# Numerical evaluation of the escape time of a classical point particle from an annular billiard

*(Estimação numérica do tempo de escape de uma partícula puntual clássica em um bilhar anular)*

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A two-dimensional annular billiard consisting of a region confined within two concentric circumferences, the outer of radius $R$ and the inner of radius $r$, is considered. The escape time of a point particle projected at a given angle within this billiard is numerically evaluated in terms of the size of the opening in the billiard. The problem is solved by means of classical mechanics and can be of interest for advanced high school physics students or undergraduate college physics students.
**Keywords:** classical mechanics, chaos, computational physics.

Consideramos neste trabalho um bilhar bidimensional que consiste em uma região circunscrita por duas circunferências concêntricas de raio externo $R$ e interno $r$, respectivamente. O tempo de escape de uma partícula projetada por um certo ângulo dentro deste sistema é calculada numericamente em termos do tamanho da abertura no bilhar. O problema é resolvido usando mecânica clássica em um nível que pode ser apresentado a estudantes do ensino médio e graduandos em física.
**Palavras-chave:** mecânica clássica, caos, física computacional.

## 1. Defining the problem

The study of reflections of point particles within a billiard of a given shape with perfectly reflecting walls is a well known topic in the literature [1-3]. Here we consider a billiard whose walls are two reflecting concentric circumferences; *i.e.*, an annular billiard. Our interest in this particular shape is given by the possibility of tracing a parallel between the escape time of the point particle moving at a constant speed $V$ within the billiard and the analogous problem of a light ray "trapped" within systems of similar shape [4].

The main question we would like to address in the present work is the following. In case we project an incoming point particle from an aperture in the billiard a given angle $\theta$ with respect to the horizontal, after what interval of time will the same particle emerge out of the billiard, assuming that the aperture has a finite size? This question is well posed, since, as we shall see, one could consider two cases. In the first case, for angles $\theta$ greater than some critical value $\theta_c$, the system behaves as if the inner circumference were not present, since the particle never hits its wall. The problem becomes thus similar to the well-known topic of circular billiards. In the second case, on the other hand, when

collisions with the inner wall are allowed ($0 < \theta < \theta_c$), either a periodic or non-periodic trajectory is followed by the point particle in case the aperture is point-like. Of course, in case the aperture has finite dimensions, there is a finite escape time and the trajectories in the configuration space cannot in general be classified in the same way.

In the case the point particle goes through a closed trajectory, it comes to the starting point after a given number of reflections, and the escape time is easily calculated even in the presence of a point-like opening. In the other case, when a non-periodic trajectory is followed, the particle will eventually come close enough to the starting point in such a way that it will escape from the area enclosed within the concentric circumferences if the intake has finite dimensions. In the latter case, a computer algorithm can allow numerical solution of the problem.

## 2. Some graphical and analytic notes

Let us start by defining the radius of the inner and outer circumferences as $r$ and $R$, respectively. Let also assume that the point particle enters the outer circum-

ference of center $O$ at $P_0$, as shown in Fig. 1, at an angle $\theta$ with respect to the horizontal. If we were just to consider the particle's trajectory inside a single circumference, as shown in Fig. 1, then the path followed is obtained in the following simple way: First draw the segment $P_0P_1$; then draw segments $P_{n-1}P_n$ all tangent to the circumference of radius $d$ equal to the distance between the point $O$ and the segment $P_0P_1$. This procedure gives the starred pattern in Fig. 1.
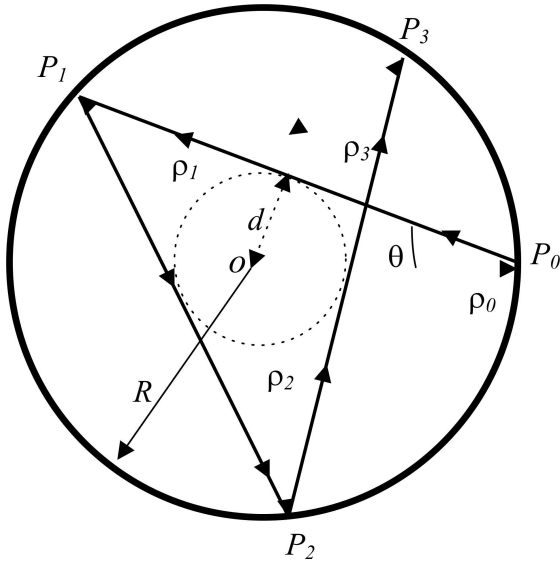


Figura 1 - Trajectory of a point particle bouncing elastically within a circumference of radius $R$ and center $O$.

We would now like to discuss the above construction from an analytic point of view. A representation of segment $P_0P_1$ can be given in terms of the difference of two complex numbers, $\rho_0$ and $\rho_1$ (see Fig. 1), in such a way that

$$P_0P_1 = \rho_1 - \rho_0 = R\,e^{i(\phi_1-\phi_0)} = R\,e^{i\varphi}, \qquad (1)$$

where $\phi_0$ and $\phi_1$ are the angles the vectors $\rho_0$ e $\rho_1$, represented on the complex plane, make with the horizontal. If we now set $\varphi = \phi_1 - \phi_0$, we can write

$$P_kP_{k-1} = \rho_k - \rho_{k-1} = R\,e^{i(\phi_k-\phi_{k-1})} = R\,e^{ik\varphi}, \quad (2)$$

where $k$ is an integer. The segment $P_kP_{k-1}$ will thus coincide with $P_0P_1$ if the following condition is satisfied after $k$ iterations of the procedure described above

$$e^{ik\varphi} = e^{i\varphi} \Rightarrow (k-1)\,\varphi = 2s\pi, \qquad (3)$$

$s$ being an integer. Eq. (3) can be written as follows

$$m\varphi = 2\,(n-2)\,\pi \qquad (4)$$

by setting $k - 1 = m$ and $s = n - 2$, with $n > 2$ and $m > 1$ integers. Notice that for $m = 2$ and $n = 3$ we have $\varphi = \pi$, so that the particle enters at $\phi_0 = 0$,

bounces once at $\phi_1 = \pi$, and then leaves the circumference at $\phi_2 = 2\pi$. In this case, the quantity $t_E = \frac{4R}{V}$, $V$ being the particle's speed, is the minimum escape time from a circumference of radius $R$. In the case $n = 3$ and $m = 3, 4, 5, \ldots$ we obtain regular polygons of perimeter $p_m$ with 3, 4, 5 $\ldots$ sides, respectively, all inscribed in the circumference of radius $R$. In this cases, the escape time is $t_E = \frac{p_m}{V}$.

In the case of non-periodic orbits, on the other hand, condition (3) is not satisfied and the point particle is seen to wander within the annular region of outer radius $R$ and inner radius $d$, never coming back to the starting point $P_0$.

## 3.    Escape time

The purpose of the present work is to find the escape time for a point particle confined within two concentric circumferences, whose trajectory is schematically shown in Fig. 2. The complex representation of the trajectory, as done in the case of a single circumference, is useful also in this case. Let us then start by considering a particle entering the region of interest for our analysis from $P_0$ as shown in Fig. 2. The partial trajectory $P_0Q_0P_1$ is a broken line, whose characteristics are shown in Fig. 2, if the angle $\theta$ ($0 \leq \theta < \frac{\pi}{2}$) at which the particle enters the annular region is such that $\theta < \tan^{-1}\left(\frac{r}{\sqrt{R^2-r^2}}\right) = \theta_c$. In the case $\tan^{-1}\left(\frac{r}{\sqrt{R^2-r^2}}\right) \leq \theta < \frac{\pi}{2}$, the particle moves as if the inner circumference were not present. Notice that the critical angle $\alpha_c$ is complementary to $\theta_c$, so that $\alpha_c = \tan^{-1}\left(\frac{\sqrt{R^2-r^2}}{r}\right)$. The relation between the angles $\alpha$ and $\theta$ can be found by trigonometry to be

$$\theta = \sin^{-1}\left(\frac{x\sin\alpha}{\sqrt{x^2 - 2x\cos\alpha + 1}}\right), \qquad (5)$$

where $x = \frac{r}{R}$.

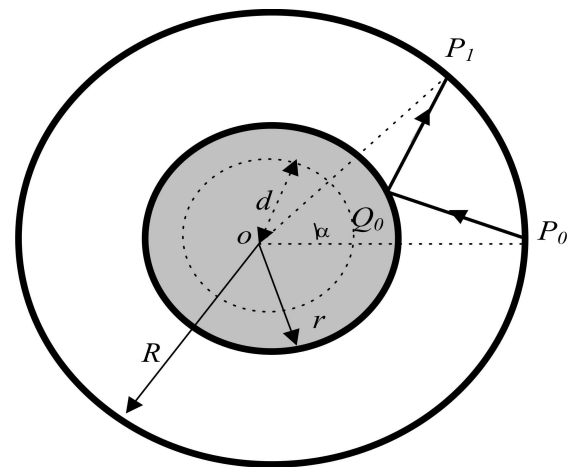

Figura 2 - Trajectory followed by a point particle, starting from $P_0$ at speed $V$, confined within two concentric circumferences of radii $r$ and $R > r$, respectively.

We shall not consider cases similar to those treated in the previous section, and will thus assume $\theta < \theta_c$. We therefore indicate with $P_k$ ($k = 0, 1, 2, \dots$) and $Q_k$ ($k =$ g, 1, 2,$\dots$) the points on which the particle impinges on the outer and inner circumference, respectively. Therefore, we may represent segments $Q_0 P_0$ and $P_1 Q_0$ by means of the three following vectors $\rho_0 = R$, $\sigma_0 = r\, e^{i\alpha}$ and $\rho_1 = R\, e^{2i\alpha}$, where $\alpha$ is the angle between $\rho_0$ and $\sigma_0$, as follows

$$Q_0 P_0 = \sigma_0 - \rho_0 = r\, e^{i\alpha} - R, \qquad (6)$$

$$P_1 Q_0 = \rho_1 - \sigma_0 = e^{i\alpha}\left(R\, e^{i\alpha} - r\right). \qquad (7)$$

In general, we may thus write

$$Q_k P_k = \sigma_k - \rho_k = e^{i2k\alpha}\left(r\, e^{i\alpha} - R\right), \qquad (8)$$

$$P_{k+1} Q_k = \rho_{k+1} - \sigma_k = e^{i(2k+1)\alpha}\left(R\, e^{i\alpha} - r\right), \qquad (9)$$

for $k =$ g, 1, 2,$\dots$ In Eqs. (7a-b), of course, we have set $\rho_k = R\, e^{i2k\alpha}$ and $\sigma_k = r\, e^{i(2k+1)\alpha}$. By now imposing that $Q_k P_k = Q_0 P_0$, or, equivalently, $P_{k+1} Q_k = P_1 Q_0$, in order to have closure of the trajectory after exactly $k$ mechanical reflections on the inner circumference, we have $e^{i2k\alpha} = 1$, so that

$$k\alpha = N\pi, \qquad (10)$$

where $N$ is an integer. Notice that Eq. (10) is only formally similar to Eq. (4). Here, in fact, one should keep in mind that the angle $\alpha$ must lie in the range $\left[0, \tan^{-1}\left(\frac{\sqrt{R^2 - r^2}}{r}\right)\right)$ and that it is different, in definition, from the angle $\varphi$ in Eq. (4). The dependence of the critical angle $\alpha_c = \tan^{-1}\left(\frac{\sqrt{R^2 - r^2}}{r}\right)$ on the positive ratio $x = \frac{r}{R} \leq 1$ is given in Fig. 3, where it can be seen that $\alpha_c$ is a monotonously decreasing function of $x$. In this way, not all possible choices of $k$ and $N$ in Eq. (10) represents a path defined in Eqs. (7a-b). In summary, we may state that if the indices $k$ and $N$ satisfy the following condition
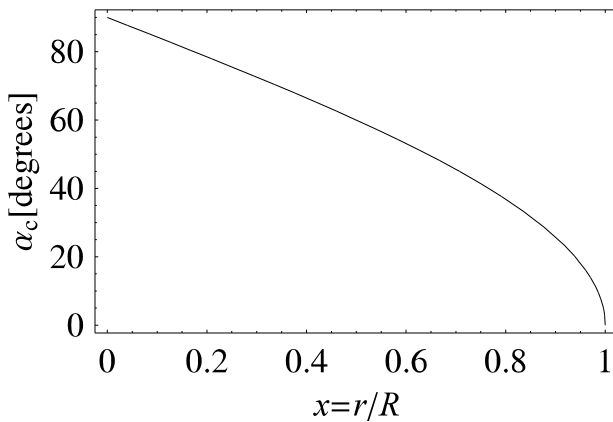
$$0 \leq \tan\left(\frac{N\pi}{k}\right) < \alpha_c, \qquad (11)$$

then the escape time can be calculated in terms of the angle $\alpha = \frac{N\pi}{k}$ as follows

$$t_E = \frac{2k\sqrt{r^2 + R^2 - 2rR\cos\alpha}}{V}, \qquad (12)$$

where $V$ is the constant speed of the particle. Eq. (5) can be used to relate the escape time $t_E$ to $\theta$. The above relation can be used, more generally, as we shall see in the following section, even when $\alpha \neq \frac{N\pi}{k}$.

## 4. General behavior of the system

Up to this point we have not considered the possibility of having a point particle in $P_0$ with an initial velocity making, with respect to the horizontal, an angle $\alpha$ being an irrational multiple of $\pi$. Therefore, in the present section we shall consider also cases in which $\alpha \neq \frac{N}{k}\pi$, $N$ and $k$ being positive integers and the angle $\alpha$ is such that $0 \leq \tan\alpha < \frac{\sqrt{R^2 - r^2}}{r}$. In this case the point particle will suffer reflection on both circular walls and will never go back to its starting point, even though it can, after a certain time, get close enough to it. The particle will thus be able to exit from the annular region in which it is temporarily confined if the opening has finite height $\varepsilon$, as shown in Fig. 4. In this way, a finite escape time $t_E$ will still exist for the system. Of course, in this case the quantity $t_E$ will go to infinity as the dimension $\varepsilon$ of the opening goes to zero and, in the absence of dissipation, the system will realize a non-periodic perpetual motion within the annular region. In what follows we shall give a numerical estimate of the escape time in the case the particle wanders within the region in between the two concentric circumferences and, at the same time, the dimension of the opening is assumed to be finite.
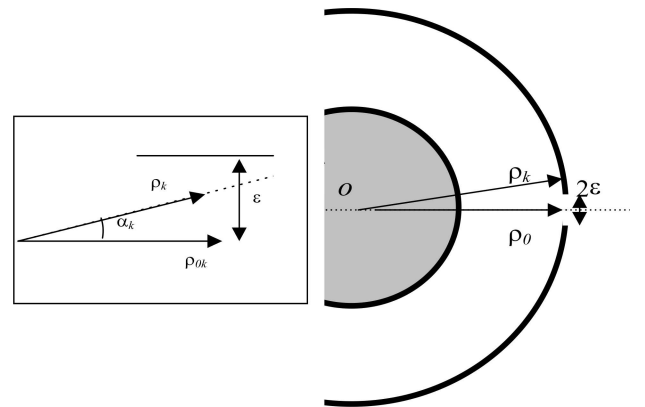
Figura 4 - After $k$ mechanical reflections on the inner circumference, the particle impinges on a region very close to the opening through which it had initially passed (on the right). If the vector $\rho_k$ is close enough to $\rho_0$, as shown in the inset on the left, then the point particle is able to escape from the annular region in which it was temporarily trapped.

Figura 3 - Critical angle $\alpha_c$ (above which the system behaves as if the inner circumference were not present) as a function of the ratio of the radii $r/R$.

By considering the inset in Fig. 4, where we set $\alpha_k = 2k\alpha$, we see that the condition for which the light ray might escape after $k$ reflections is simply the following

$$0 < |\sin(2k\alpha)| < \frac{\varepsilon}{R} \ , \qquad (13)$$

provided $\cos(2k\alpha) > 0$. The above relation can readily be used in a numerical algorithm, so that the escape time may be computed for varying values of $\theta$ , recalling Eq. (5), in the interval $0 \leq \theta < \theta_c$. In Fig. 5 we report a numerical evaluation of an averaged escape time $T$. In order to find $T$, we start from a list of values of $t_E$ as given by Eq. (12), found by a numerical algorithm based on condition (13) for various values of the dimensions of the opening $\frac{\varepsilon}{R}$, choosing $\frac{r}{R} = 0.75$ and $V = 10 \ \frac{m}{s}$. We then average the outcome of the escape times calculated for specific values of the angle $\theta$ over an angular range of about $\Delta\theta = 1.21$ degrees.
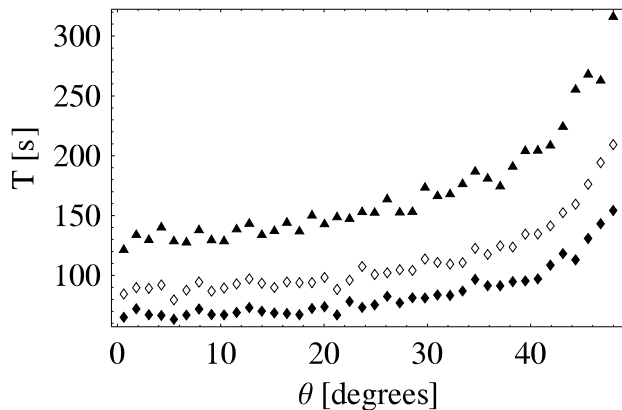


Figura 5 - Escape time of a particle trapped within an annular billiard for $V = 10$ m/s, $r/R = 0.75$ and $\varepsilon/R = 0.0010$ (triangles), 0.0015 (empty diamonds), 0.0020 (full diamonds). The critical angle for $r/R = 0.75$ is $48.6°$.

This averaging process is necessary in order to show, in an experimentally significant fashion, the outcome of our analysis, given the highly scattered nature of plots obtained when evaluation of $t_E$ is performed. A large amount (10.000 in our case) of $t_E$ values in the angular range $0 < \theta < \theta_c$ are then grouped in such a way to form intervals of amplitude $\Delta\theta = 1.21$ degrees . Considering the outcome of this averaging process, in Fig. 5 we notice that the quantity $T$ lies below the value of 200 s for most projection angles $\theta$ for the choice of parameters given. However, as the angle $\theta$ increases, the average escape time rises in the vicinity of $\theta_c$. Naturally, the obvious result that for increasing values of the dimension of the opening the escape times are, on the average, smaller for a given angular range, is confirmed by the present simple analysis.

## 5.   Conclusions

We have considered the problem of the motion of a classical point particle confined within an annular region with perfectly reflecting walls. Assuming finite dimensions of the hole from which the point particle is projected within the billiard with speed $V$ at an angle $\theta$ with respect to the horizontal, we evaluate, numerically, by means of a rather simple algorithm, the escape time from the billiard. The escape time is seen to increase, on the average, with increasing values of the projecting angle $\theta$. The analysis of the problem can be proposed to advanced high school students or undergraduate college students. Future work will involve light propagation within a region enclosed between two coaxial cylinders, the external one presenting a circular opening from which a light ray may enter or exit.

## Acknowledgements

## References

[1] M.V. Berry, Eur. J. Phys. **2**, 91 (1981).

[2] R.W. Robinett, Am. J. Phys. **67**, 67 (1999).

[3] M.A. Bandres and J.C. Gutiérrez-Vega, Am. J. Phys. **72**, 810 (2004).

[4] R. De Luca, F. Romeo and P. Zozzaro, Eur. J. Phys. **27**, 347 (2006).