{\nu+2}) \cdot (-a(1+\nu)\zeta^{*\nu} + (\nu-1)\zeta^{*\nu+2}) = 0. \quad (44)$$

Thus, $(-a(1+\nu)\zeta^{\nu} + (\nu-1)\zeta^{\nu+2}) = 0$ or $(-a(1+\nu)\zeta^{*\nu} + (\nu-1)\zeta^{*\nu+2}) = 0$. Therefore, in order to gain a better understanding of the equation, let us consider some specific cases. For instance, if we take $a = 1$ and $\nu = 1$, we obtain:

$$-2(X \pm iZ)^1 = 0, \quad (45)$$

whose only solution would be $(0, 0)$.

Now, for $\nu \neq 1$ and any $a$:

$$\zeta = \zeta^* = \pm\sqrt{\frac{a(1+\nu)}{(\nu-1)}}. \quad (46)$$

When $a = 1$ is fixed and $\nu$ is varied, the following singularities arise:

- For $\nu = 1.0$, we have a single singularity at $\zeta = (0, 0)$;
- For $\nu = 1.2$, we have three singularities at $\zeta = (0, 0)$ and at $\zeta = (\pm\sqrt{11}, 0)$;
- For $\nu = 1.6$, we have three singularities at $\zeta = (0, 0)$ and at $\zeta = \left(\pm\sqrt{\frac{13}{3}}, 0\right)$;
- For $\nu = 2.0$, we have two singularities at $\zeta = (0, 0)$ and at $\zeta = (\pm\sqrt{3}, 0)$.

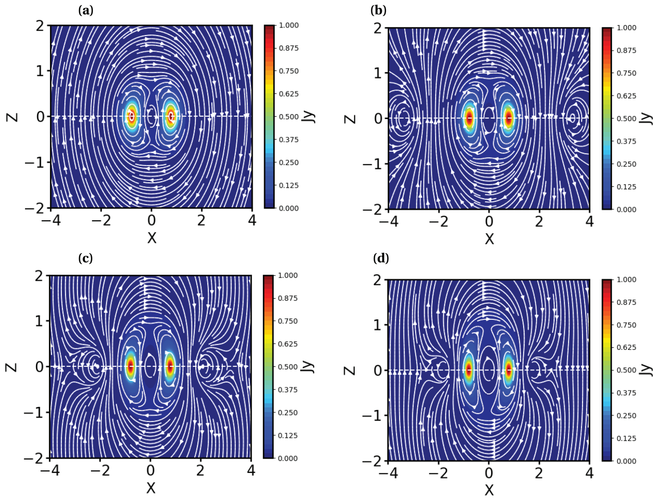


**Figure 3:** The plot below represents the current density of the proposed solution, given by equation (39). To generate the figures, we kept the value of $a$ constant at 1, while varying the value of $\nu$ in each of the four images, corresponding to $\nu = 1.0$, $\nu = 1.2$, $\nu = 1.6$, and $\nu = 2.0$, respectively from *a)* to *d)*. The singular points are indicated in each of the images at the locations $(0, 0)$, $(\pm\sqrt{11}, 0)$, $\left(\pm\sqrt{\frac{13}{3}}, 0\right)$, and $(\pm\sqrt{3}, 0)$, corresponding to the singularities found for each value of $\nu$. These singular points are important for the analysis of the behavior of the proposed solution in specific regions of the domain and help to understand the behavior of the solution in different regimes.

## 6. Discussion

Figure 3 displays the solution(39) and illustrates the location of the previously mentioned singularity points.

The proposed solution in (39) presents singularities at specific points in the domain, which are associated with physical characteristics of the solution. Comparing with the Yoon-Lui-1 and Yoon-Lui-2 solutions, we can notice that the new solution has singular points similar to Yoon-Lui-2, but with the axes reversed, located on the $X$ axis. The double islands are also present in the proposed solution, but with less rounding compared to Yoon-Lui-2.

In addition, the proposed solution does not exhibit the formation of a current ring, a characteristic present in the Yoon-Lui-1 solution, even when the parameters $a$ and $\nu$ are varied. It is important to highlight that these characteristics are relevant for understanding the physical behavior of the solution in different regimes, such as in regions near the singular points and in areas where double magnetic islands are formed.

The proposed solution presents a valuable contribution to understanding the physical characteristics in confined plasma regions. As a result of the investigation carried out in this work, it was found that the new solution can present a unique combination of characteristics, possibly leading to a greater plasma confinement efficiency. In Figure 3, we observe that as the parameter $\nu$ increases, the two singular points located above the $X$ axis move closer to the magnetic islands, resulting in more effective confinement of each magnetic island against singular point fixed at the origin of the coordinate system. This is because, in the magnetic configuration presented, when a magnetic island is located between two singular points on each edge of the island that interact with the magnetic field of the singularity, the fields are directed in the same direction and thus do not undergo magnetic reconnection. In this situation, the magnetic island becomes more tightly confined between the singular points, which is why we say that it becomes more and more confined as we increase the value of the parameter $\nu$. Furthermore, the new solution allows for the possibility of changing the

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position of the external singular points, while keeping the origin singular point and magnetic islands fixed. In this way, our hypothesis was confirmed, and the results obtained show that the new solution may have potential for application in confined plasma systems, with possible benefits for plasma confinement efficiency.

Through this study, it is possible to highlight the importance of bibliographic review and critical analysis of scientific literature for obtaining new results and innovations in physics. By following this example, readers can learn to use scientific literature to solve practical problems and theoretical activities, providing a richer and more in-depth experience in learning physics.

This article is just the beginning of an exciting series that explores theoretical solutions for confined plasmas. In the continuation of this series, in part 2, we will discuss new solutions that were obtained by manipulating the solutions of Yoon and Lui (2005). Specifically, part 2 will present a solution based on the Yoon-Lui-1 and Yoon-Lui-3 solutions, which promises to offer an even deeper understanding of the physical characteristics of confined plasmas. If you are interested in expanding your knowledge in this fascinating area, keep reading and discover what part 2 has to offer!

## 7. Conclusion

In this work we performed a literature review of relevant studies in the areas of plasma physics, electromagnetism, and space physics, demonstrating how study of scientific literature can be used as a source of inspiration to promote innovation and obtain new results. The importance of the Grad-Shafranov equation in modeling confined plasmas was emphasized, and different solutions to this equation proposed in the scientific literature were discussed.

The article by Yoon and Lui (2005) was the main focus of this work, which presented two new solutions to the simplified Grad-Shafranov equation. A new solution was developed from these solutions through algebraic manipulations, presenting unique characteristics of plasma confinement.

A comparison was made between the new solution and the Yoon-Lui-1 and Yoon-Lui-2 solutions. The new solution has singular points on the X-axis, while the previous solutions have singular points above the Z-axis, for the case of $a > 0$. In addition, the new solution presents magnetic islands with less rounded shapes and does not have a current ring, which is observed in the Yoon-Lui-1 solution.

We can conclude that the study of these solutions is crucial to understanding the physical characteristics of confined plasmas, which can lead to the development of more efficient solutions for obtaining high-quality plasmas in fusion reactors. This work illustrates the importance of bibliographic research in the search for new theoretical solutions in physics, emphasizing how scientific literature serves as a valuable tool for advancing knowledge at the forefront of scientific progress.

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## References

[1] P.H. Yoon and A.T.Y. Lui, Journal Geophys. Res. **110**, A01202 (2005).
[2] A. Ojeda-González, J.J. González-Avilés, V. De La Luz and L. Nunes dos Santos, Rev. Bras. Ensino Fis. **43**, e20200329 (2021).
[3] R. Fitzpatrick, *Maxwell's Equations and the Principles of Electromagnetism* (Jones & Bartlett Learning, Massachusetts, 2008).
[4] A. Ojeda-González, A. Prestes and A.N. Laurindo-Sousa, Braz. J. Phys. **46**, 408 (2016).
[5] H. Grad and H. Rubin, in: *Proceedings of the 2nd United Nations International Conference on the Peaceful Uses of Atomic Energy* (Geneva, 1958).
[6] V.D. Shafranov, Rev. Plasma Phys. **2**, 103 (1966).
[7] J.R. Kan, Journal Geophys. Res. **78**, 3773 (1973).
[8] J.R. Kan, Plan. Space Sci. **27**, 351 (1979).
[9] B.U.Ö. Sonnerup, H. Hasegawa, W.L. Teh and L.N. Hau, Journal Geophys. Res. **111**, A09204 (2006).
[10] A. Ojeda-González, M.O. Domingues, O. Mendes, M.K. Kaibara and A. Prestes, Braz. J. Phys. **45**, 493 (2015).
[11] Q. Hu, Sci. China Earth Sci. **60**, 1466 (2017),
[12] A. Ojeda-Gonz'alez, V. Klausner, O. Mendes, M.O. Domingues and A. Prestes, Sol. Phys. **292**, 160 (2017).
[13] A.V. Khrabrov and B.U.Ö. Sonnerup, ISSI Sci. Rep. Ser. **1**, 221 (1998).
[14] L.N. Hau and B.U.Ö. Sonnerup, Journal Geophys. Res. **104**, 6899 (1999).
[15] Q. Hu and B.U.Ö. Sonnerup, Geophys. Res. Lett. **28**, 467 (2001).
[16] B. Sonnerup and M. Guo, Geophys. Res. Lett. **23**, 3679 (1996).
[17] S.B. Zheng, A.J. Wootton and E.R. Solano, Phys. Plasmas **3**, 1176 (1996).
[18] P.J. Mc Carthy, Phys. Plasmas **6**, 3554 (1999).
[19] C.V. Atanasiu, S.Günter, K. Lackner and I.G. Miron, Phys. Plasmas **11**, 3510 (2004).

[20] K. Schindler, *Physics of Space Plasma Activity* (Cambridge University Press, Cambridge, 2006).

[21] M. Cristaldo-Oliveira, A. Ojeda-González, L.N. Santos, A. Prestes and A.N. Laurindo-Sousa, Rev. Bras. Ensino Fis. **42**, e20200297 (2020).

[22] D. Biskamp, Phys. Fluids **29**, 1520 (1986).

[23] G.W. Walker, Proc. R. Soc. Lond. A. **91**, 410 (1915).

[24] E. Harris, II Nuovo Cim. **23**, 115 (1962).

[25] V. Fadeev, I. Kvabtskhava and N. Komarov, Nucl. Fusion **5**, 202 (1965).

[26] A. Manankova and M. Pudovkin, Geomagn. Aeron. **36(4)**, 426 (1996).

[27] M. Brittnacher and E. Whipple, JGR Space Phys. **107**, 1022 (2002).

[28] A. Manankova and M. Pudovkin, Geomagn. Aeron. **36**, 40 (1999).

[29] A. Manankova, Ann. Geophys. **21**, 2259 (2003).

[30] E. Curiel, *Singularities and Black Holes*, available in: https://plato.stanford.edu/entries/spacetime-singularities/

[31] E.L. Rempel, T.F.P. Gomes, S.S.A. Silva and A.C.L. Chian, Phys. Rev. E. **99**, 043206 (2019).

[32] S.A. Orszag and C.M. Tang, J. Fluid Mech. **90**, 129 (1979).

[33] B. Van Waeyenberge, A. Puzic, H. Stoll, K.W. Chou, T. Tyliszczak, R. Hertel, M. Fähnle, H. Brückl, K. Rott, G. Reiss et al., Nature **444**, 461 (2006).

[34] T. Schäpers, *Semiconductor Spintronics* (De Gruyter, Berlin, 2021).

[35] V. Génot, JGR Space Phys. **110**, A09214 (2005).

[36] P. Yoon and A. Lui, JGR Space Phys. **110**, A09215 (2005).

[37] W.B. Thompson, in: *An Introduction to Plasma Physics (Second Edition)*, edited by W.B. Thompson (Pergamon, Oxford, 1962).