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Relativistic mean proportion

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The so called modern physics is at present included in the curricula of high school. Therefore, we often organize in-depth lessons on quantum mechanics and on the special theory of relativity (SR). In the following sections, we want to show a lesson given by the first author of this paper. He proposes to students a proportion to deduce relativistic time dilation and length contraction without knowing the Lorentz transformations.

Keywords: Special relativity, speed of light, mean proportion.

1. Introduction

Special and general relativity are currently part of the high school curricula in almost all countries [1–13]. Teaching is a much more important branch of physics than one might think. Indeed, it may also happen that concepts of classical physics, well known to specialists, require further study from a didactic point of view due to the young age of the students [14–29]. Generally during the last year of high school, students learn that SR is founded on the principle of relativity and on the fact that the speed of light is a fundamental constant of nature and it does not follow the Galilean law of addition. Therefore, the students learn that the Galilean transformations must be replaced. Since a rigorous derivation of the new relativistic transformations is not possible for their knowledge, generally the textbooks show, without proof, the final form of the so-called special Lorentz transformations

$$\begin{cases} t = \dfrac{t' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \\ x = \dfrac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}} \\ \quad y = y' \\ \quad z = z' \end{cases} \tag{1}$$

where (x, y, z, t) and (x', y', z', t') are the coordinates of an event in two frames, c is the speed of light and relative velocity v is confined to the $x - x'$ direction. Through relations (**??**) some relativistic effects such as the contraction of lengths and time dilation are deduced [30, 31]. In the following sections, we want to deduce these relativistic predictions without using Lorentz transformations.

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2. Mean Proportions

As pointed out in a previous work [8], even simple newcomers to physics participate as speakers in the in-depth lessons. For example, we want to describe a lesson given by the first author of this paper who is a lawyer with a passion for science. He explains to students how to deduce length contraction and time dilation without using Lorentz transformations but with a new postulate that he hypothesized. Similarly to what we saw in [8], we consider a train of length L and a ray of light that starts from the rear of the train arrives at the front and is reflected back to the starting point. The relative velocity between train and ground is v. Obviously, from a train point of view, the flight time is

$$\Delta t = \frac{2L}{c}. \tag{2}$$

Whereas $\Delta t'$ and L' are the flight time and train length in the ground reference frame. They are still unknown and, reasoning in Newtonian terms, we expect that $\Delta t' = \Delta t$ and $L' = L$. The author of the lecture, for reasons of symmetry and reasoning with classical physics, hypothesizes that the space gained in the direction of motion and the space lost in the opposite one must be in proportion to the length of the train. For this reason, he postulates: *the double of the train length, from the point of view of an observer, is always mean proportional between the space that the light would travel in the direction of motion and that in the opposite direction during the flight time interval measured by the other observer's clock.* Mathematically speaking, we postulate

$$(c + v)\,\Delta t' : 2L = 2L : (c - v)\,\Delta t' \tag{3}$$

or

$$(c + v)\,\Delta t : 2L' = 2L' : (c - v)\,\Delta t. \tag{4}$$

We already want to underline that, as we will see in the next section, this proportion is actually a consequence of the invariance of spacetime interval in Minkowsky chronotope. From (3) we get

$$4L^2 = (c+v)\,(c-v)\,\Delta t'^2. \tag{5}$$

Therefore

$$2L = \sqrt{c^2 - v^2}\Delta t' = c\sqrt{1 - \frac{v^2}{c^2}}\Delta t' \tag{6}$$

getting

$$\Delta t' = \frac{2L}{c\sqrt{1-\frac{v^2}{c^2}}}. \tag{7}$$

From (2), we can finally write the following relation between the proper and improper time interval

$$\Delta t' = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{8}$$

Similarly, from the proportion (4) we deduce

$$4L'^2 = (c+v)\,(c-v)\,\Delta t^2 \tag{9}$$

getting

$$2L' = \sqrt{c^2 - v^2}\Delta t = c\sqrt{1 - \frac{v^2}{c^2}}\Delta t. \tag{10}$$

From (2) we can write

$$2L' = \sqrt{1 - \frac{v^2}{c^2}}2L \tag{11}$$

finally getting the length contraction

$$L' = \sqrt{1 - \frac{v^2}{c^2}}L. \tag{12}$$

3. Physical Meaning of Proportions

The physical meaning of these proportions was given by Carmine Serio, an Italian physicist, and Antonio Rita an Italian mathematician [32]. We start by the fundamental invariant of SR, that is, the space-time distance

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{13}$$

Let us consider a clock in relative motion with respect to an inertial system (x', y', z', t') and a reference system (x, y, z, t) at rest with respect to the moving clock. In the latter frame the clock is at rest and we have

$$ds^2 = c^2dt^2 \tag{14}$$

while in the former

$$ds'^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2. \tag{15}$$

Therefore

$$c^2dt^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 \tag{16}$$

and it is possible to write

$$c^2\frac{dt^2}{dt'^2} = c^2 - \frac{dx'^2 + dy'^2 + dz'^2}{dt'^2} = c^2 - v^2 \tag{17}$$

getting

$$c^2dt^2 = c^2dt'^2 - v^2dt'^2. \tag{18}$$

From (18) we deduce the following well known relation

$$dt' = \frac{dt}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{19}$$

Now we can rewrite the relation (18) in the following manner

$$cdt \cdot cdt = (c-v)\,dt'\,(c+v)\,dt' \tag{20}$$

obtaining

$$\frac{(c+v)\,dt'}{cdt} = \frac{cdt}{(c-v)\,dt'} \tag{21}$$

and that is the proportion (3) in the following infinitesimal form

$$\frac{(c+v)\,dt'}{dl} = \frac{dl}{(c-v)\,dt'}. \tag{22}$$

Thanks to (19), the previous relation can be written as

$$\frac{(c+v)\,dt}{dl\sqrt{1-\frac{v^2}{c^2}}} = \frac{dl\sqrt{1-\frac{v^2}{c^2}}}{(c-v)\,dt} \tag{23}$$

and that is the proportion (4) in the following infinitesimal form

$$\frac{(c+v)\,dt}{dl'} = \frac{dl'}{(c-v)\,dt}. \tag{24}$$

From what we have seen, we can say that the proportions (3) and (4) are the mathematical expression of the invariance of the speed of light.

4. Conclusions

In this manuscript, starting from some in-depth lessons held at high schools, we present a lesson proposed by the first author of this paper. The showed proportions admit a simple interpretation in Newtonian three-dimensional space. We do not claim that this approach can replace the common deductions explained in the introductory texts using bouncing light clocks and relative velocity symmetry but, perhaps due to its simplicity, it aroused a strong interest and a great participation of the students. A demanding of symmetry in Newtonian framework, allowed to deduce relativistic deformations without resorting to more elaborate concepts such as four-dimensional space-time and the relative metrics induced by the invariance of the speed of light.

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