Revista Brasileira de Ensino de Física, vol. 46, e20240069 (2024)
www.scielo.br/rbef
DOI: https://doi.org/10.1590/1806-9126-RBEF-2024-0069

Articles

Deformed exponential function for complex argument: writing roots of polynomial equations with complex coefficients

Função exponencial deformada para argumento complexo: escrevendo raízes de equações polinomiais com coeficientes complexos

Paulo S. Adami*[1,2], Olavo H. Menin[1], Alexandre S. Martinez[2,3]

[1]Instituto Federal de Educação, Ciência e Tecnologia de São Paulo, Sertãozinho, SP, Brasil.
[2]Universidade de São Paulo, Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto, Ribeirão Preto, SP, Brasil.
[3]Instituto Nacional de Ciência e Tecnologia de Sistemas Complexos, Maringá, PR, Brasil.

Received on February 27, 2024. Accepted on April 20, 2024.

The generalization of the logarithm and exponential functions proposed by Tsallis in 1988 has brought contributions in several areas in the last decades. Recently, we introduced a deformed exponential function that may assume negative or even complex values, but with the constrain of admitting only real arguments. We also showed the possibility of writing the roots of polynomial equations with real coefficients by using our generalization. Here, we present an extension of our deformed exponential function that admits complex argument. We show that our new generalization can address all the roots of polynomial equations up to the third degree even with complex coefficients.
Keywords: Complex systems, Generalized functions, Family of functions, Analyticity in the complex plane, Polynomial roots.

A generalização das funções exponencial e logarítmica proposta por Tsallis em 1988 tem trazido contribuições em várias áreas nas últimas décadas. Recentemente, introduzimos uma função exponencial deformada que pode assumir valores negativos ou mesmo complexos, mas com a restrição de admitir apenas argumentos reais. Também demonstramos a possibilidade de escrever as raízes de equações polinomiais com coeficientes reais usando nossa generalização. Aqui, apresentamos uma extensão de nossa função exponencial deformada que admite argumento complexo. Mostramos que nossa nova generalização pode abordar todas as raízes de equações polinomiais até o terceiro grau, mesmo com coeficientes complexos.
Palavras-chave: Sistemas complexos, Funções generalizadas, Famílias de funções, Analiticidade no plano complexo, Raizes de polinômios.

1. Introduction

Generalization of mathematical functions creates families of functions capable of capturing characteristics of systems that would not be possible by using the original ones. In 1988, for example, Tsallis introduce an interesting generalization of the entropy within the framework of nonadditive thermodynamics [1]. The aim was to broaden the applications of Boltzmann-Gibbs statistical mechanics to complex systems. Few years later, he also presented the generalized logarithmic function and its inverse, the generalized exponential function [2]. The applicability of these generalized functions has been recognized in different areas, such as population dynamics [3, 4], econophysics [5], and optimization and inverse problems [6, 7].

The generalization of the exponential function proposed by Tsallis [2] maintains the most prominent properties of the standard exponential function: it is real, it has nonnegative values and it increases monotonically. Nevertheless, unlike the traditional exponential function, it may diverge for finite values of the argument, which can be efficiently used to describe population dynamics models [8].

Recently, we expanded this generalization to the real domain region where it would vanish, allowing the function to assume negative or even complex values and to present loss of monotonicity [9, 10]. Although one of the main properties of the exponential function is missed, one can write this generalization as a deformed exponential function with real arguments, which is analytical in the complex plane. Also, we can write the roots of algebraic equations with real coefficients using this function with real arguments.

Notwithstanding, not all polynomial equations can have their roots written with our deformed exponential. For example, to write the roots of cubic equations with three real roots, our formulation would have to

*Correspondence email address: adami.paulo@ifsp.edu.br

admit complex arguments, which is not possible with the current proposal. Furthermore, algebraic equations with complex coefficients cannot also be solved. In this context, we present here an expansion of our deformed exponential function that makes it capable of admitting complex arguments. We also show its potential applicability in writing the roots of polynomial equations with complex coefficients.

Our paper is organized as follows. In Sec. 2 we review previous generalizations of the exponential function and introduce our new deformed exponential function for complex arguments. In Sec. 3 we discuss the application of our new proposal in solving polynomial equations with complex coefficients up to third degree. In Sec. 4 we present our final remarks and point out avenues of further investigations. Finally, in the Appendix we present codes in Python for solving the cubic equations discussed in this study by applying the deformed exponential function for complex arguments.

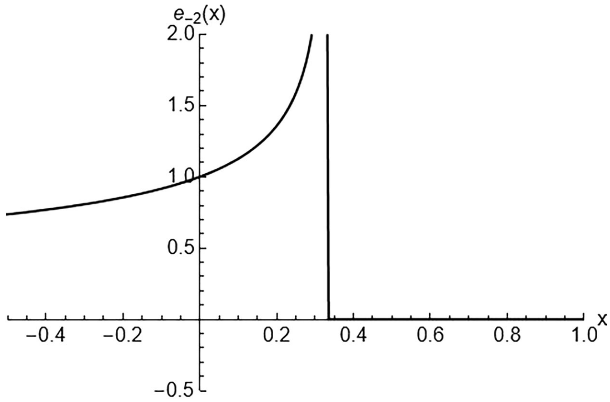

Figure 1: Graphical representation of the generalized exponential function $e_\lambda(x)$ for $\lambda = -3$. In the interval $x < 1/3$, the function has properties similar to those of the usual exponential function, such as being equal to 1 for $x = 0$, being positive and strictly increasing, but unlike the usual function, $\lim_{x\to 1/3-} e_\lambda(x) = +\infty$, that is, it diverges at a finite value of the domain. For $x > 1/3$, the function $e_\lambda(x)$ takes null values.

2. Deformed Exponential Function for Complex Argument

In this section, we initially present the definitions and discuss some characteristics and properties of the Tsallis's generalized exponential function, $e_\lambda(x)$, and our deformed exponential function for real argument, $\tilde{e}_\lambda(x)$. Next, we present our new deformed exponential for complex argument, $\tilde{E}_\lambda(z)$.

2.1. Generalized exponential function $e_\lambda(x)$ and deformed exponential function for real argument $\tilde{e}_\lambda(x)$

The generalization of the exponential function proposed by Tsallis [2] is

$$e_\lambda(x) = \lim_{\lambda'\to\lambda} (1+\lambda' x)^{1/\lambda'}\, \theta(1+\lambda' x), \qquad (1)$$

with $\theta(1+\lambda' x)$ being the Heaviside function. This function is very well defined for real arguments and vanishes for $1+\lambda x < 0$. This is a true generalization of the exponential function, since it maintains the most prominent properties of the standard exponential function: it is real, it has nonnegative values, and it is monotonically increasing. Unlike the usual exponential function, the generalized function $e_\lambda(x)$ diverges at a finite value of the domain, as shown in Fig. (1). This feature can be useful in several applications [11], it can be thought of an order parameter[4], such as the dynamical population model [8].

The vanishing of the Tsallis's generalized exponential in some regions of the domain also presents drawbacks, such as limitations in its application in integral transforms [9, 12]. This characteristic also brings issues in its invertibility, which is an important property of the usual exponential function. One way to overcome these limitations is to assign values to the domain region where the Tsallis's exponential vanishes. To address these issues, we have proposed a deformed version of the Tsallis's exponential function, given by [9, 10]

$$\tilde{e}_\lambda(x) = \lim_{\lambda'\to\lambda} |1+\lambda' x|^{1/\lambda'} e^{i\pi/\lambda'[1-\theta(1+\lambda' x)]}, \qquad (2)$$

which we rather not to call it as "exponential" since it loses many of the properties of the standard exponential function. For instance, assigning values to $1+\lambda x < 0$, the function may have negative or complex values, lose monotonicity, become null or present inflection points, as shown in Fig. (2).

Although these contrasts with the standard exponential function may seem like a handicap, there are silver linings in considering this deformed function. It may be mapped to a standard exponential with a complex argument, proving that it is analytical in the complex plane, since $\tilde{e}_\lambda(x) = \lim_{\lambda'\to\lambda} e^{\ln|1+\lambda' x|^{1/\lambda'} + \imath\frac{\pi}{\lambda'}[1-\theta(1+\lambda' x)]} = e^z$. Besides having an inverse in the whole domain, it can be used to write the roots of polynomial equations with real coefficients [10].

The benefit of the formulation provided by Eq. (2) is also observable in the generalization of other structures such as integral transforms and trigonometric, hyperbolic, and log-periodic functions [9]. However, despite the consistency and applicability of this deformed function, it does not admit complex arguments.

2.2. Deformed exponential function for complex argument $\tilde{E}_\lambda(z)$

We now introduce a new deformed exponential function that admits complex arguments, $\tilde{E}_\lambda(x+yi)$. Using the generalized product $a \otimes_\lambda b = (a^\lambda + b^\lambda - 1)^{1/\lambda}$,

 DOI: https://doi.org/10.1590/1806-9126-RBEF-2024-0069

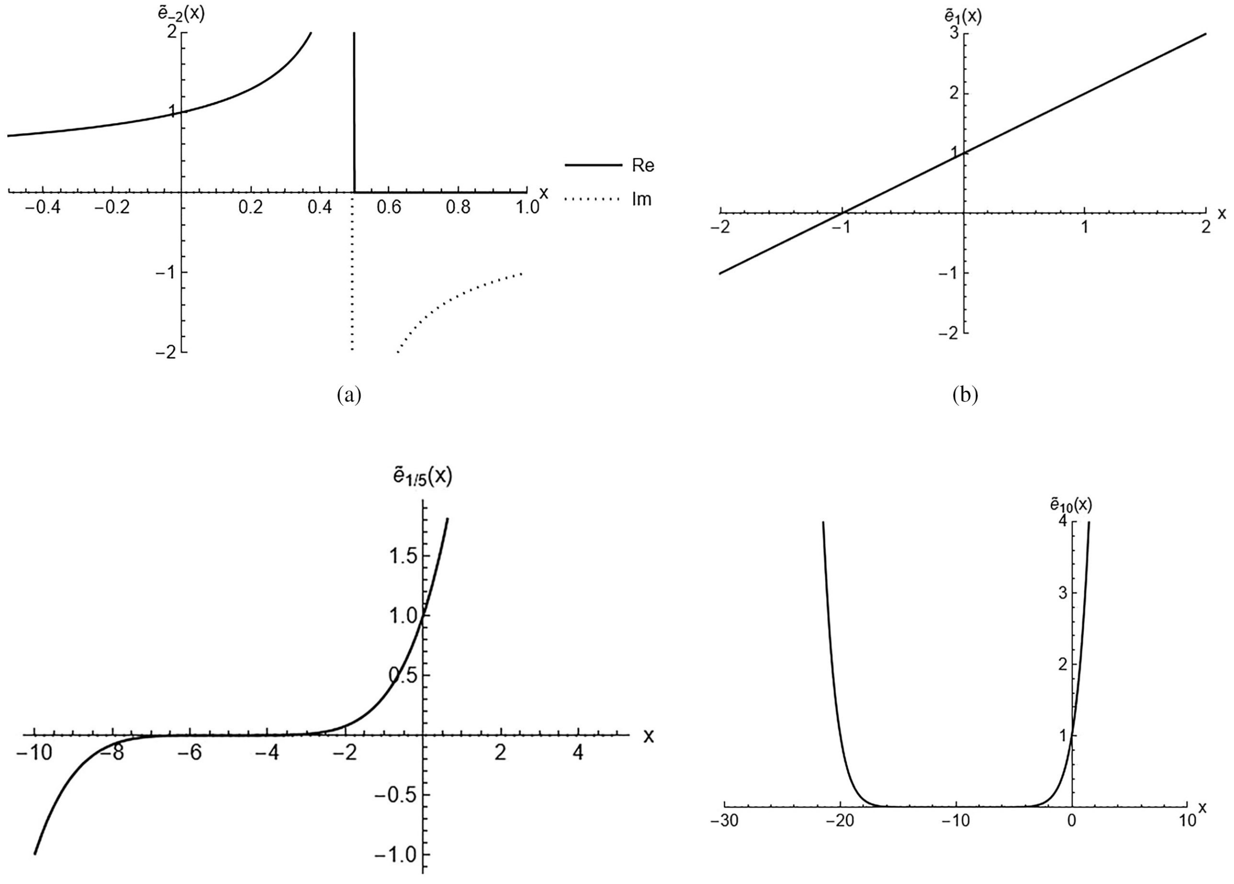

Figure 2: Graphical representations for the deformed exponential function $\tilde{e}_\lambda(x)$, where the real part (Re) is represented by the solid line and the imaginary part (Im) by the dashed line. In (a), for $\lambda = -2$, unlike the Tsallis function, the deformed exponential admits nonzero values in the interval of $x > 1/2$, assuming complex values. For values of $x < 1/2$ the function only admits real values, behaves like the Tsallis function. It can be seen that the function diverges at a finite value of the domain, that is, $\lim_{x\to 1/2-} \tilde{e}_{-2}(x) = +\infty$. In (b), for $\lambda = 1$, the function behaves like a linear function, assuming negative values for $x < -1$. For $\lambda = 1/5$, Fig. (c) highlights the existence of an inflection point at $x = -5$ and the fact that $\lim_{x\to-\infty} \tilde{e}_{1/5}(x) = -\infty$. Such characteristics are not observed in the usual exponential function. In (d), for $\lambda = 10$ the loss of monotonicity can be observed.

proposed by Nivanen et al. [13] and Borges [14], and considering $e^{x+yi} = e^x e^{yi} \to \tilde{E}_\lambda(x+yi) = \tilde{e}_\lambda(x) \otimes_\lambda \lim_{\lambda'\to\lambda} \left[|1+\lambda' yi| e^{i\arctan(\lambda' y)}\right]^{1/\lambda'}$, we have

$$\begin{aligned}\tilde{E}_\lambda(x+yi) &= \lim_{\lambda'\to\lambda}\left[|1+\lambda' x| e^{i\pi[1-\theta(1+\lambda' x)]} + 1 + \lambda' yi - 1\right]^{1/\lambda'} \\ &= \lim_{\lambda'\to\lambda}\left[|1+\lambda' x| e^{i\pi[1-\theta(1+\lambda' x)]} + \lambda' yi\right]^{1/\lambda'}, \end{aligned} \tag{3}$$

which assure that $\tilde{E}_0(z) = \tilde{E}_0(x+yi) = e^{x+yi} = e^z$, $\forall\, z \in \mathbb{C}$, and $\tilde{E}_\lambda(0) = 1$, for $\forall\, \lambda \in \mathbb{R}$.

This new proposal preserves the characteristics of the previous generalization [9] for $y = 0$. It can also be noted that $\tilde{E}_\lambda(0) = 1$, $\forall\, \lambda \in \mathbb{R}$, and $\tilde{E}_0(yi) = e^{yi}$, for $\lambda = 0$. Since the function $\tilde{E}_\lambda(z)$ admits complex arguments, it may be applied to write the roots of polynomial equations with complex coefficients, as we show in the next section.

3. Solving Polynomial Equations with the New Deformed Exponential Function for Complex Arguments

In this section, we apply our new deformed exponential function, given by Eq. (3), to write the roots of complex coefficient polynomial equations. With our new formulation, it becomes possible to find complex roots.

3.1. First-degree polynomial equations

Consider a polynomial equation of the first degree in the variable x, $ax + b = 0$, with $a, b \in \mathbb{C}$ and $a \neq 0$. Its root is given by $\tilde{x} = -b/a$. Rewriting $\tilde{x}$ conveniently and considering $\gamma = -b/a - 1$, we have

$$\tilde{x} = 1 + \left(-\frac{b}{a} - 1\right) = 1 + \gamma = \tilde{E}_1(\gamma). \tag{4}$$

Thus, the solution of a first-degree polynomial equation with complex coefficients can be written from the deformed exponential function for complex argument with $\lambda = 1$.

In fact, since the complex $z_1 = -b/a$, we have $\tilde{x} = \mathrm{Re}(z_1) + \mathrm{Im}(z_1)i$. As $\gamma = -b/a - 1 = z_1 - 1$, we can verify this same solution by applying the deformed exponential function for the complex argument given by Eq. (3). So,

$$\begin{aligned}\tilde{E}_1(\gamma) &= |1 + [\mathrm{Re}(z_1) - 1]| e^{i\pi\{1-\theta[1+\mathrm{Re}(z_1)-1]\}} + \mathrm{Im}(z_1)i \\ &= |\mathrm{Re}(z_1)| e^{i\pi\{1-\theta[\mathrm{Re}(z_1)]\}} + \mathrm{Im}(z_1)i \\ &= \mathrm{Re}(z_1) + \mathrm{Im}(z_1)i = \tilde{x}.\end{aligned} \tag{5}$$

3.2. Second-degree polynomial equations

For a quadratic equation, $ax^2 + bx + c = 0$, with $a, b \neq 0$ and $a, b, c \in \mathbb{C}$, its roots are given by $\tilde{x}_\pm = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$. Rewriting $\tilde{x}_\pm$ in a convenient way and setting $\alpha = -b/2a$ and $\beta = -2ac/b^2$, we have

$$\begin{aligned}\tilde{x}_\pm &= -\frac{b}{2a}\left(1 \pm \sqrt{1 - \frac{4ac}{b^2}}\right) \\ &= \alpha\left(1 \pm \sqrt{1 + 2\beta}\right) = \alpha\left[1 \pm (1 + 2\beta)^{1/2}\right] \\ &= \alpha[1 + \mathrm{Re}(z_\pm) + \mathrm{Im}(z_\pm)i],\end{aligned} \tag{6}$$

where $z_\pm$ are the square roots of $(1 + 2\beta)$. Applying the definition of deformed exponential function to complex argument, we also have

$$\begin{aligned}\tilde{x}_\pm &= \alpha\left[1 \pm (1 + 2\beta)^{1/2}\right] = \alpha\left[1 \pm \tilde{E}_2(\beta)\right] \\ &= \alpha\left\{\tilde{E}_1[\pm\tilde{E}_2(\beta)]\right\}.\end{aligned} \tag{7}$$

Note that β and $\pm\tilde{E}_2(\beta)$ may assume imaginary values. It would not be possible to completely find the roots of the equation if the generalized exponential function did not admit complex arguments. In this way, the deformed exponential function for complex arguments expands the possibilities for writing the roots of polynomial equations.

Similarly to what we did in the study of first-degree polynomial equations, we can writte the roots $\tilde{x}_\pm$ of the quadratic equation, given by Eq. (6), by applying the definition of the deformed exponential function given by Eq. (3). So, we have

$$\begin{aligned}\tilde{E}_2(\beta) &= [1 + 2\,\mathrm{Re}(\beta) + 2\,\mathrm{Im}(\beta)i]^{1/2} \\ &= (1 + 2\beta)^{1/2} = z_\pm.\end{aligned} \tag{8}$$

It can also be verified that

$$\tilde{E}_1[\tilde{E}_2(\beta)] = 1 + \mathrm{Re}(z_\pm) + \mathrm{Im}(z_\pm)i. \tag{9}$$

From there, we can conclude that

$$\alpha\{\tilde{E}_1[\tilde{E}_2(\beta)]\} = \alpha[1 + \mathrm{Re}(z_\pm) + \mathrm{Im}(z_\pm)i] = \tilde{x}_\pm. \tag{10}$$

To validate our analytic approach, we provide a Python code for solving polynomial equations in the Appendix.

3.3. Third-degree polynomial equations

The method presented by Adami et al. [10] guarantees at least one correct root for third-degree polynomial equations. However, with our new approach we can write all roots, including the case where the polynomial equation coefficients are complex or when all three roots are real, which implies necessarily using complex numbers as arguments for the generalized exponential function. These cases are only possible by applying the new proposal to generalize the exponential function for complex arguments.

Consider a cubic equation, $ax^3 + bx^2 + cx + d = 0$, $a \neq 0$. Setting $\gamma = b/3a$, $p = c/a - 3\gamma^2$, $q = 2\gamma^3 - \gamma c/a + d/a$ and $\phi = 2p^3/(27q^2)$, we note that the roots $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ of the cubic equation can be found by applying the function $\tilde{E}_\lambda(z)$ for $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$, from the expressions

$$\Delta_\pm = \tilde{E}_3\left\{\frac{-\frac{q}{2}\tilde{E}_1\left[\pm\tilde{E}_2(\phi)\right] - 1}{3}\right\}, \tag{11}$$

and finally, the roots are given by

$$\begin{aligned}\tilde{x}_1 &= -\gamma + \Delta_+ + \Delta_-, \\ \tilde{x}_2 &= -\gamma + \chi_+\Delta_+ + \chi_-\Delta_-, \\ \tilde{x}_3 &= -\gamma + \chi_-\Delta_+ + \chi_+\Delta_-.\end{aligned} \tag{12}$$

Additional information about determining the roots of the equation can be found in Refs. [10, 15].

When we calculate the cube root of a complex number with the help of a computer algebra system, the provided root is one of the three possible ones. Therefore, we need to adjust, in the code provided in the Appendix, the ways to obtain the results given by Eq. (12).

4. Conclusion

We present a new deformed exponential function that admits complex arguments, $\tilde{E}_\lambda(z)$, given by Eq. 3. With this new generalization with complex arguments it is possible to write the roots of polynomial equations that were not possible with the previous generalizations with real arguments. We show the solutions of polynomial equations from the first to the third degree, applying our generalization proposal, and we note their efficiency in this sense.

Our next step is to investigate the implications of this new deformed exponential function for complex arguments in the generalization of other structures, such as trigonometric functions. Thus, we hope to analyze the log-periodic behavior of these generalized trigonometric functions and associate them with new possible generalizations of integral transforms and Fourier series.

Acknowledgments

A.S.M. acknowledges Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) (Grant No. 0304972/2022-3). P.S.A. Fundação Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) CAPES-PRINT – 88887.717368/2022-00.

Supplementary Material

The following online material is available for this article:

Appendix: Solving polynomial equations of the first degree.

References

[1] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).

[2] C. Tsallis, Química Nova **17**, 468 (1994).

[3] A.S. Martinez, R.S. González and A.L. Espíndola, Physica A: Statistical Mechanics and its Applications **388**, 2922 (2009).

[4] B.C.T. Cabella, A.S. Martinez and F. Ribeiro, Phys. Rev. E **83**, 061902 (2011).

[5] N. Destefano and A.S. Martinez, Physica A: Statistical Mechanics and its Applications **390**, 1763 (2011).

[6] O.H. Menin, A.S. Martinez and A.M. Costa, Applied Radiation and Isotopes **111**, 80 (2016).

[7] O.H. Menin and C.T. Bauch, Physica A **490**, 1513 (2018).

[8] H. Von Foerster, P.M. Mora and L.W. Amiot, Science **132**, 1291 (1960).

[9] A.H. Martini, *Analiticidade da função exponencial generalizada para argumentos complexos e suas implicações*. Doctoral Thesis, Universidade de São Paulo, São Paulo (2020).

[10] P.S. Adami, S.M. Mendonça, O.H. Menin, A.H. Martini and A.S. Martinez, Revista Brasileira de Ensino de Física **45**, e20230095 (2023).

[11] R.F. Vigelis, L.H.F. De Andrade and C.C. Cavalcante, IEEE Transactions on Information Theory **66**, 2891 (2019).

[12] E.K. Lenzi, E.P. Borges and R.S. Mendes, Journal of Physics A: Mathematical and General **32**, 8551 (1999).

[13] L. Nivanen, A. Le Méhauté and Q.A. Wang, Reports on Mathematical Physics **52**, 437 (2003).

[14] E.P. Borges, Physica A **340**, 95 (2004).

[15] E.L. Lima, *A Matemática do Ensino Médio: Volume 1* (Sociedade Brasileira de Matemática, Rio de Janeiro, 2016).