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Cartas ao Editor

On the derivation of the terminal velocity for the falling magnet from dimensional analysis

(Sobre a derivação da velocidade terminal do ímã em queda via análise dimensional)

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Dimensional analysis was employed to develop a predictive formula for the terminal velocity for a magnet dropped down a metallic tube. In this particular application, the technique succeeded in generating the same formula theoretically derived and that has been published by others. The analysis thus presented suggests other applications that can be developed for motivating in the use of the technique.
Keywords: Lenz's law, electromagnetic braking, magnets.

Análise dimensional foi utilizada na derivação de uma fórmula de predição da velocidade terminal de um ímã em queda no interior de um tubo metálico. Nesta aplicação em particular, a técnica conseguiu gerar a mesma fórmula derivada teoricamente e que foi publicada por outros autores. A análise aqui apresentada sugere outras aplicações que podem ser desenvolvidas para motivar na utilização da técnica.
Palavras-chave: lei de Lenz, freio eletromagnético, ímãs.

1. Introduction

A rather popular experiment demonstrating Lenz's law, which has been analyzed in many works, is the falling magnet dropped down a metallic tube, in which, the eddy currents induced in the tube wall, produce a upward drag force that brings the magnet to its terminal velocity. The result is that the time of falling of the magnet is much longer than an otherwise identical nonmagnetic object dropped through the same tube.

Saslow [1], revisiting Maxwell's 1872 theory of eddy currents, presented a discussion of Lenz's law followed by a calculation of the drag force on a magnetic dipole falling down a long conducting tube. MacLatchy *et al.* [2] described methods of calculating and measuring the terminal velocity and magnetic forces in the magnetic braking experiment. Hahn *et al.* [3] reported the results of precise measurements of the motion and damping of the magnet with variations of pipe composition, length, thickness, radius, and position. They also presented the calculation of electromagnetic damping for any pipe configuration, which is coaxial with the magnet's motion. More recently, Levin *et al.* [4] presented a calculation that quantitatively accounts for the terminal velocity of a cylindrical magnet falling through a long copper or aluminum pipe. In all these previous published works, the same dependence of the terminal velocity on magnetic dipole moment, mass, conductivity, pipe wall thickness, and pipe wall radius was found.

In the paper of Pelesko *et al.* [5], an attempt has been made to find the dependence of the terminal velocity on these same variables via dimensional analysis. However, these authors argued that such analysis was not possible as posed, and have proceed in a combination of elementary physics and dimensional arguments, as they put it, to uncover the dependence of the terminal velocity on the various variables. The result was that the approaches taken, failed to show the dependence of the pipe wall thickness. Later on, Roy *et al.* [6] complement this work, by considering the effect of the thickness of the tube, by curve-fitting experimental data taken from tubes with different thickness. This can be considered an out of order approach to fix the previous solution because, as will be shown later, dimensional analysis is capable to find the correct dependence of all related variables. It is argued here that both works have missed the opportunity to show the full potential of dimensional analysis to deal with problems of this kind, and the present work tries to redeem the correct use of the technique.

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Dimensional analysis is a powerful technique that based on dimensional arguments and physical insight, can be used to find the dependence on the variables that control a particular phenomenon. It is very useful to uncover the relationship amongst the pertinent variables written in terms of dimensionless parameters.

The basic principle of dimensional analysis was known since Newton, and has been used by many scientists thenceforth, including Stokes, Maxwell, Fourier and Rayleigh. Eventually, it was formalized in the Buckingham π theorem [7], which describes how every physically meaningful equation involving n variables can be equivalently rewritten as an equation of $n - m$ dimensionless parameters, where m is the number of fundamental dimensions used. Furthermore, and most importantly, it provides a method for computing these dimensionless parameters from the given variables.

2. Construction of dimensionless parameters for the falling magnet

The basic idea for constructing dimensionless parameters, following the procedure put forward by Buckingham, is to choose amongst the variables that control the physical phenomena being addressed, those which can be used as base to write the dimensions of all the variables involved in the problem. First of all, these variables should be measured in a consistent system of units, such as the SI system, in which the basic units are the meter (dimension of length L), kilogram (dimension of mass M), second (dimension of time T), ampere (dimension of electrical current I). For the construction of the dimensionless parameters this base of dimensions suffices. There are physical phenomena which would be necessary to include other dimensional elements such as: the kelvin (K), the mole (mol), and the candela (cd).

The identification of the variables involved, requires a physical analysis of the problem at hand. The falling magnet is a phenomenon controlled by two forces: gravity and magnetic drag. It can be shown that the magnetic drag is directly proportional to the terminal velocity $k \cdot v_t$, where k is the magnetic damping constant [k] = kg $\times$ s^{-1} and v_t is the terminal velocity. From theory [1], it is known that the variables that control the magnetic damping constant are, in turn: radius of the tube [a] = m, the magnetic permeability in vacuum $[\mu_0]$ = henry $\times$ m^{-1} = kg $\times$ m $\times$ A$^{-2}\times$ s^{-2}, the magnetic dipole moment $[M_0]$ = A $\times$ m^2, the conductivity of the tube material $[\sigma]$ = siemens $\times$ m^{-1} = A$^2\times$ s$^3\times$ kg $\times$ m^{-3}, and the thickness of the tube wall $[w]$ = m. Note that the influence of the geometrical parameters of the magnet are supposedly already included in the magnetic dipole moment. These variables would be related to an unknown function $f(k,\ a,\ \mu_0,\ M_0,\ \sigma,\ w) = 0$.

The listed variables can be measured with the reduced base of dimensions: (M, L, T, I). Six variables have been listed ($n = 6$), which needs four fundamental elements to write their dimensions ($m = 4$). The number of dimensionless parameters involved in this problem is then given by n - m = 6 - 4 = 2 (two), which would be related to a still unknown function $\phi(\pi_1, \pi_2) = 0$, where π_1 and π_2 are the two dimensionless parameters to be constructed. Note that the original function, with six primitive variables, is replaced by a function with only two dimensionless parameters, and as we shall see, without any loss of information. The reduction of the number of variables is the main motive for applying the dimensional analysis technique, particularly when the function relating the dimensionless parameters (new derived variables) must be found experimentally.

Once the variables have been identified, and the base to write the dimensions of these variables has been selected, the next step is to choose the so-called 'new base' of dimensions. The elements of this new base are selected amongst the variables that have been listed, which will substitute the elements of the original base of dimensions. Since the original base has four elements, four variables should be selected to represent the elements of the original base. There is considerable freedom allowed in the choice. The two most important rules to follow are: a) of course, the chosen variable should contain in its dimensions, the element of the original base which it will be substituted for; b) the elements of the new base must not form a dimensionless group. Here the variables $\mu_0,\ a,\ \sigma,\ M_0$ have been chosen to represent the dimensions M, L, T, I, respectively.

The two dimensionless parameters will be constructed from the two variables that were left, namely the magnetic damping constant k, and the thickness of the tube wall w. These variables will turn out dimensionless by combine them with the elements of the new base, each of them elevated to exponents to be determined according to the following procedure.

As the π parameters are all dimensionless *i.e.* they have dimensions M^{0}L^{0}T^{0}I^0, we can use the principle of dimensional homogeneity to equate the dimensions for each π parameter.

For the first π parameter $\pi_1 = \mu_0^x \cdot a^y \cdot \sigma^z\ \cdot M_0^t \cdot k$, which in terms of the SI units can be written as $1 = (\mathrm{kg} \times \mathrm{m} \times \mathrm{A}^{-2} \times \mathrm{s}^{-2})^x \cdot \mathrm{m}^y \cdot (\mathrm{A}^2 \times \mathrm{s}^3 \times \mathrm{kg}^{-1} \times \mathrm{m}^{-3})^z \cdot (\mathrm{A} \times \mathrm{m}^2)^t \cdot (\mathrm{kg} \times \mathrm{s}^{-1})$, and in terms of dimensions can be written as $[\pi_1] = (\mathrm{M} \times \mathrm{L} \times \mathrm{I}^{-2} \times \mathrm{T}^{-2})^x \cdot \mathrm{L}^y \cdot (\mathrm{I}^2\times \mathrm{T}^3\times \mathrm{M}^{-1}\times \mathrm{L}^{-3})^z\cdot (\mathrm{I} \times \mathrm{L}^2)^t\cdot (\mathrm{M} \times \mathrm{T}^{-1}) = \mathrm{M}^0\times \mathrm{L}^0\times \mathrm{T}^0\times \mathrm{I}^0$.

For each dimension (M, L, T or I) the powers must be equal on both sides of the equation, so that

$$\left\{\begin{array}{l}\text{for M} : x - z + 1 = 0,\\ \text{for L} : x + y - 3z + 2t = 0,\\ \text{for T} : -2x + 3z - 1 = 0, \text{ and}\\ \text{for I} : -2x + 2z + t = 0\end{array}\right\} \Rightarrow x = -2, y = 3, z = -1, t = -2,$$

giving π_1 as $\pi_1 = \mu_0^{-2} \cdot a^3 \cdot \sigma^{-1} \cdot M_0^{-2} \cdot k$, or

$$\pi_1 = \frac{ka^3}{\mu_0^2 \sigma M_0^2}, \tag{1}$$

and a similar procedure is followed for the second π parameter π_2.

$$\pi_2 = \mu_0^x \cdot a^y \cdot \sigma^z \cdot M_0^t \cdot w$$

$$[\pi_2] = (\mathrm{M} \times \mathrm{L} \times \mathrm{I}^{-2} \times \mathrm{T}^{-2})^x \cdot \mathrm{L}^y \cdot (\mathrm{I}^2 \times \mathrm{T}^3 \times \mathrm{M}^{-1} \times \mathrm{L}^{-3})^z \cdot (\mathrm{I} \times \mathrm{L}^2)^t \cdot (L) = \mathrm{M}^0 \times \mathrm{L}^0 \times \mathrm{T}^0 \times \mathrm{I}^0$$

$$\left\{\begin{array}{l}\text{M} : x - z = 0\\ \text{L} : x + y - 3z + 2t + 1 = 0\\ \text{T} : -2x + 3z = 0\\ \text{A} : -2x + 2z + t = 0\end{array}\right\} \Rightarrow x = 0, y = -1, z = 0, t = 0,$$

giving π_2 as $\pi_2 = \mu_0^0 \cdot a^{-1} \cdot \sigma^0 \cdot M_0^0 \cdot w$, or

$$\pi_2 = \frac{w}{a}. \tag{2}$$

3. Derivation of an expression for the terminal velocity from the dimensionless parameters π_1 and π_2

Thus the problem of the falling magnet dropped down a metallic tube may be described by the following function of the two dimensionless parameters that have been constructed,

$$\phi(\pi_1, \pi_2) = 0 \Rightarrow \phi(\frac{ka^3}{\mu_0^2 \sigma M_0^2}, \frac{w}{a}).$$

Once identified, manipulation of the π parameters is permitted. These manipulations do not change the number of parameters involved, but may change their appearance drastically.

Taking the defining equation as: $\phi(\pi_1, \pi_2, \pi_3 \ldots \ldots \ldots \pi_{n-m}) = 0$. Then the following manipulations are permitted.

1. Any number of groups can be combined by multiplication or division to form a new group which replaces one of the existing. *E.g.* π_1 and π_2 may be combined to form $\pi_{1a} = \pi_1/\pi_2$ so the defining equation becomes $\phi(\pi_{1a}, \pi_2, \pi_3 \ldots \ldots \ldots \pi_{n-m}) = 0$
2. The reciprocal of any dimensionless group is valid. So $\phi(\pi_1, 1/\pi_2, \pi_3 \ldots \ldots \ldots 1/\pi_{n-m}) = 0$ is valid.
3. Any dimensionless group may be raised to any power. So $\phi[(\pi_1)^2, (\pi_2)^{1/2}, (\pi_3)^3 \ldots \ldots \ldots \pi_{n-m}] = 0$ is valid.
4. Any dimensionless group may be multiplied by a constant.
5. And, according to the Implicit Function Theorem, any group may be expressed as a function of the other groups, *e.g.* $\pi_2 = \phi(\pi_1, \pi_3 \ldots \ldots \ldots \pi_{n-m})$.

The magnetic damping constant k is proportional to the eddy currents induced in the tube, and since these are proportional to the thickness of the tube wall w, then according to manipulations 1 and 2 above, $\phi(\pi_1, \pi_2) = 0$ may be written as

$$\phi(\pi_1 \cdot \pi_2^{-1}) = 0 \Rightarrow \phi(\frac{ka^4}{\mu_0^2 \sigma M_0^2 w}) = 0,$$

or

$$\frac{ka^4}{\mu_0^2 \sigma M_0^2 w} = c \tag{3}$$

where c is a constant.

The manipulations performed are physically plausible since the eddy currents should be proportional to the product σw, as has been theoretically found by Saslow [1].

It has been shown in the paper of Pelesko *et al.* [5] that in the equilibrium $mg = kv_t$, or $k = mg/v_t$, where m is the mass of the magnet and g is the gravity. This allows writing Eq. (3) in terms of the terminal velocity as

$$v_t = c\frac{mga^4}{\mu_0^2\sigma\, M_0^2 w}. \tag{4}$$

With c =1024/45, Eq. (4) is the same expression obtained by MacLatchy *et al.* [2] and by Levin *et al.* [4]. Of course, the value of the constant c cannot be provided by dimensional arguments only. Nonetheless, a single experiment can fix the value of c and the derived formula for the terminal velocity [Eq. (4)] becomes predictive.

In the experimental demonstrations of Levin *et al.* [4], they used a copper pipe (conductivity $\sigma = 5.71 \times 10^7$ siemens $\times$ m^{-1}) of length $L = 1.7$ m, radius $a =$ 7.85 mm, and wall thickness $w = 1.9$ mm; a neodymium cylindrical magnet of mass $m = 6$ g, radius $r =$ 6.35 mm, and height $d = 6.35$ mm, for which the measured time of fall T was equal to 22.9 s and that gives $v_t = L/T = 1.7$ m/22.9 s ≈ 0.074 m/s.

The magnetic dipole moment for the magnet was calculated from the effective magnet charge q_m multiplied by the magnetic height d as $M_0 = q_m \cdot d$. According to Levin *et al.* [4], the effective magnet charge can be estimated from

$$q_m = \frac{2\pi B r^2\sqrt{d^2+r^2}}{\mu_0 d}, \tag{5}$$

where B is the intensity of the magnetic field, which was measured as $B = 393$ mT.

With the aid of Eq. (5), the magnetic dipole moment for the magnet that was used in the demonstrations Levin *et al.* [4] was estimated as $M_0 = 711 \times 10^{-3}$ A $\times$ m^2 (with $\mu_0 = 4\pi \times 10^{-7}$ henry $\times$ m^{-1}).

By isolating c in the first member of Eq. (4), and substituting the above given numerical values for the variables that appear in this equation, gives $c \approx 28.76$ which seems to be in reasonable agreement with the theoretical value of $c = 1024/45 \approx 22.76$ obtained by MacLatchy *et al.* [2] and by Levin *et al.* [4].

4. Conclusion

Given the complexity of theoretically analyzing the problem, the use of dimensional analysis seems to be a simpler and straightforward approach to reveal the relationship amongst the variables that control a given phenomenon. In this application, with the data collected in just a single experiment, it succeeded in providing a predictive formula for the terminal velocity for a magnet dropped down a metallic tube. The analysis thus made, suggests numerous extensions that can be developed for motivating in the use of the technique.

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