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History of Physics and Related Sciences

Isaac Newton's early documents on circular motion: can the dynamic reasoning in the "Principia" be found in them?

Os documentos iniciais de Isaac Newton sobre o movimento circular: O raciocínio dinâmico no "Principia" pode ser achado neles?

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Some historians claim that Isaac Newton had already formulated the categories in the *Principia* twenty years earlier. This is based on two extant documents (1664–1665 and 1669). Derek Whiteside claims that second order differentials are essential categories in the book, and that Newton had not yet mastered them in the 1660s. Isaac Bernard Cohen states a desideratum: "I am certain it was Hooke's method of analyzing curved motion that set Newton on the right track". I show that differentials in the early documents are introduced by geometric arguments, lacking dynamic meaning; however in hindsight can a dynamic meaning be recognized in them. The drawing of an orbit in a letter from Newton to Hooke on December 13, 1679 indicates the breakthrough toward the *Principia*: I argue that the orbit was drawn by Hooke's method, which proves Cohen's desideratum.
Keywords: Mechanic orbits, Hooke's method to draw orbits, Newton's treatment of motion under central forces.

Alguns historiadores consideram que Isaac Newton formulou as categorias no *Principia*, vinte anos antes. Isso é baseado em dois documentos (1664–1665 e 1669). Derek Whiteside considera que diferenciais de segunda ordem são essenciais às categorias no livro e que Newton ainda não os dominava nos anos 1660. Isaac Bernard Cohen formula um desiderato: "Eu estou certo que foi o método de Hooke para analizar curvas que colocou Newton nos trilhos corretos". Neste artigo, mostro que os diferenciais nos documentos iniciais são introduzidos por argumentos geométricos, dos quais argumentos dinâmicos estão ausentes; entretanto, uma leitura em retrospecto, anacrônica, pode enxergar conteúdos dinâmicos nesses diferenciais. O desenho de uma órbita em uma carta de Newton para Hooke datada de 13 de dezembro de 1679 revela a inspiração levando ao *Principia*: Argumento que a órbita foi feita usando o método de Hooke, o que demonstra o desiderato de Cohen.
Palavras-chave: Órbitas mecânicas, Método de Hooke para desenhar órbitas, O tratamento de movimentos sob forças centrais feito por Newton.

1. Introduction

In two manuscripts written, respectively, *c.* 1664–1665 and *c.* 1669 (at most 1671), Isaac Newton made two calculations of the "conatus recedendi à centro", as he called the tendency of bodies in rotation to move away from the center of rotation. These calculations led many scholars to claim that as early as the 1660s Newton had already developed the dynamic concepts later found in the *Philosophiæ Naturalis Principia Mathematica*. Newton himself might have helped to spread this belief: in order to emphasize his priority on the law of gravity, he placed this discovery some twenty years earlier [1][1]; in this context, he also mentions some "method of curvature" for drawing orbits [2]. Isaac Bernard Cohen and Derek Thomas Whiteside oppose these claims. Their criticisms disclose the conceptual background on which the earlier documents should be analyzed; these are: a geometry of second order differentials and a method to draw curves.

Whiteside recognizes that second order geometric differentials are the essential mathematical tools on which Newton develops his dynamic thinking in the *Principia* ([3], p. 108):

> [...] the continuous growth during the period 1664–84 of Newton's expertise with the various orders of the infinitely small was a significant conditioning factor on the effective expression and forceful pursuance of his dynamical research.

Then he observes that Newton did not have these tools in the 1660s ([3], p. 110):

> [...] I am not sure [Newton] could then [in 1665] have differentiated successfully

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[1] Cohen shows that Newton's reasoning leading to the law of gravitation invokes the third law, which is first stated in *De Motu Corporum in medijs regulariter cedentibus*, a draft of the *Principia*.

> between the various orders of the infinitely small which are involved.

Bernard Cohen [1, 4] argues that the method to draw orbits in proposition 1 in the *Principia* is identical to the method presented to Newton by Robert Hooke in a letter on November 24, 1679. In the letter, Hooke invites Newton to comment on the following problem ([5], v.2, p. 297): "[...] compounding the celestiall motions of the planetts of a direct motion by the tangent & an attractive motion towards the centrall body [...]"; Newton answered on November 28, 1679 that he had never heard about the method. Bernard Cohen makes the conjecture ([1], p. 169):

> The exact progression of Newton's ideas in the time between his correspondence with Hooke and the completion of the first draft of *De Motu* is not documented. Nevertheless, I am certain it was Hooke's method of analyzing curved motion that set Newton on the right track.

Later, in *A Guide*, Cohen states ([6], p. 77):

> What Hooke did for Newton, therefore, was not to tell him how to analyze curved motion into components, but rather to reverse the direction of his concept of displacement in orbital motion, to shift from an outward to an inward displacement.

This change of mind is motivated by a reconstruction of an orbit drawn by Newton in a letter to Hooke (December 13, 1679); it has been claimed by Michael Nauenberg [7], Bruce Brackenridge [8] and Brackenridge and Nauenberg [9] that Newton used some "method of curvature" to draw it, not Hooke's method. Nevertheless, Bernard Cohen does not seem to have been entirely convinced, and keeps a guarded opinion on the "method of curvature" (*A Guide*, in [6], p. 75, n. 85):

> [t]his reconstruction explains a number of aspects of the development of Newton's thought but does have some gaps. For example, Nauenberg must assume that documents some years apart refer to the identical methods.

In this paper, I analyze the early documents according to the categories disclosed by Whiteside and Cohen. In Sect. 2, I investigate the role of second order differentials in Newton's *Principia*: they are geometric entities that appear in curves generated by points in motion, and are distances moved in (virtual) dynamic motions. The same structure is also found in the *De Motu*, the draft of the *Principia* (Sect. 3)[2]. The early manuscripts are discussed in Sect. 4: I show that those manuscripts fit in a conceptual framework, different from the one in the book, as observed by Cohen; furthermore, Newton does not investigate in them (dynamic) processes by which mechanic curves are generated. In Sect. 5, I answer the question in the title; it is impossible not to be sympathetic to Whiteside's and Cohen's respective claims. However — with the benefit of hindsight — the calculations in the documents can be made consistent with dynamic results; perhaps Newton is in the way to his mature work, but some essential piece is still missing: this is the piece that made possible to Newton to associate the infinitesimal segments in the static curves in the early documents with the dynamic differentials in the *Principia*. The missing piece is discussed in Sect. 6: it is the same curve in the December 13 letter. Hermann Erlichson [13] makes a hermeneutic analysis of the letter: by taking Newton at his own words, Erlichson reproduces the construction of the curve; but he does not comment that the reconstruction reproduces Hooke's method. Erlichson's reconstruction show that the letter would be the documentary evidence that Cohen thought to be non existent. That the curve was drawn by Hooke's method is enhanced by numerical computation: P. M. Cardozo Dias and T.J. Stuchi [14] translated Hooke's method and the "method of curvature" in the language of numerical computation, and compared their solutions with the solution found using a more precise method of computation; Hooke's is the solution closest to the curve in the letter. Cohen's contention that Hooke's method "set Newton on the right track" gains strong support[3].

2. The Mature Work: the Foundations of a Mechanic Orbit

In the introduction to the first edition of the *Principia*, Newton defines mathematical principles of natural philosophy (translation in [8], p. 230–231):

> [...] the description of straight lines and circles, on which geometry is founded, belongs

[2] The story of how Newton came to write the *Principia* has been told in many places [1, 4, 10]; it is traced back to Edmond Halley's visit to Newton in August 1684. In his *Introduction to the 'Principia'* [11], Bernard Cohen reconstructs the development of Newton's ideas on dynamics from the various manuscripts to the copy to the printer, as well modifications in the editions. He lists the following manuscripts: several versions of the tract *De Motu*; two fragments, *De Motu Corporum in medijs regulariter cedentibus* ([10], pp. 188–194; [12], pp. 304–308) and *De Motu Corporum: definitiones* ([10], pp. 92–96; [12], pp. 315–317); the *Lucasian Lectures*, ([11], p. 61) "closely resembling the manuscript [...] used for printing the *Principia* [...]", composed of a set of manuscripts that Cohen ([11], p. 85) reconstructs as *De Motu Corporum Liber Primus*. The manuscript written between Halley's two visits is called "The original De Motu" by Whiteside ([10], pp. 30–74); it is the document "De Motu Corporum in Gyrum", in John Herivel's edition ([12], pp. 257–274).

[3] The whole argument is better stated as "overwhelmingly probable" that Newton used Hooke's method. As it is going to be shown, Newton did not draw the whole curve, but only a small part where various integration methods coincide. However, together with other evidences presented in this paper, the adverb 'overwhelmingly' is justified.

> to Mechanics. Geometry does not teach the drawing of these lines, but requires it. For it requires that the beginner learn to describe these accurately before he reaches the threshold of Geometry; then it teaches how problems may be solved through these operations. To describe straight lines and circles are problems, but not geometric ones. The solution of these is required from Mechanics; the use of the solutions is taught in Geometry. [...]. Therefore Geometry is founded on mechanical practice and is nothing else than that part of universal Mechanics that proposes and demonstrates the art of measuring accurately. But since the manual arts are chiefly employed in the moving of bodies, it happens that Geometry commonly refers to their magnitudes, and Mechanics to their motion. In this sense rational Mechanics will be the science of motions that result from any forces whatsoever, and also of the forces required to produce any motions, accurately proposed and demonstrated. [...]. I offer this work as the mathematical principles of philosophy. For the whole burden of philosophy seems to consist in this: from the phenomena of motions to investigate the forces of nature, and them from these forces to demonstrate the other phenomena; [...]

In other words, "[...] geometrical objects [are conceived] as generated by mechanical devices. Most notably, curves are generated by tracing mechanisms" ([2], p. 15). Colin MacLaurin recognizes in the separation of these two methods, the geometric and the mechanic, the foundations of Newton's "method of fluxion" ([15], p. 1–2):

> In the method of indivisibles, lines were conceived to be made up of points, surfaces of lines, and solids of surfaces; [...]. But as this doctrine was inconsistent with the strict principles of geometry [...] others in the place of indivisible, substituted infinitely small divisible elements, of which they supposed all magnitudes to be formed [...]. There were some, however, who disliked the making much use of infinites and infinitesimals in geometry. Of this number was Sir *Isaac Newton* [...]. In demonstrating the grounds of the method of fluxion, he avoided them, establishing it in a way more agreeable to the strictness of geometry. He considered magnitudes as generated by a flux or motion [...].

In Book I of the *Principia* are given the foundations of the mechanic geometry of central orbits:

- Identification of second order differentials in orbits drawn by moving points. This is made in lemma 9, lemma 10 and its corollary 4, lemma 11 and its corollary 3.
- These differentials define virtual motions. The first order segment on the tangent represent inertial motion. The second order segments represent uniformly accelerated motions toward a center of force (either the center of the osculating circle or the center of force in central orbits). They also fix a mechanism to draw orbits.
- Proposition 1 gives the drawing mechanism, which is Hooke's method[4]; accordingly, a central orbit is obtained by the composition of two virtual motions: a uniform motion along the tangent, and a motion similar to a gravitational fall to the center of force.

2.1. The second order differentials in general non uniform motions

In this section it is shown: Lemma 9 identify a second order area; lemma 10 interprets this area as distance moved in uniformly accelerated virtual motions; lemma 11 identifies this motions: it is a "fall" from the tangent to the curve on a line through the center of force, which is either the center of the osculating circle or the center of force in orbital motion.

2.1.1. Lemma 9: second order areas

Lemma 9 is stated in Fig. 1. In the figure, Abc is a first order arc of an arbitrary curve; ABC is an arc of another curve, tangent to Abc at A; $\overline{AFGfg}$ is the tangent common to both curves at A. The curved areas Ace e $Abd, \ldots$, and the curved areas ACE e $ABD, \ldots$ are successive figures in the limit that c tends to A and C tends to A, respectively. The right triangles Ace e $Abd, \ldots$, and the right triangles ACE e $ABD, \ldots$ are successive figures in the limit that c tends to A and C tends to A, respectively.

It is demonstrated:

$$\frac{\text{cuved area } Abd}{\text{curved area } Ace} \approx \frac{\text{area } \triangle Afd}{\text{area } \triangle Age} = \frac{\text{area } \triangle AFD}{\text{area } \triangle AGE} \approx \frac{\text{curved area } ABD}{\text{curved area } ACE};$$

on the other hand, exactly:

$$\frac{\text{area } \triangle Afd}{\text{area } \triangle Age} = \left(\frac{\overline{Ad}}{\overline{Ae}}\right)^2 = \left(\frac{\overline{df}}{\overline{eg}}\right)^2$$

and

$$\frac{\text{area } \triangle AFD}{\text{rea } \triangle AGE} = \left(\frac{\overline{AD}}{\overline{AE}}\right)^2 = \left(\frac{\overline{DF}}{\overline{EG}}\right)^2.$$

[4] That the drawing mechanism in Proposition 1 is given by Hooke's method is not disputed. The contention is whether Newton had a different method before 1679.

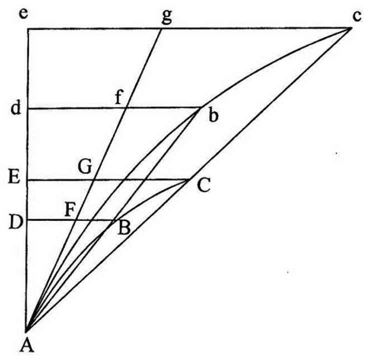

Figure 1: Second order areas. These are the areas of the triangles Age, and the curved areas ACE, ...

Lemma 9 means that the curves tend to the common tangent, or that cg, bf, ... and CG, BF, ... respectively approach the tangent, in the same ratio that the curved areas tend to the areas of the right triangles $\triangle Age$, $\triangle Afd$, ... and $\triangle AGE$, $\triangle AFD$, ..., respectively.

Furthermore, the areas are in the same ratio as the square of the sides of the right triangles. The parameter that measures how much a point on the tangent goes away from A is the length of the segment on the tangent, or $\Delta\lambda$; but the hypotenuses of the right triangles of sides, AD and FD, ad and fd, etc. are on the tangent, so that (a side being proportional to the hypotenuse for a fixed angle $\widehat{cAb}$) it follows that the areas are proportional to $\Delta\lambda^2$.

2.1.2. Lemma 10: existence of a uniformly accelerated virtual motion

Lemma 10 is obtained from lemma 9 by an appropriate interpretation of the axes $\overline{ce}$ and $\overline{eA}$. Calling 'time' the direction $\overline{Ae}$, and 'speed' the direction $\overline{ce}$, or vice-versa, the right triangles $\triangle Age$, $\triangle Afd$, $\triangle AGE$ and $\triangle AFD$ are the graphic velocity versus time in a uniformly accelerated motion. Therefore, the curved areas in Fig. 1 are interpreted as distances moved in uniformly accelerated motions ([6], p. 437):

> Lemma 10. The spaces which a body describes when urged by any finite force whether that force is determinate and immutable or is continually increased or continually decreased, are at the very beginning of the motion in the squared ratio of the times.

Corollary 4 to lemma 10 introduces the force ([6], p. 438):

> And thus the forces are as the spaces described at the very beginning of the motion directly and as the squares of the times inversely.

The spaces "urged ... at the very beginning of the motion" are the segments cg, bf, ..., and CG, BF, ..., in Fig. 1.

2.1.3. Lemma 11: identification of the uniformly accelerate virtual motion

Lemma 11 is stated in Fig. 2. In the figure, arcAB is a first order arc of some curve. $\triangle ABG$, $\triangle Abg$, ... are successive triangles in the limit that B tends to A; in the limit, G and g tend to J, the lower end of the diameter of the osculating circle.

Lemma 11 states that for small arcs ($\overline{AD} \propto \Delta t$; $\overline{AB} \approx {}_{\mathrm{arc}}AB \propto \Delta t$):

$$\text{lemma 11:} \qquad \overline{BD} \propto \frac{\overline{AB}^2}{2R}, \tag{1}$$

where R is the radius of the osculating circle.

The proof starts from the assumption that $\triangle AGB$, $\triangle agb$ are respectively similar to the right triangles $\triangle ABC$ and $\triangle abc$. Therefore, $\triangle AGB$ and $\triangle agb$ must be triangles inscribed in semi circles. Therefore lemma 11 introduces the osculating circle and the curvature. But AB is meant to be an arc of an arbitrary curve, so that Newton is taking the circle as a first order approximation to the curve. Lemma 11 associates the curvature with a uniformly accelerated motion: the "bending" of the tangent results from a uniformly accelerated motion, in which points on the tangent "fall" to the center of the osculating circle, "wrapping" the tangent around it. Once $\triangle AGB$, $\triangle agb$, ... are recognized as right triangles, the remaining of the proof is "non-illuminating".

Newton uses the Latin word for arrow — *sagitta* — to name the arrow that bisects the chord $\overline{AB}$ (and the

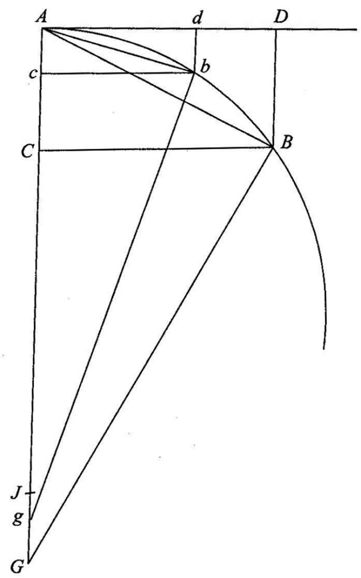

Figure 2: Second order segments. These are the segments $DB, db, \ldots$, whatever their inclination with respect to the tangent.

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arc ${}_{\text{arc}}AB$). It is a corollary that the inclination of $\overline{BD}$ is arbitrary, so that lemma 11 can be applied to the *sagitta*, as well. This can be explained: when the point is fixed, the angle between $\overline{BD}$ and the *sagitta* (which goes through the center of the circle) is also fixed, so that the angle between them (and their circular functions) can be taken as constants (at the point). Then:

$$\begin{array}{llcl} & \text{sagitta} & \propto & \overline{BD} \\ \text{Lemma 11:} & \overline{BD} & \propto & \frac{\left(\overline{AB}\right)^2}{2R} \\ \text{then:} & \text{sagitta} & \propto & \frac{\left(\overline{AB}\right)^2}{2R}. \end{array}$$

Then corollary 3 to lemma 11 gives dynamic meaning to the sagitta ([6], p. 440):

> Corollary 3 [to lemma 11]. And thus the sagitta is in the squared ratio of the time in which a body describes the arc with a given velocity.

In mathematical notation, sagitta $\propto (\Delta t)^2$; in fact by the law of inertia, $\overline{AB} \propto \Delta t$.

Lemma 11 can be stated:

$$\text{distance of “fall”} = sagitta \propto (\Delta t)^2 . \qquad (2)$$

This result means that the uniformly accelerated virtual motion found in lemmas 9 and 10 is the “fall” from the tangent to the osculating circle, along a line through the center of the osculating circle. The force is proportional to the *sagitta*, according to Galileo’s law for the uniformly accelerated motion.

The language of proportions is essential in the argumentation. It also allows to conflate a line through the center of force and a line through the center of the osculating circle, as already mentioned. Therefore, Equation 2 can be applied to a central orbit, as in proposition 1.

2.2. Method to trace central orbits by moving points

The method of tracing orbits is Hooke’s method, illustrated in Fig. 3. A body at A moves uniformly to B, in an interval of time Δt. If the body continues in its uniform motion, in an equal time interval it moves a distance $\overline{Bc} = \overline{AB}$. However, if the body receives at B a push toward S (the center of force), it goes to C in Δt, and not to c. And so successively, so that the body describes the polygonal line $ABCDEF\ldots$. In the process a curved motion is obtained by the composition of two motions:

1. A uniform motion on the tangent (AB).
2. A uniformly accelerated motion along BV, according to the second order nature of BV. In Fig. 3, BV is toward the center of force, so that the polygonal orbit is a central orbit[5].

[5] The center of the osculating circle also is a center of force, so that the decomposition holds for any curved motion.

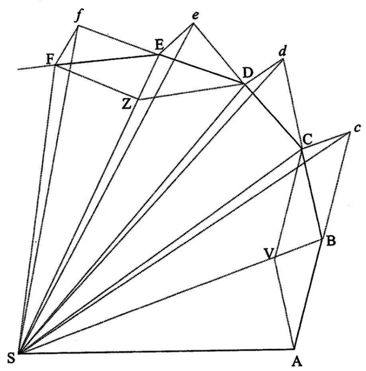

Figure 3: Drawing a central orbit by the motion of a point (proposition 1). S is the center of force. The orbit is the polygonal line $ABCDEF\ldots$. $\overline{AB} = \overline{Bc}$, $ABCV$ e $BcCV$ are parallelograms. $\vec{AV} = \vec{AB} + \vec{BV} = \vec{BC} = \vec{Bc} + \vec{cC}$.

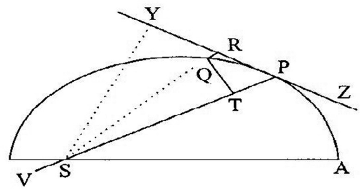

Figure 4: Parameters of an orbit (proposition 6). S is the center of force, SP is the radius vector, $\overline{QT} \perp \overline{SP}$, area $\triangle SPQ \approx \overline{SP} \times \overline{QT} \propto \Delta t$.

These motions occur in an infinitesimal instant Δt. They are “instantaneous (virtual) motions”.

The central force acts along $\overline{BV}$. Clearly, sagitta $\propto \overline{BV}$, and using Equation 2,one obtains the fundamental equation that caracterizes the uniformly accelerated motion:

$$\text{distance} = sagitta \propto \text{force} \times \Delta t^2,$$

or

$$\overline{BV} \propto \text{central force} \propto \frac{sagitta}{\Delta t^2}. \qquad (3)$$

Fig. 4 defines the parameters involved in a central motion. The segment $\overline{QR}$ is (proportional to) the *sagitta* of an arc twice arcPQ with P at the middle. From Equation 3, and the geometry of the segments in Fig. 4, Newton proves a relation between the force and the parameters of the curve:

$$\text{force} \propto \frac{\overline{QR}}{\left(\overline{SP} \times \overline{QT}\right)^2}. \qquad (4)$$

This is the solution to the “direct problem” — find the force, given the curve.

3. The Intermediate Work: “De Motu” (August–December 1684)

Although the story of the *Principia* starts with Edmond Halley’s visit to Newton in August 1684, as already said

Table 1: "De Motu" versus "Principia". Equivalence between propositions and theorems.

De Motu	*Principia*	Meaning
theorem 1	proposition 1	proof of the law of areas
theorem 2	proposition 4	expression of the centripetal force
theorem 3	proposition 6	proof of the direct problem

(footnote 1), the "real beginning" of the book is the treat *De Motu*[6], placed by Bernard Cohen close to Halley's second visit in December 1684 ([11], p. 61):

> [...] when we talk of the real beginning of the *Principia*, we cannot go back much before Halley's second visit: true first steps toward the *Principia* as a treatise must be dated in November or December of 1684.

The structure of the treat is similar to the structure of the *Principia*. Tab. 1 compares theorems in the *De Motu* and in the *Principia*.

Hypothesis 2 introduces the law of inertia, the other two laws are absent. The content of lemma 10 is introduced as a hypothesis ([10], p. 33):

> *Hypothesis 4.* The space which a body, urged by any centripetal force, describes at the very beginning of its motion is in the doubled ratio of the time.

Instead of the general lemma 11, theorem 2 introduces the centripetal force; the proof of the theorem and the identification of the force are done with the aid of Fig. 5. The 'centripetal force' is then defined ([10], p. 31):

> *Definition 1.* A 'centripetal' force I name that by which a body is impelled or attracted towards some point regarded as its centre.

The centripetal force has the same dynamic meaning of the segments in lemma 11, Fig. 2 ([10], p. 39):

> [t]he centripetal forces are those which perpetually drag the bodies back from the tangent to the circumferences and hence are to each other as distances CD, cd surmounted by them [...].

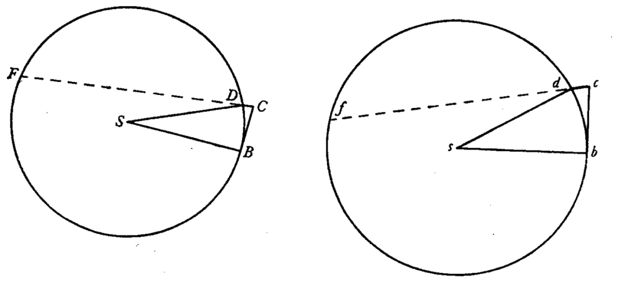

Figure 5: The centripetal force. The force is along cd and CD and tend to the radius of the circle.

The figure corresponding to theorem 3 in *de Motu* (direct problem) is identical to Fig. 4; hypothesis 4 defines $\overline{QR}$ as describing a uniformly accelerated fall, so that $\overline{QR} \propto (\Delta t)^2$; also, $\overline{QR}$ is proportional to the centripetal force, so that $\overline{QR} \propto (\text{force}) \times (\Delta t)^2$. Equation 4 is proved in the same way as it is proved in the book. The proof points to a subtlety also found in lemma 10: Newton conflates central and centripetal forces; this is possible for infinitesimal segments, because at a fixed point the angle between them can be considered constant, so the forces are proportional.

4. The Early Documents

The documents present two different calculations of the "conatus". The first manuscript is dated between 1664 and 1665. Herivel [12] places the second manuscript in 1669, and Rupert Hall ([16], p. 64) places it earlier than 1671.

4.1. The first document (*c.* 1664–1665)

The calculus of the "conatus" is approached as follows ([12], pp. 128–132). A small sphere (b) moves inside a circle (Fig. 6); it collides with the circle, rebounds, collides again, and so forth, so that it describes the square *abcd*. Newton proves:

$$\frac{\text{"pressure" of the sphere on the circle}}{\text{"force of motion"}} = \frac{\text{perimeter of the circle}}{\text{radius of the circle}}.$$

The proof is geometric, and involves only the similarity of $\triangle abd$ and $\triangle afb$, and the a definition of "force of motion" (mv) and "pressure" ($\overline{bn}$)[7].

The terms 'force of motion' and 'pressure' do not necessarily have dynamic meaning, they could as well have kinetic meaning (respectively momentum and acceleration)[8]. Newton's "pressure" is as reminiscent of the

[6] Bernard Cohen [11] reconstructs the development of Newton's ideas on dynamics from the various manuscripts to the copy to the printer, as well modifications in the editions. He lists the following manuscripts: several versions of the tract *De Motu*; two fragments, *De Motu Corporum in medijs regulariter cedentibus* ([10], pp. 188–194; [12], 1965, pp. 304–308;) and *De Motu Corporum: definitiones* ([10], pp. 92–96; [12], pp. 315–317;); the *Lucasian Lectures*, ([11], p. 61) "closely resembling the manuscript [...] used for printing the *Principia* [...]", composed of a set of manuscripts that Cohen ([11], p. 85) reconstructs as *De Motu Corporum Liber Primus*. The manuscript written between Halley's two visits is called "The original De Motu" by Whiteside ([10], pp. 30–74); it is the document "De Motu Corporum in Gyrum", in Herivel's edition ([12], pp. 257–274).

[7] From $\triangle abd \sim \triangle afb$: $\frac{db}{ab} = \frac{ad}{af} = \frac{ab}{fb}$. Or $\frac{\text{pressure}}{\text{force of motion}} = \frac{ad}{af} = \frac{ab}{fb}$. But $\frac{ad}{af} = \frac{ab}{fb} = \frac{\text{side}}{\text{radius}}$, or $\frac{\text{pressure}}{\text{force of motion}} = \frac{\text{side}}{\text{radius}}$. After four collisions, $\frac{\text{pressure}}{\text{force of motion}} = \frac{\text{perimeter}}{\text{radius}}$. In the limit in which the number of sides tends to infinity, $\frac{\text{pressure}}{\text{force of motion}} = \frac{2\pi r}{r} = 2\pi$.

[8] A similar construction is made to introduce the centripetal acceleration, in introductory physics courses.

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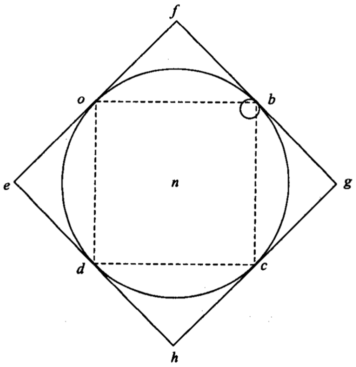

Figure 6: First calculation (1664–1665). A sphere hits the circle from the inside, describing the square *abcd*. The "force of motion" is defined by $\overline{ab}$ ($\propto mv$); the "pressure" of the sphere on the circle is along the diameter *bnd*.

modern concept of 'force' as Renè Descartes's centrifugal tendency or "effort" in circular motions ([17], pp. 45–47; [18], pp. 131–133[9].

Furthermore, as indication that the "pressure" is centrifugal, the sphere in Fig. 6 is inside the circle. It can be objected (Brackenridge, 1995, p. 47–48):

> Implicit in this statement is Newton's understanding that the impulsive force exerted by the body b on the side of the square fg at a reflection is equal to the impulsive force exerted on the bdy by the square (i.e. action and reaction).

However, the third law was first stated in *De Motu Corporum in medijs regulariter cedentibus* [1], which was written after the *De Motu*.

Nevertheless this calculation points to the dynamic process in Hooke's method. Fig. 6 is similar to Fig. 3; completing $\triangle abc$ in Fig. 6, its sides have the same meaning as the sides of $\triangle ABV$ in Fig. 3:

1. In $\triangle abc$: $\vec{bc} - \vec{ab} = \vec{bd} = 2\vec{bn}$. But $m\vec{v}_{ab} = \vec{ab}$ and $m\vec{v}_{bc} = \vec{bc}$, so that $\Delta\left(m\vec{v}\right) = 2\vec{bn}$.
2. In $\triangle ABC$: $\vec{BC} - \vec{AB} = \vec{BV}$. But $m\vec{v}_{AB} = \vec{AB}$ and $m\vec{v}_{BC} = \vec{BC} = \vec{AV}$, so that $\Delta\left(m\vec{v}\right) = \vec{BV}$.

Once Newton learned Hooke's method, he might have recognized that his earlier construction was similar to Fig. 3, and could be similarly interpreted. This implies the composition of motions expressed by $\vec{bc} - \vec{ab} = \vec{bd}$, which demands that the "pressure" be inwards.

The claim that prior to the *Principia*, Newton already had a method to draw orbits (and that it was the "method of curvature") is based on a corollary to the calculation ([12], p. 130):

> If the body b moved in an Ellipsis that its force in each point (if its motion in that point bee given) [will?] bee found by a tangent circle of Equall crookednesse with that point of the Ellipsis.

This corollary does not necessarily indicate a "method of curvature" to draw orbits. Newton introduced the concept of 'curvature' in the October 1666 version of the *Tract on Fluxions*; in the quotation, he only recognizes that the circle in Fig. 6 can be the osculating circle at some point of an ellipse, in which case the "pressure" on the ellipse is similarly calculated. This does not add anything else to the construction of a dynamics, that is not already in Fig. 6; it at most shows how the calculation can be applied to planetary motions. Nevertheless, I agree that if Newton had a method of curvature, this was it.

4.2. The second document (*c.* 1669)

In the document ([12], pp. 192–198; [16]), Newton considers a body moving on a circle with uniform speed v (Fig. 7). In an infinitesimal time t, it moves ${}_{\text{arc}}AD$; if the body leaves the circle at A, it moves in the same time a distance $\overline{AB} = vt$ on the tangent; also ${}_{\text{arc}}AD \approx \overline{AB}$. The radial distance away from the circle is $\overline{BD}$. In a time equal to the period of the circular motion (τ), the body moves the whole circumference, $\mathcal{C} = 2\pi r$. The problem stated by Newton is to find x such that

$$\frac{\overline{BD}}{x} = \frac{\left({}_{\text{arc}}AD\right)^2}{\mathcal{C}^2} \qquad \text{or} \qquad \frac{\overline{BD}}{x} \approx \frac{\overline{AB}^2}{\mathcal{C}^2}. \tag{5}$$

The proof starts from a geometric property of circles:

$$\left(\overline{AB}\right)^2 = \left(\overline{BD}\right) \times \left(\overline{BE}\right). \tag{6}$$

For small arcs, $\overline{BE} \approx \overline{DE} = 2r$, and the geometric Equation 6 can be written

$$\left(\overline{AB}\right)^2 \approx \left(\overline{BD}\right) \times \left(\overline{DE}\right) \tag{7}$$

[9] Descartes explains the "natural tendency" of bodies to go away from the center of a circle [19]. A stone rotates in a sling, and is released. Then it moves uniformly on a straight line tangent to the circle (gravity absent). Taking the center of the circle described by the sling as the center of (plane polar) coordinates, Descartes argues that the uniform velocity is everywhere decomposed in two components: one is along the radius, and the other is perpendicular to the radial component; circular motion occurs, when the radial motion is hindered by an obstacle ("empesché", in Descartes's words). Descartes then imagines ([18], p. 131–133) an infinite rule rotating around an end, and an ant moving on it; in order that the ant be seen moving (uniformly) on a straight line as the ruler rotates (as is seen by an observer external to the cylinder, of course), the ant "fait an effort" (p. 132) , which is conceptually the same as the "effort" made by the stone in the sling to move away from the center of the circle. In other example, a small sphere moves inside an infinite cylinder that rotates around an extremity; then (p. 133) "[. . .] la pierre qui est dans une fonde, fait tendre la corde d'autant plus fort qu'on la fait tourner plus vite".

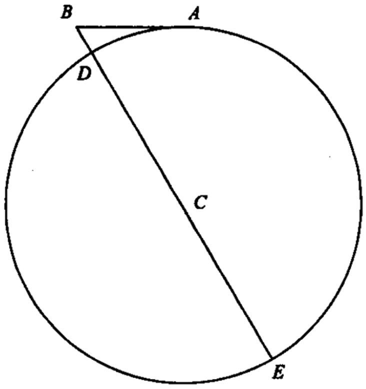

Figure 7: Second calculation (1669). The body leaves the circle at A. The segment AB represents a uniform motion with the speed at A. The segment $\overline{BD}$ is the distance moved away from the circumference.

hence:

$$\frac{(\overline{AB})^2}{\mathcal{C}^2} \approx \frac{(\overline{BD}) \times (\overline{DE})}{\mathcal{C}^2} \equiv \frac{(\overline{BD})}{\frac{\mathcal{C}^2}{(\overline{DE})}}. \tag{8}$$

Comparing Equation 8 with Equation 5: $x \equiv \frac{\mathcal{C}^2}{(\overline{DE})}$[10].

Equation 6 is geometric; it involves the square of $\overline{AB}$, therefore $\overline{AB}^2$ is to be compared with $\mathcal{C}^2$. The variable "time" is introduced by the law of inertia: $\overline{AB} = vt$ and $\mathcal{C} = v\tau$; this is also the time to move $\overline{BD}$, hence from Equation 6, $\overline{BD} \propto t^2$. Therefore the parameter t^2 need not be introduced by a dynamic principle: it can be justified from the motion on the tangent, which is uniform. Furthermore, as in the first calculation, the circle is static, and not traced by the motion of a point, so that $\overline{BD}$ (Fig. 7) does not belong in the same framework of thinking as segments $\overline{BV}$ (Fig. 3), $\overline{QR}$ (Fig. 4), and $\overline{CD}$ and $\overline{cd}$ (Fig. 5).

Once $\overline{BD}$ is recognized to be proportional to t^2, it is possible to make an analogy with a uniformly accelerated motion ([12], p. 195):

> Now since the endeavour, provided it were to act in a straight line in the manner of gravity, would impel bodies through distances which are as the square of the times: [then Newton states the problem].

This analogy is not entirely a novelty: it had already been made by Christiaan Huygens. He proves in the *De Vi Centrifuga* [20] many theorems on the centrifugal force[11], and treats the problem dynamically from the beginning: he proves that the centrifugal "tendency" is analogous to a fall, then segments such as $\overline{BD}$ are as t^2[12]; Newton in the above passage reasons in the inverse direction: the analogy with a weight is only formal, and is based on the t^2 dependence, but this dependence need not represent an actual motion of fall by which the circle is drawn, as commnmeted in Sect. 4.1.

Textual evidence that Newton does not think in terms of a centripetal *"conatus"* is given by the words with which he frames two corollaries to the calculation: he refers to the Moon as "receding" from the Earth and planets as "receding" from the Sun. The first is ([12], p. 196; italics are mine):

> And so the force of gravity [as at the surface of the Earth] is 4000 and more times greater than the *endeavour of the Moon to recede from the centre of the Earth.*

In the second, using Kepler's third law, Newton finds an expression for the "endeavour" to recede from the Sun ([12], p 197; italics are mine):

> Finally, since in the primary planets the cubes of their distances from the Sun are reciprocally as the squares of the numbers of revolutions in a given time the *endeavours of receding from the Sun* will be reciprocally as the squares of the distances from the Sun.

The two corollaries have been taken as evidence that Newton already had the law of universal gravitation. In view of the analysis of Newton's conceptual framework, this seems not well founded, because the "pressure" is centrifugal.

5. Answer to the Question in the Title

A signature of the mature dynamics described from *De Motu* on is that the geometric segments are "animated" in the sense that they are associated with a dynamic motion described by a moving point mass. It means that an orbit can actually be drawn by a graphic program, such as GeoGebra [23], or by numerical computation [14], or by hand (as certainly did Newton). The treat *De Motu* contains this structure, but not yet in the sophisticated form it attains in the *Principia.*

Of course, one can always claim that a t^2 dependence means that a motion is uniformly accelerated; but there

[10] Anachronistically, $\overline{BD} \approx \frac{1}{2}\left(\frac{v^2}{r}\right)t^2$.

[11] Although the book, written in 1673, was published only in 1703, much after Huygens's death, the theorems were stated at the end of *Horologium Oscillatorium* [21], published in 1673. Huygens sent a copy to Newton, through Oldenburg, as documented in a letter from Oldenburg to Newton, on June 4, 1673 ([5], v. 1, p. 284). Newton answers to Oldenburg on June 33, 1673 ([5], v. 1, p. 290): "I received your letters wth M. Hugens kind present, wch I have viewed wth great satisfaction, finding it full of very subtile & usefull speculations very worthy of ye Author".

[12] Huygens argues [22] that the centrifugal tendency on a small sphere held by a person standing at the top of a wheel is canceled by the weight of the sphere. Therefore, the centrifugal tendency is equivalent to a weight. When released, the sphere moves uniformly on the tangent, but at its beginning, the motion away from the center of the wheel is similar to the free fall.

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is not a hint in the early documents that the segments proportional to t^2 are associated with a motion: the dependence is entirely justified on geometric and kinematic arguments[13].

The association of $\overline{bn}$ (Fig. 6) and $\overline{BD}$ (Fig. 7) with a fall to the curve relies on the change from a centrifugal to a centripetal "pressure"; as seen in Sect. 4.1 and in Sect. 4.2, Newton thinks in terms of a centrifugal "pressure", which has been pointed by many authors. In the Introduction, I mentioned that Cohen weakened his initial claim. Acccording to the new claim, what Newton learned was the change from a centrifugal to a centripetal force, not the composition of motions; I could not disagree more: the circle in Fig. 6 is static, and the composition of motions is what turns a static circle into a dynamic one; of course, for the composition to be possible, the "pressure" must be centripetal.

The comparison between the early and the later documents corroborates the claim that the structure of Newton's mature thinking is not found in the early documents. Sometime in the intervening twenty years between the early manuscripts and the *De Motu*, Newton made a breakthrough. According to Bernard Cohen, it was triggered by Hooke's method to draw orbits (Fig. 3).

6. The Missing Piece: the Curve in the Letter on December 13, 1679:

Answering on November 28, 1679 to an invitation made by Hooke (November 24, 1679) to comment on his method, Newton proposes a new problem ([5], v. 2, p. 300–303); leaving aside Newton's motivations, the problem can be given an anachronistic statement: to find the orbit of a body moving in a central field of force of constant magnitude, $\vec{F}(r) = mg\hat{r}$. The solution in the letter is a spiral. The spiral was criticized by Hooke (December 9, 1679), who proposed instead an "elliptueid" ([5], v. 2, pp. 304–306). On December 13, 1679 Newton sent Hooke a new solution, the curve in Fig. 8 ([5], v. 2, pp. 307–308).

The correct solution to the problem for an orbit near the circular orbit is in Fig. 9 [14]. It was obtained using a method of numerical integration, which is sufficient precise to be taken as a pattern.

6.1. Numerical computation helps to decide the method of drawing the orbit

Nauenberg [7] shows that Newton drew the curve by the method of curvature up to the axis of symmetry (CO), and then reflected it; the curve so obtained is similar to the curve in the letter. Furthermore, as observed by many historians, the pericenter in Newton's drawing is displaced downwards, i.e., the angle ACO in the drawing

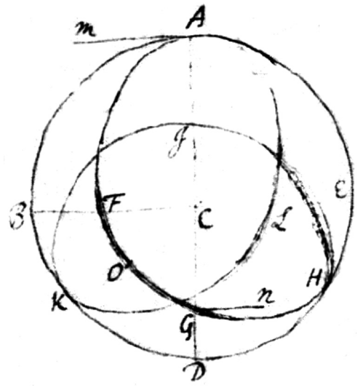

Figure 8: Newton's solution. The orbit drawn on 12/13/1679.

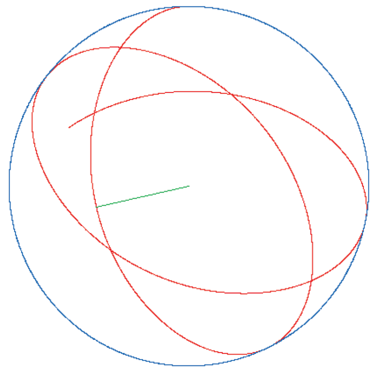

Figure 9: The correct solution. Solution of the equation $m\frac{d^2\vec{r}}{dt^2} = mg\hat{r}$ for an orbit close to the circular orbit, obtained by numerical computation. Its similarity with Fig. 8 is striking.

is bigger than it should be; this is accounted for by an error in reflecting, as Nauenberg correctly shows.

Cardozo Dias and Stuchi [14] solved the problem using three different methods other than the method used in Fig. 9: Hooke's method, the "method of curvature" and a third method called "generic"[14]; these methods are rephrased in the language of numerical computation, and the solutions are then compared with the pattern in Fig. 9. For orbits close to the circular orbit, the three methods agree with each other up to some point (not far from the pericenter). Beyond the point, Hooke's method closely follows the pattern, but the other two methods diverge from it: (1) they leave the circle, then go back inside it; (2) they display higher eccentricities. Therefore, of the three methods, Hooke's is the one that come the closest to the orbit by Newton, and to the orbit in Fig. 9. As to the position of the pericenter, adding Newton's error to the pericenters found by the three methods, the pericenter in Hooke's method is the one that leads to a value closest to the value in Fig. 8, measured with a protractor.

[13] Although acceleration is proportional to force, Newton's second axiom first appears in the *Principles*. However it is meant to be a definition, as shown by Cohen [24].

[14] The "generic" method is called "Euler's method" in numerical computation. To avoid the anachronistic name and unnecessary criticisms, the name has been changed.

That Newton used the "method of curvature" cannot be entirely disclaimed on the matching of the curves, because in each of the three methods the curves match up to a certain point: it happens that Newton drew the matching segment, and reflected it; unhappily many historians are mislead into believing in the "method of curvature" ([25], p. 236). However, the "method of curvature" does not account for the position of the pericenter in Newton's drawing, as does Hooke's method. Furthermore, the method of curvature is awkward. It suffices to try to draw the curve following the script in Nauenberg [7] and in Brackenridge and Nauenberg [9]; the method demands algebraic calculations of the centripetal forces and the measure of angles.

6.2. Newton's explanation of the construction in the December 13 letter

In his letter (December 13), Newton gives an explanation of the curve ([5], v. 2, p. 307–308)[15]. Herman Erlichson [13] shows that Newton's words fit the construction in Fig. 10.

The curve is the polygonal $ARSTUV, \ldots$, so built:

1. Each side is extended by a segment equal to it: $RR' = AR, SS' = RS, TT' = ST, UU' = TU, \ldots$. The extensions respectively represent uniform motions with the speed at the beginning of the segment, respectively $R, S, T, U, \ldots$.

2. The points $R, S, T, U, \ldots$ result from "falls" from $R', S', T', U', \ldots$, respectively. The "fall" is equal at each point, so that $A'R = R'S = S'T = T'U = U'V = \cdots \propto mg$, but is directed to the center of force C, so that $A'R \parallel CA$, $R\prime S \parallel CR$; $S'T \parallel CS$, $T'U \parallel CU$; $U'V \parallel CU$.

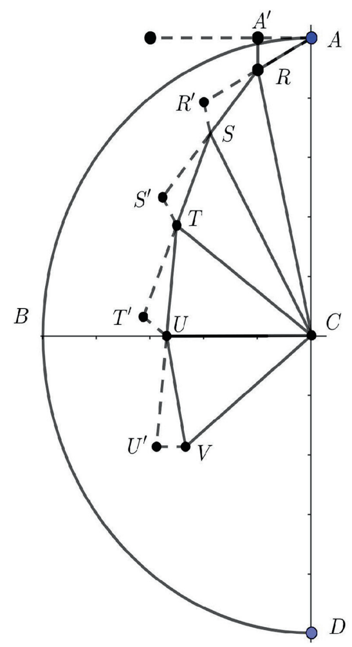

Figure 10: Erlichson's reconstruction. Remake of the figure in [13].

Erlichson dores not mention that this is the method in Fig. 3, nor that this is Hooke's method; but he is not discussing priorities.

7. Comments

In the Introduction, I quote Bernard Cohen's comments that Newton's ideas on mechanics in the period between the exchange of letters with Hooke and the *De Motu* are not documented. However, reconstructive analysis also produces evidences: it shows "tools for thinking" (I borrow the expression from Mathias Schemmel [26]), and it makes possible to decide whether documents belong in a common thinking framework, or whether not. These evidences are much in need, when documentary evidences are misunderstood, scarce, and historiography too contentious. The reconstruction of the documents made in this paper is such a kind of evidence[16], and so is Erlichson's hermeneutic reconstruction of Newton's letter[17].

[15] Suppose A ye body, C ye center of ye earth, $ABDE$ quartered wth perpendicular diameters AD, BE, wch cut ye said curve in F & G; AM ye tangent in wch ye body moved before it began to fall & GN a line drawn parallel to yt tangent. When ye body descending through ye earth (supposed pervious) arrives at G, the determination of its motion shall not be towards N but towards ye coast between N & D. Two motions are compounded at each instant: a motion parallel to AM, and a "converging motion" generated by gravity. For ye motion of ye body at G is compounded of ye motion it had at A towards M & of all ye innumerable converging motions successively generated by ye impresses of gravity in every moment of it's passage from A to G: the motion AM takes the body to a place parallel to GN. The motion from A to M being in a parallel to GN inclines not ye body to verge from ye line GN. The other motion, takes the body at G to a point below G, and away from D: The innumerable & infinitly little motions (for I here consider motion according to ye method of indivisibles) continually generated by gravity in its passage from A to F incline it to verge from GN towards D, & ye like motions generated in its passage from F to G incline it to verge from GN towards C. But these motions are proportional to ye time they are generated in, & the time of passing from A to F (by reason ofye longer journey & slower motion) is greater then ye time of passing from F to G. And therefore ye motions generated in AF shall exceed those generated in FG & so make ye body verge from GN to some coast between N & D. The nearest approach therefore of ye body to ye center is not at G but somewhere between G & F as at O. And indeed the point O, according to ye various proportions of gravity to the impetus of ye body at A towards M, may fall any where in ye angle BCD in a certain curve wch touches ye line BC at C & passes thence to D. Thus I conceive it would be if gravity were ye same at all distances from ye center.

[16] In this paper many arguments already presented in [27] are improved and new arguments are added.

[17] This is not the actual trend in the history of physics. The preference is for narratives that cover a large period of time (perhaps too large), placing primary sources in general contexts. The result is that the meaning of the papers is not taken into account.

The sequence of letters establishes a chain of events leading to the mature conceptual structure shown in the *De Motu*:

- Hooke's letter on November 24 possibly brought Newton's interest to the investigation of the drawing of mechanic orbits in specific physical problems, and called attention to the centripetal "pressure". The similarity between Fig. 6 and the method (such as Fig. 3 in proposition 1) might have been a motivation. Newton started experimenting with Hooke's method.
- The spiral (November 28) is a puzzle. It is an exact, integrable solution for a central force of magnitude proportional to r^{-3} (which Newton proves in the *Principia*).
  A hypothesis is that Newton used some systematic method, such as Hooke's, but made a mistake. For instance, if the steps of integration in Hooke's method are large, in two or three steps the curve goes far inside the circle, as in Fig. 10 above, which may suggest a spiral, misleading Newton.
  Hooke's criticism led Newton to review his drawing, and he realized that he had to consider "motion according to ye method of indivisibles" in which a motion is composed in small steps by "innumerable & infinitly little motions", as he states in the letter with the reviewed curve (December 13).
- In the four days between December 9 and December 13, Newton made advances in mastering Hooke's method, as shown by Erlichson's analysis. Methods of numerical computation allow a more precise reproduction of the method in Fig. 10 and Fig. 3.

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