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# Close relation between the integrating factor and the Green function for first-order differential equations

**Íntima relação entre o fator integrante e a função de Green para equações diferenciais de primeira ordem**

Antonio S. de Castro[*1]

[1]Universidade Estadual Paulista "Júlio de Mesquita Filho", Departamento de Física, Guaratinguetá, SP, Brasil.

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For first-order ordinary differential equations, it is shown by a very simple and straightforward approach how the integral representation of the particular solution with the integrating factor as part of the integrand can be manipulated in favor of an integral representation in terms of the Green function.
**Keywords:** Integrating factor, Green function, Linear velocity-dependent friction.

Para equações diferenciais ordinárias de primeira ordem, é mostrado por uma abordagem muito simples e direta como a representaçã o integral da solução particular com o fator integrante como parte do integrando pode ser manipulada em favor de uma representação integral em termos da função de Green.
**Palavras-chave:** Fator integrante, Função de Green, Atrito linearmente dependente da velocidade.

## 1. Introduction

The Green function method is an elegant mathematical technique particularly useful for solving nonhomogeneous ordinary differential equations, as well as homogeneous partial differential equations with nonhomogeneous boundary conditions, and has many applications in diverse fields of science and engineering. The Green function method is widely used in classical mechanics to solve problems related to waves, oscillations, and scattering (see, e.g. [1–4]), to solve the wave equation for electromagnetic fields in different media, such as conductors, dielectrics, and plasmas (see, e.g. [5, 6]), to solve the heat equation for temperature distributions in different geometries, such as spheres, cylinders, and slabs (see, e.g. [7, 8]), to solve the elasticity equation for stresses and strains in different materials, such as beams, plates, and shells (see, e.g. [9, 10]), to solve the Navier-Stokes equation for fluid flows in different geometries, such as channels, pipes, and cylinders (see, e.g. [11, 12]), to solve the Poisson equation for gravitational potentials in different astrophysical systems, such as galaxies, clusters, and black holes (see, e.g. [13, 14]), to solve various problems in quantum mechanics related to scattering, bound states, and time evolution (see, e.g. [15–17]). The method of Green functions is a fundamental tool in quantum field theory and is widely utilized for the calculation of correlation functions and scattering amplitudes (see, e.g. [18–20]). Concerning ordinary differential equations, the Green function itself satisfies a problem that is similar to the original equation but the Dirac delta symbol replaces the nonhomogeneous term. The Green function method, typically taught in advanced-level courses using second-order ordinary differential equations, is present in several textbooks (see, e.g. [21–26]) and didactic papers (see, e.g. [27–32]). However, Ref. [24] is a notable exception, as Butkov focuses on the special case of a first-order differential equation for a particle subjected to a time-dependent driving force plus a possible linear velocity-dependent friction to introduce the concept of the Green function. In this paper, we demonstrate how to obtain an integral representation of the particular solution in terms of the Green function, even for first-order ordinary differential equations with a non-constant coefficient. Specifically, we show how this integral representation can be obtained from the more commonly known integral representation of the particular solution in terms of the integrating factor. The concept of integrating factor is commonly taught in the first years of science and engineering courses and is often covered in introductory calculus courses. By introducing the Green function at an earlier stage, students can develop problem-solving skills that will be useful for tackling more complex problems in the future.

## 2. From the integrating factor to the Green function

Consider the first-order nonhomogeneous ordinary differential equation

$$\left[\frac{d}{dx} + Q\left(x\right)\right] y\left(x\right) = R\left(x\right) \tag{1}$$

[*]Correspondence email address: antonio.castro@unesp.br

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defined on $(a, b)$, where the interval may be infinity at either end or both ends. In terms of the so-called integrating factor

$$\mu(x, \widetilde{x}) = \exp\left[\int_{\widetilde{x}}^{x} Q(\zeta)\, d\zeta\right], \tag{2}$$

that satisfies the equation

$$\left[\frac{\partial}{\partial x} + Q(x)\right] \frac{1}{\mu(x, \widetilde{x})} = 0, \tag{3}$$

the general solution of (1) can be written as (see, e.g. [33])

$$y(x) = \frac{1}{\mu(x, \widetilde{x}_0)} y(\widetilde{x}_0) + \frac{1}{\mu(x, \widetilde{x}_0)} \int_{x_0}^{x} \frac{1}{\mu(\widetilde{x}_0, \widetilde{x})} R(\widetilde{x})\, d\widetilde{x} \tag{4}$$

where $x_0$ and $\widetilde{x}_0$ are arbitrary constants. The integral representation of the particular solution in (4) can also be written as

$$y_p(x) = \int_{x_0}^{x} \frac{1}{\mu(x, \widetilde{x})} R(\widetilde{x})\, d\widetilde{x} \tag{5}$$

which can satisfy homogeneous Dirichlet condition at either end of the interval $(a, b)$ depending on the choice of $x_0$. Another integral representation of the particular solution can be found by changing the limits of integration:

$$y_P(x) = \int_{a}^{b} G(x, \widetilde{x}) R(\widetilde{x})\, d\widetilde{x}, \tag{6}$$

where $G(x, \widetilde{x})$ is the Green function. Using the following properties of the Dirac delta symbol $\delta(x)$ (see, e.g. [23–26])

$$\begin{gathered} \delta(x) = 0 \text{ for } x \neq 0, \\ \textstyle\int_{-\infty}^{+\infty} f(x)\, \delta(x - x_0)\, dx = f(x_0), \end{gathered} \tag{7}$$

for any function $f(x)$ continuous at $x_0$, it can be shown that $G(x, \widetilde{x})$ obeys the following equation

$$\left[\frac{\partial}{\partial x} + Q(x)\right] G(x, \widetilde{x}) = \delta(x - \widetilde{x}). \tag{8}$$

Integrating (8) from $\widetilde{x} - |\varepsilon|$ to $\widetilde{x} + |\varepsilon|$, we see that the Green function has a jump discontinuity at $x = \widetilde{x}$ given by

$$[G(\widetilde{x} + |\varepsilon|, \widetilde{x}) - G(\widetilde{x} - |\varepsilon|, \widetilde{x})] \underset{|\varepsilon| \to 0}{\rightarrow} 1. \tag{9}$$

Note that $G(x, \widetilde{x})$ and $1/\mu(x, \widetilde{x})$ satisfy the same homogeneous equation for all $x$ except $x = \widetilde{x}$, so that on each side of the singular point $\widetilde{x}$ one of these functions is proportional to the other. If we set

$$G_{\gtrless}(x, \widetilde{x}) = \frac{1}{\mu(x, \widetilde{x})} \times \begin{cases} \theta(x - \widetilde{x})\, C_{>} \\ \theta(\widetilde{x} - x)\, C_{<} \end{cases} \tag{10}$$

where $\theta(x)$ is the unit step function ($\theta(x) = 1$ for $x > 0$, and $\theta(x) = 0$ for $x < 0$), and $C_{>}$ and $C_{<}$ are constants ($C_{>} - C_{<} = 1$ because of (9)), we can write

$$G(x, \widetilde{x}) = \frac{1}{\mu(x, \widetilde{x})} [\theta(x - \widetilde{x})\, C_{>} + \theta(\widetilde{x} - x)\, C_{<}] \tag{11}$$

and

$$\begin{aligned} y_P(x) \;=\; & C_{>} \int_{a}^{x} \frac{1}{\mu(x, \widetilde{x})} R(\widetilde{x})\, d\widetilde{x} \\ & + C_{<} \int_{x}^{b} \frac{1}{\mu(x, \widetilde{x})} R(\widetilde{x})\, d\widetilde{x}. \end{aligned} \tag{12}$$

From (8), if $G(a, \widetilde{x}) = 0$ then the derivatives of all orders of $G_{<}(x, \widetilde{x})$ vanishes. As a consequence, $G_{<}(x, \widetilde{x}) = 0$ ($C_{<} = 0$) so that

$$y_P(x) = \int_{a}^{x} \frac{1}{\mu(x, \widetilde{x})} R(\widetilde{x})\, d\widetilde{x}, \quad y_P(a) = 0. \tag{13}$$

Similarly, the boundary condition $G(b, \widetilde{x}) = 0$ makes $G_{>}(x, \widetilde{x}) = 0$ ($C_{>} = 0$), which gives

$$y_P(x) = -\int_{x}^{b} \frac{1}{\mu(x, \widetilde{x})} R(\widetilde{x})\, d\widetilde{x}, \quad y_P(b) = 0. \tag{14}$$

## 3. Final remarks

To summarize, we have established a close relationship between the integrating factor and the Green function. We have demonstrated a straightforward approach for obtaining the Green function from the integrating factor by adjusting the limits of integration for the integral representation of the particular solution. This relationship provides a valuable opportunity for students to learn about the Green function method in their introductory calculus courses.

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