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Artigos Gerais

A measurement of g with a ring pendulum

(Medida de g com pêndulo anular)

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A classical problem in mechanics, in which a solid ring of negligible thickness may oscillate around two axes, is studied. Simple and accurate measurements of the periods of oscillation about two different axes provide both a convincing check of the main features of this problem-solving example and a measurement of g. A simple photogate system and "Creative Wave Studio", an improperly used software, allows accurate measurements of the periods of oscillation. The experiment is designed as an undergraduate student's laboratory activity, but it can be equally used and discussed as a classroom experiment.
Keywords: classical mechanics, measurement of g, ring pendulum.

Estudamos neste artigo um problema clássico da mecânica na qual um anel sólido, de espessura desprezível, pode oscilar em torno de dois eixos. Medidas simples e precisas dos períodos de oscilação em torno dos dois diferentes eixos fornecem não apenas uma comprovação convincente das características principais deste exemplo de problema solúvel como também propiciam uma medida de g. Um simples sistema de porta de luz junto ao "Creative Wave Studio", um software comumente usado de maneira imprópria, permitem que façamos medidas precisas dos períodos de oscilação. O experimento é feito pensando em uma atividade de laboratório para estudantes de graduação, mas pode ser igualmente usado e discutido em sala de aula.
Palavras-chave: mecânica clássica, medida de g, pêndulo anular.

1. Introduction

There is a problem in classical mechanics [1], involving a thin solid ring supported by a knife edge at a point, in which the ring may oscillate *in twofold way.* The solution is aimed to finding the two periods of oscillation, showing that the ratio of these periods is a constant. A determination of the acceleration due to gravity g follows, within the small oscillations approximation, from the analytic expression of the periods of oscillation. The problem has a pedagogical value because it involves several concepts as the so called "*parallel axis theorem*" and a *direct* evaluation of the moment of inertia as a useful Trigonometry exercise. In addition, if a simple photogate system is used through the PC sound card and an audio software as "Creative Wave Studio" is improperly used as an *accurate* times datalogger, a low cost experiment follows. By this experimental setup one can concretely show that small oscillations are not isochronous compared to large oscillations of the ring pendulum. Moreover, in the special case of small oscillations, a good measurement of g can be given. The experiment can be considered as a students' Lab activity or used and discussed as classroom experiment complementary to problem-solving practice. The time resolution of the software, which in our case is improperly used, provides a rather effective mean to show to students the difference in the periods when large and small oscillations are considered.

2. The analytical problem

In Fig. 1 the geometry of the problem is shown. The z axis is taken orthogonal to the paper sheet and oriented outward. In the first problem the point A is the knife edge suspension, in such a way that the ring may oscillate in the $z = 0$ plane about point A (namely, around an axis orthogonal to the paper sheet through A). In the second problem the ring is pivoted by an axis PP' lying in the paper sheet plane and executes oscillations in and out this plane. In practice, a knife edge suspension at A may be used in both cases using a little care in the second case, in order to avoid undesired rotational effects (ring oscillations around PP' axis alone).

Let solve the first problem. The moment of inertia I_{zO} around the z axis can be easily evaluated to be

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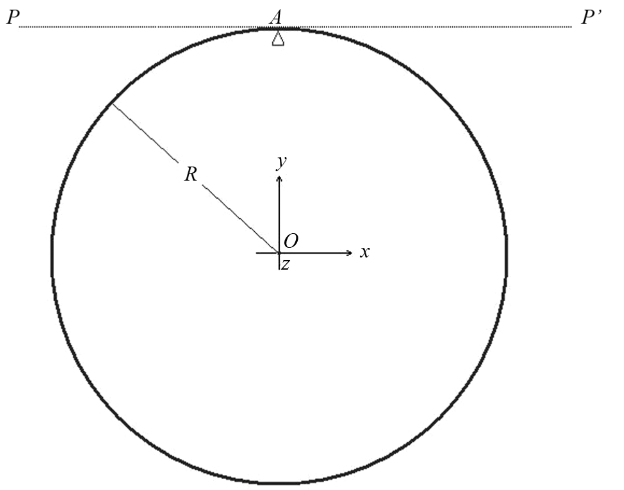

Figura 1 - The geometry of the oscillation problem. In the first problem the ring suffers a knife edge suspension at point A and oscillates in the $z = 0$ plane about point A (namely around an axis orthogonal to the paper sheet through A). In the second problem the ring (still suspended at A) oscillates around an axis PP' lying in the paper sheet, executing oscillations out the $x - y$ plane and orthogonally to it. In practice, a knife edge suspension at A may be used in both cases using a little care in the second case to avoid undesired rotational effects.

$$I_{zO} = \lambda R^3 \int_0^{2\pi} d\phi = MR^2, \quad (1)$$

being λ the linear mass density and $Rd\varphi$ the infinitesimal length element. By the *parallel-axis theorem* [2], the moment of inertia I_{zA} around an axis parallel to z through A is

$$I_{zA} = I_{zO} + MR^2 = 2MR^2. \quad (2)$$

Let us allow an angular displacement ϑ out of the equilibrium of the ring. The fundamental equation of a rigid body dynamics applied to this special case gives

$$I_{zA}\ddot{\vartheta} = -MgR\sin\vartheta \quad (3)$$

giving the linear equation

$$\ddot{\vartheta} + \frac{g}{2R}\vartheta = 0 \quad (4)$$

in the special case of small oscillations. Hence the period T_1 of small oscillations is given by

$$T_1 = 2\pi\sqrt{\frac{2R}{g}} \quad (5)$$

A student may observe that a direct calculus of the moment of inertia considering A *as a fixed polar axis* is easily obtained by taking into account the geometry in Fig. 2. In fact, with reference to the triangle in Fig. 2 one can write

$$r = 2R\cos\vartheta, \quad (6)$$

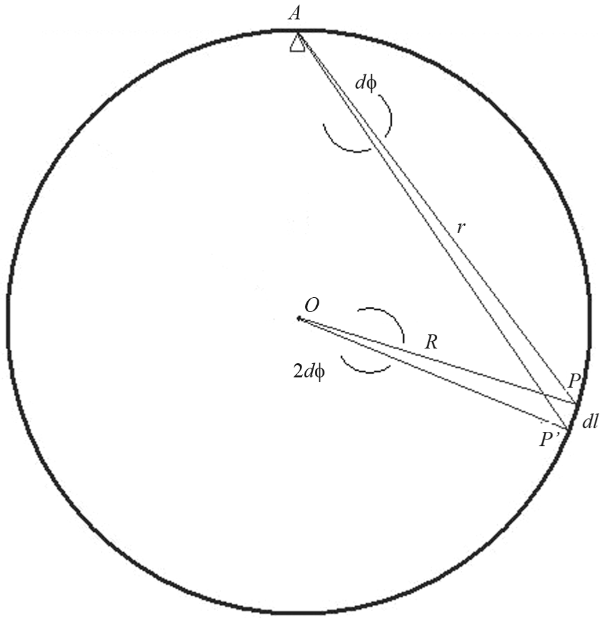

Figura 2 - The geometry for a direct calculating the moment of inertia around a pivot at A (see Eqs. (6) and (7)).

being r the distance from A to an infinitesimal element dl on the right half circumference. Since $dl = Rd(2\vartheta)$ (see geometric relation between $d\vartheta$ and $d(2\vartheta)$ in Fig. 2), it is easy to evaluate the moment of inertia (*with respect to a fixed polar axis A*) of the half circumference. The calculation gives

$$\frac{1}{2}I_{zA} = 2\lambda R^3 \int_o^{\pi/2} \cos^2\vartheta d\vartheta = MR^2. \quad (7)$$

Finally, by doubling this results for the entire circumference, Eq. (2) follows. Using the parallel-axis theorem it follows that the moment of inertia around PP' axis is

$$I_{PP'} = I_{xO} + MR^2, \quad (8)$$

being $I_{xO} = MR^2/2$ the moment of inertia referred to a rotation around the x axis as follows by direct evaluation. Hence the moment of inertia around the axis PP' is

$$I_{PP'} = \frac{3}{2}MR^2. \quad (9)$$

Considering again Eq. (3), the period T_2 of small oscillations around PP' axis becomes

$$T_2 = 2\pi\sqrt{\frac{3R}{2g}}. \quad (10)$$

Finally, the constant ratio of the two periods is

$$\frac{T_1}{T_2} = 2\frac{\sqrt{3}}{3}. \quad (11)$$

3. The experiment

The experiment can be performed using an iron ring (in our case 360 mm ± 1.3 mm outer diameter) and a cutter blade as knife edge suspension, in order to obtain oscillations with low damping. The experimental setup assembled for the special case of $z = 0$ plane oscillations (in-plane oscillations) is shown in Fig. 3. Oscillations related to the first problem do not require particular care in maintaining a stable plane of oscillation; only little care is required when the ring is released from a position out of equilibrium. A satisfying solution may be realized with a small electromagnet or, more simply, with a sewing tread holding: the ring can be, in this way, *gently* released from its equilibrium position. Oscillations starting from a big initial angular displacement (about 45°) show a stability of about half an hour; after this interval of time the amplitude of oscillations reduces to that of a small angle approximation (about 5°). A photogate system can be made with a reverse-biased photodiode and a laser pointer illuminating it.[2] At each light interruption event, the signal of *few millivolts* appearing across the photometric circuit resistance is fed into the "audio card" jack socket. A suitable audio software, as "Creative Wave Studio", will thus record a peak in correspondence to each light interruption event. In this way, we may collect a series of peaks, one for each half oscillation time interval. "Creative Wave Studio", improperly used in this way, may provide time measurements with uncertainties of few milliseconds.[3] Moreover, using a suitable zoom, it is possible to measure a set of 40-60 peaks taken in various time regions of the oscillation history (characterized as great and small amplitude regions), thus obtaining a set of periods of oscillations. From a set of 50 or more measurements, a significant value of the period T_1 and the standard deviation associated to this set of measurements may be found. In Fig. 3 the experimental setup for the first problem is shown. Recording of the oscillation periods relative to the first problem has been interrupted when the oscillation amplitude was under a few degrees.

The same cutter blade as knife edge suspension has also been used for the out-of-plane oscillation mode (*i.e.*, when ring displacement is aligned with the knife edge but the plane containing the ring is obviously unstable in time). A practical solution in stabilizing the oscillations around the PP' axis is given by a pair of linear magnets attached at a pole to the cutter blade and near the ring arc suspended by the knife edge as shown in Fig. 4. Because the ring is made of iron, the light attraction exerted by the linear magnets maintains a sufficient stability in the oscillation but introduces some additional damping. In this second case damping gives about 5 minutes of measurable small amplitude oscillations. Fig. 4 shows the way two "Geomag" magnets are placed on a large cutter blade.

Figura 3 - The experimental setup assembled on a workbench. A laser pointer coupled with photodiode system on the lab jack is shown. In the in-plane oscillating pattern the beam of the laser pointer impinges on the ring when it passes through it equilibrium position at a distance equal to the ring's radius from the center on the x-axis.

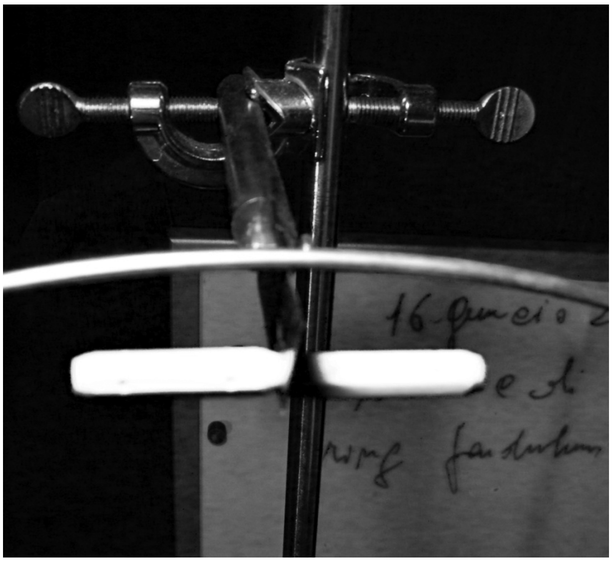

Figura 4 - The detail of two "Geomag" magnets assembled at the knife edge (a large cutter blade) to stabilize out-of-plane oscillations in a direction orthogonal to the xy plane of the Fig. 1. The magnets exerts a little attraction on the ring near the knife edge. This attraction avoids undesired deviation in time of the axis PP' in Fig. 1, but introduces additional damping in the system.

[2] A *BX 65* Silicon photodiode in series with a resistance R of about 20 k. A 9 V battery is connected to a reversed biased photodiode. A low power 670 nm Laser pointer is assembled in front of the photodiode. The signal pick up from the resistor R is fed into the PC's MIC socket.

[3] *Creative Wave Studio* is one of the various software furnished with any *Creative Sound Blaster* package. It is freely downloadable at `http://files2.europe.creative.con/Applications/AVP/10609/0x52EA7DAF/WAWESTD_PCAPP_LB_7_10_24.exe`. If the "sound" (any variable electrical signal under 2 V) is recorded in a time interval Δt, the total uncertainty is $dt = \pm 2$ ms.

4. Check between theory and practice

For the special case of the first problem oscillation starts from an angle of about 45° and the laser pointer beam is periodically interrupted by the oscillating ring. The signal across the resistance in series with the photodiode is fed into the audio card with the software "Creative Wave Studio" running. In a typical experimental run, large oscillations survive for about a half-hour period. By selecting a convenient zoom factor on the software and by recording about 40-60 complete "electric peaks" relative to complete oscillations, a measurement of the period of the oscillation T_1 is possible with a maximum uncertainty of 2 ms per group. A set of 50 groups of "electric peaks" can be selected in the domain of the oscillations having great amplitude (around 40°) and 50 groups of "electric peaks" can be selected in the domain of small amplitude (under 5°). In a typical measure a significant difference in the period T_1 is detected. The mean value of T_1 found for great oscillations is $T_1 = 1.231$ s $\pm$ 0.004 s, where the uncertainty is assumed to be the standard deviation on the series of 50 measurements. Analogously, the mean value of T_1 found for small oscillations is $T_1 = 1.206$ s $\pm$ 0.005 s, where the uncertainty is again the standard deviation on the series of 50 measurements in the domain of small oscillations. So, the so called "small oscillation" approximation has a concrete checking in the measurements done.

If a measurement is made for small oscillations (under 5°) around the PP' axis by still selecting groups of 50 groups of "electrical peaks", the period T_2 found in a typical measurement is T_2= 1.037 s $\pm$ 0.005 s, where uncertainty is still the standard deviation on the series of the 50 measurements considered. The experimental ratio is thus $T_1/T_2 = 1.16 \pm 0.01$ and agrees with the theoretical value given by Eq. (11) within the uncertainty width.

The value of the acceleration due to gravity g following from Eq. (5) is found to be

$$g \approx 9.8 \text{ m/s}^2 \pm 0.1 \text{ m/s}^2. \tag{12}$$

On the other hand, the value of g following from Eq. (10) is found to be

$$g \approx 9.9 \text{ m/s}^2 \pm 0.1 \text{ m/s}^2. \tag{13}$$

In both cases the relative uncertainty is calculated as follows

$$\frac{dg}{g} = \frac{dR}{R} + 2\frac{dT}{T}, \tag{14}$$

by taking $dR = 1.3$ mm and $dT = 0.005$ s.

5. Correction due to the ring thickness

In Section 2 we have considered a very thin ring. In order to see how the measurement of g are affected by the mass distribution of the torus, we may use the following expressions of the moments of inertia as calculated for the in-plane (I_{zO}) and the out-of-plane (I_{xO}) oscillations

$$I_{zO} = Mc^2\left(1 + \frac{3}{4}\lambda^2\right), \tag{15a}$$

$$I_{xO} = \frac{Mc^2}{8}\left(4 + 5\lambda^2\right), \tag{15b}$$

where $c = R + a$, R and a being the inner radius and the cross section radius of the torus, respectively, and where $\lambda = \frac{c}{a}$. By adding $MR^2 = Mc^2\left(1-\lambda\right)^2$ to both terms, as prescribed by the parallel axis theorem, we have

$$I_{zA} = Mc^2\left[\left(1 + \frac{3}{4}\lambda^2\right) + \left(1-\lambda\right)^2\right], \tag{16a}$$

$$I_{xA} = Mc^2\left[\left(\frac{1}{2} + \frac{5}{8}\lambda^2\right) + \left(1-\lambda\right)^2\right]. \tag{16b}$$

To first order in λ, we can therefore write

$$I_{zA} = 2Mc^2\left(1-\lambda\right), \tag{17a}$$

$$I_{xA} = Mc^2\left(\frac{3}{2} - 2\lambda\right). \tag{17b}$$

Recalling now Eq. (3), for small oscillations we have

$$T_1 = 2\pi\sqrt{\frac{I_{zA}}{gR}} = 2\pi\sqrt{\frac{2c}{g}}, \tag{18a}$$

$$T_2 = 2\pi\sqrt{\frac{I_{xA}}{gR}} = 2\pi\sqrt{\frac{\left(3-4\lambda\right)c}{2\left(1-\lambda\right)g}}. \tag{18b}$$

In this way, by calculating g from the above expressions, we obtain

$$g = 9.7m/s^2 \pm 0.2m/s^2; \tag{19a}$$

$$g = 9.8m/s^2 \pm 0.2m/s^2, \tag{19b}$$

where the uncertainty has been calculated by considering the sum of the relative uncertainties as follows

$$\frac{dg}{g} = \frac{dc}{c} + 2\frac{dT}{T} = \frac{dR}{R} + \frac{da}{a} + 2\frac{\Delta T}{T}, \tag{20}$$

with $c = R - a = 178.7$ mm, $dT = 0.005$ s, $dR = 1.0$ mm and $da = 0.05$ mm.

6. Conclusions

We have performed an experiment on the period of oscillations of a ring pendulum. Both the in-plane period T_1 and the out-of-plane period T_2 were measured by acquiring data through an improper use of "Creative Wave Studio" software. This data acquisition system is coupled to a photometric resistance across which a voltage signal of few millivolts appears each time an opportunely positioned photogate system is obscured by the oscillating ring. Measurement of the periods T_1 and T_2 in the limit of small oscillations provide a mean to measure the acceleration due to gravity g. Moreover, by analyzing the problem from an analytic point of view, we may derive, for a very thin ring, the expression for the ratio of the two periods, which is in good agreement with experimental findings. The value of g is found within the uncertainty width estimated for it.

Correction due to the ring thickness has also been considered. When this problem is tackled, one finds that the relative uncertainty is greater than the one found for the "ideal" problem of a very thin ring. This last step adds some pedagogical value to the experiment. In fact, the student may concretely see how, by the refinement of the analysis of the problem, an increase of the uncertainty due to correction factors is introduced.

Referências

[1] A.M. Alpern, *3000 Solved Problems in Physics* (McGraw-Hill, New York, 1988), p. 236-237.

[2] D. Halliday and R. Resnick, *Fundametals of Physics* (John Wiley & Sons, New York, 1981), § 12.5.