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# A note on nonholonomic systems

*(Uma nota sobre sistemas não-holonômicos)*

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This note describes a question that deals with nonholonomic systems, a subject that has been gradually fading away from textbooks and even treated somewhat incorrectly as holonomic.
**Keywords:** non-holonomic systems, Appell's equations, Newton's law.

Esta nota descreve uma questão sobre sistemas não-holonômicos, um assunto que tem desaparecido de livros-texto e que até mesmo tem sido tratado incorretamente como holonômico.
**Palavras-chave:** sistemas não-holonômicos, equações de Appell, lei de Newton.

## 1. Introduction

There are two popular ways to teach a course. Traditionally one can prepare a set of topics to be dealt with and delivered lecture style with little student participation. Alternatively, one can pose one or more carefully chosen problems for a debate (questions and answers between the teacher and the students and among the students) which gradually solves the questions. This second way of teaching can be very stimulating both to the students and to the teacher.

One of us (FABC) has used this method in an undergraduate Mechanics course for physics students at the University of São Paulo. While using this method, it was noticed that the students are very easily led to consider problems that have been treated superficially or left out completely from modern textbooks. This is entirely understandable since most books tend to emphasize those parts of classical mechanics that are essential to quantum mechanics.

Usually the answer to the questions can be found in older treatises such as Whittaker [1] or Appell [2]. The objective of this note is to describe the answers for one such a question that has only an obscure and long forgotten treatment in the literature.

The question deals with nonholonomic systems. This subject has been gradually fading away from textbooks because, as mentioned, it is not so important for quantum mechanics oriented courses. However, although less important for physicists, the subject is extremely important for engineers in general, and control engineering in particular [3].

Other books [4] treat nonholonomic systems only when the constraints are expressed as non integrable relations of the type

$$\sum_j A_{ij}\,\dot{q}_j + B_i = 0 \quad \begin{array}{l} i = 1, \ldots, m \\ j = 1, \ldots, n \\ m < n \end{array} \tag{1}$$

where $A_{ij}$ and $B_i$ are functions of the coordinates $q_j$.

The questions that are almost immediately posed by the students are: i) What happens when the constraint is non-linear in the $\dot{q}_j$? ii) Can higher order derivatives appear in the constraint equation? iii) How may we treat such cases?

These kind of questions are sometimes answered, somewhat vaguely, by saying [5] that if the constraint is not of the form (1), then one should go back to Newton's law. Sometimes they are correctly although incompletely answered by stating that one should use the so-called Appell's equations [6].

A more complete answer, given a long time ago by Delassus [7], appeared in a French publication. His presentation is however tedious and perhaps the guide that follows can be useful.

He classifies the constraints according to the order of the highest time derivative of the coordinates that appears in the constraint equation. Furthermore, a constraint equation is linear if it is linear in its highest time derivative of the coordinates that appear in it. So, linear first order constraints can be reduced to second

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order linear constraints by time derivation. Equation (1) represents therefore a first order linear constraint.

He then divides the possible constraints into two classes. The first class comprises first and second order linear constraints and therefore comprises also nonlinear first order constraints. The second class comprises non-linear second order constraints and all higher order constraints.

His central result can then be expressed as follows: systems of the second class do not represent well posed mechanical problems in the sense that their solutions are not uniquely determined by the applied forces and the initial conditions ($q_j(0)$ and $\dot{q}_j(0)$) and so this class does not constitute possible problems. The demonstration of the result is complicated but interesting because it challenges the student to construct an auxiliary massless physical system which realizes the constraint equation. He shows that if the constraint equation is of the second class, the motion of the system depends on how one physically constructs the auxiliary system.

Delassus' theorem does not teach us how to solve the problem and this subject has been treated incorrectly many times. Therefore, we give a summary below and refer to Ray [8], Flannery [9] and Gatland [10]. As to constraints of the first class, it can be shown that first order linear constraints can be treated by means of Lagrange multipliers [4, 9], this is because the constraints are assumed to do no virtual work. Nonlinear first order constraints can be treated using the formalism developed by Saletan and Cromer [11] if the condition discussed by Ray [8] - his Eq. (4.10) - is obeyed. On the other hand, second order linear constraints may be treated by Newton's equations or, more economically, using Appell's equations that are equally applied without modification to holonomic and nonholonomic systems [6]. This is so because for such systems in general can no longer be uniquely characterized by the kinetic energy and the virtual work of the external applied forces[2], that is, the constraint forces have to be included. The fact that Appell's equation is based on a "energy of acceleration" function defined by

$$S = \sum_{\nu} \frac{1}{2} m_{\nu} \left( \ddot{\mathbf{r}}_{\nu} \right)^2 \qquad (2)$$

that is, by a function of the accelerations $\ddot{\mathbf{r}}_{\nu}$ of the $\nu$-particles, explains why in ref. [5] it is said that one has to go back to Newton's law in such cases.

To conclude this article, we should mention that one can use the method of Lagrange multipliers to nonlinear first order constraint developed by [12, see errata [5]]. However, in this case, the resulting equations of motion in general are not the physical equations of motion as shown by [5, 10]. In fact, the systems that obey these equations are called vakonomic systems described, for example, by [13]. In some cases, the physical equations are the same as the vakonomic equations [14].

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