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Noncommutative derivation of the Planck's radiation law

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The Planck's radiation law for the black body radiation spectrum was capable to explain the experimentally-measured black body spectrum. In order to get this result, Planck proposed his radiation law in a two-fold way: 1) by an *ad hoc* modification of the assumed connection between energy and entropy for thermal radiation; 2) by assuming that the calculation of the entropy of an oscillator in thermal equilibrium with radiation is carried out by discrete units of energy. As a consequence, the energy quantization, linear in frequency, was thus introduced into physics. However, the energy quantization of the simple harmonic oscillator was originally postulated by Planck in an incomplete way, i.e., the ground state energy was assumed to be null. Of course, this issue has been solved in different ways over time, even by him. Despite this, we propose an alternative way to restore the non-null ground state energy of the harmonic oscillators at noncommutative(NC) framework, where, how will be shown, the non-null ground state energy naturally arises as a NC contribution. With this approach, the Planck's quantum theory is updated and, consequently, becomes compatible with the Quantum Mechanics inaugurated in 1925.

Keywords: Blackbody radiation, Planck's radiation law, Planck's postulate, ⋆-product, Noncommutative theory.

1. Introduction

In the second half of the 19th century, the electromagnetic theory, founded on the Maxwell's equations [1], had consolidated the idea of the nature of light as a wave. This idea had been established previously by the results of the double slit experiment carried on in 1803 by T. Young [2–6]. These results corroborated the Huyngen's principle in detriment to the corpuscular theory presented almost two centuries before by I. Newton [7]. By combining the electromagnetic theory with the also recently developed kinetic theory of gases, by Boltzmann, Rayleigh and Jeans tried to elucidate the interaction of radiation with matter, to deduce an expression for the spectral radiation of the black body in order to, satisfactorily, adjust the experimental data obtained more recently [2–6]. This attempt conducted to the so called ultraviolet catastrophe, i.e., a disagreement between theory and experiment in the range of high frequencies of the electromagnetic radiation. Then Planck, in a heuristic way, proposed that, in the Boltzmann distribution, the energy should not behaves like a continuous variable but as a discreet function linearly proportional to the frequency [2–6]. As it turns out, this redounded in an excellent adjustment to the experimental data for an specific value of the proportionality constant, which was afterwards called the Planck's constant. Subsequently, Einstein and Compton [2–6] used the concept of Planck's energy quantum, or photon, to obtain good agreements to the experimental data of the photoelectric and Compton effects, respectively. These agreements could not be achieved by a purely classical approach, in which the energy is a continuous variable. An effort to reach a conciliation between so antagonistic interpretations for the nature of light, an apparent paradox, lead to the formulation of the wave-particle duality hypothesis. This proposal is central in the Copenhagen interpretation, school leaded by Niels Bohr. According to this interpretation, light propagates as a probability wave, preserving the effects of interference and diffraction of the double slit Young's experiment. On the other hand, when detected, this wave function collapses in a particle – the energy quantum or photon –, preserving the results observed in the photoelectric and Compton effects.

Later developments arising from this interpretation like de Broglie's matter waves, Heisenberg's uncertainty principle and Schrödinger's equation redounded on the foundations of the modern Quantum Mechanics in the mid 1920's.

The quantum concept – energy quanta linear in frequency – usually appears in modern physics textbooks [2–6] through a theoretical derivation of the black body energy spectrum for thermal radiation, demonstrating the failure of Classical theory and in proposing the formulation of a new mechanics, i.e., Quantum Mechanics. Of course, Quantum Mechanics has been developed far beyond the original modifications of Classical theory demanded for the derivation of the Planck's radiation law. Nevertheless, we revisit the black body radiation problem and present an alternative approach to Planck's quantum theory.

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A few years after the publication of Planck's original paper, he extended his quantum theory [8], starting from entropy, and introduced the non-zero ground state energy of the harmonic oscillator. Despite this, some interesting papers [9–12] present derivations of the experimentally-measured black body energy spectrum and the non-null ground state energy without quantum assumptions (discrete energy), but requiring in addition to the usual ideas of Classical physics some classical additional, for example: Lorentz-invariant electromagnetic radiation at the absolute zero of temperature, or exploring some connections between Classical and Quantum theories for the harmonic oscillator or applying purely thermodynamic theory to the classical simple harmonic oscillator.

On the other hand, the desire to describe Quantum Mechanics on phase space instead of Hilbert space is as old as Quantum Mechanics itself [13–17] and others have contributed to the subject [18–21]. In this scenario, we propose to obtain the Planck's radiation law for the full black body energy spectrum. This is accomplished by using the $\star$-product instead of the scalar product among phase space variables. Due to this, the ground state energy $\hbar\omega/2$ arises as a NC contribution and, as a consequence, the Planck's postulate for simple harmonic oscillators and the Planck's radiation law are updated to also embrace this ground state energy.

2. A Brief Review of Canonical Quantization

As it is well known from analytical mechanics [22], an unconstrained system has its dynamics described by a Hamiltonian function, $H(Q_i, P_i)$, written in terms of the phase space of generalized coordinates Q_i and their respective momenta P_i ($i = 1, \ldots, n$, where n is the number of the degrees of freedom). On the other words, the state of the system is specified as a point in the 2n-dimensional phase space M, which is a smooth manifold. In canonical coordinates, a point ξ in M is written as $\xi = (\mathbf{Q}, \mathbf{P}) = (Q_1, \ldots, Q_n, P_1, \ldots, P_n)$ and the observables of the system, such as the Hamiltonian function, are smooth real-valued functions on this phase space M. Further, a new function on M can be obtained by the scalar product – point wise way – of two smooth real-valued functions on the phase space M, $f(\xi)$ and $g(\xi)$, read as

$$(f \cdot g)(\xi) = f(\xi)\ g(\xi), \tag{1}$$

which is also a smooth real-valued function on the phase space M and the scalar product presents a commutative algebra – $(f \cdot g)(\xi) = (g \cdot f)(\xi)$. In the context of Classical Mechanics, we have a commutative classical algebra of observables. In the Hamiltonian formalism, a new smooth real-valued function on the phase space M can be obtained by the Poisson brackets between two functions $f(\xi)$ and $g(\xi)$ on M, read as

$$\{\, f, g\,\}(\xi) = \frac{\partial f(\xi)}{\partial \xi_\alpha}\omega^{\alpha\beta}\frac{\partial g(\xi)}{\partial \xi_\beta},\ \alpha, \beta = 1, \ldots, 2n \tag{2}$$

with the Poisson tensor ω being

$$\omega = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{3}$$

where the non-null Poisson brackets, among the phase space (Q_i, P_i) coordinates, are

$$\{\, Q_i\ ,\ P_j\,\} = \delta_{ij}. \tag{4}$$

The dynamics of the system is given by the Hamilton's equations of motion – $\dot{\xi}_\alpha = \{\xi_\alpha, H\}$ –, read as

$$\begin{aligned} \dot{Q}_i &= \{Q_i, H\} \\ \dot{P}_i &= \{P_i, H\}. \end{aligned} \tag{5}$$

Given this, all the physical quantities $\mathcal{A}$ are supposed to be expressible as a function dependent on generalized phase space coordinates, i.e., $\mathcal{A} \equiv \mathcal{A}(Q_i, P_i)$.

The word quantization means the construction of the quantum theory of a certain system according to a corresponding classical theory. However, Classical and Quantum Mechanics are essentially distinct due to the Heisenberg's uncertainty relation, because states of quantum system can no longer be represented as points in phase space, indeed, the state of quantum system is specified by the vectors ψ of an abstract Hilbert space $\Re$. Further, the Heisenberg's uncertainty is a consequence of the noncommutativity of the quantum mechanical observables. Therefore, the commutative classical algebra of observables must be replaced by a noncommutative quantum algebra of observables. In the canonical quantization approach, which maps Classical to Quantum Mechanics, this noncommutativity is implemented by representing the quantum mechanical observables by linear operators in $\Re$. That is, each physical quantity $\mathcal{A}$, written as $\mathcal{A}(Q_i, P_i)$ in the classical theory, must be assigned, in quantum theory, to a certain operator $\widehat{\mathcal{A}}$ that acts in a space $\Re$ as $\widehat{\mathcal{A}} \equiv \mathcal{A}(\widehat{Q}_i, \widehat{P}_i)$, where the operators of the generalized coordinates $\widehat{Q}_i$ and momenta $\widehat{P}_i$ are postulated to obey the following non-null canonical commutation relation:

$$\{Q_i, P_j\} = \delta_{ij} \quad \to \quad \left[\widehat{Q}_i, \widehat{P}_j\right] = i\,\hbar\ \delta_{ij} \tag{6}$$

where $\hbar$ is the Planck's constant divided by 2π, $[\ ,\ ]$ is the quantum commutator and $\widehat{Q}_i$ and $\widehat{P}_j = -\imath\hbar\partial/\partial Q_j$ are quantum operators in the Hilbert space. Then, the time evolution of the state $\psi(\mathbf{Q}, t)$ is described by the Schrödinger equation is

$$\imath\hbar\frac{\partial \psi}{\partial t} = \widehat{H}(\widehat{Q}_i, \widehat{P}_i)\psi, \tag{7}$$

where

$$\widehat{H}(\widehat{\mathbf{Q}}, \widehat{\mathbf{P}}) = \frac{\widehat{\mathbf{P}}^2}{2m} + V(\widehat{\mathbf{Q}}, t). \tag{8}$$

3. A Brief Review of Moyal ⋆-product

In a classical point of view, a state of a system is specified as a point ξ in a 2n-dimensional phase space M, any measurable quantities f – observables – are function of ξ and possibly of the time, so $f \equiv f(\xi, t)$, and the equations of motion can be expressed in terms of Poisson brackets, given in Eq. (2). In order to quantize this same system, the real quantities f are replaced by Hermitian operators $\widehat{f}$, which now represent the observables, while the Poisson brackets, in the equations of motion, are replaced by commutators, see Eq. (6). The problem is to find the correspondence $f \to \widehat{f}$. The noncommutativity of Eq. (6) imposes that the classical quantities $f(\xi)$ cannot unambiguously be replaced by $\widehat{f}(\widehat{\xi})$, where the ambiguity is of the order of $\hbar$. The classical quantities $f(\xi)$ can be assumed as approximation to the quantum operators $\widehat{f}(\widehat{\xi})$ for $\lim \hbar \to 0$, in a way that the former gives us the guidelines to get on path of the latter. Due to the noncommutativity in Eq. (6), it is not possible to measure simultaneously Q_i and P_i without errors, then it is meaningless to represent states of a system as point in the phase space M. However, it is possible do develop a phase space formulation for the Quantum Mechanics based on the Wigner's quasi-distribution function [13] and Weyls's correspondence [14, 15] between quantum operators and ordinary c-number phase space functions. Groenewold and Moyal, independently, understood the problem, initially tackled by Wigner and Weyl, and proposed a deformation in the scalar product between two functions, usually called Moyal ⋆-product, so that noncommutativity arises in the phase space M, since Q_i and P_i no longer commute: $Q_i \star P_i \neq P_i \star Q_i$. Further, the Moyal ⋆-product of two function, f and g, at the point $\xi \in M$ involves f and g at ξ and, as well as, the all higher derivatives of these functions at ξ. Since f and g and their all derivatives are well define at ξ, then f and g are well define on the entire M. As a consequence, it is possible to departure from Classical Mechanics to the corresponding Quantum Mechanics by using the same quantities used to describe classical systems. Roughly speaking, the observables of the system are described by the same functions on phase space M as their classical counterparts.

After a brief historical summary of the role played by noncommutativity, first in the break between Classical and Quantum Mechanics, and then in the extension of the former into the latter through the deformation of the product between functions defined in phase space, we will present the basic foundations of this deformation, i.e., the Moyal ⋆-product and its relations with the canonical quantization.

The Moyal ∗-product [16, 17] which, written in a more general way, is

$$(f \star g)(\xi) \equiv f(\xi)\, exp\left[\frac{i\hbar}{N}\left(\frac{\overleftarrow{\partial}}{\partial Q_i}\,\frac{\overrightarrow{\partial}}{\partial P_i} - \frac{\overleftarrow{\partial}}{\partial P_i}\,\frac{\overrightarrow{\partial}}{\partial Q_i}\right)\right] g(\xi), \tag{9}$$

where $\frac{\overleftarrow{\partial}}{\partial Q_i}$ e $\frac{\overrightarrow{\partial}}{\partial P_i^{\prime}}$ mean, respectively, derivative to the left and to the right, and N is a deformation parameter to be inferred to reproduce both experimental and theoretical results. Note that the Moyal ⋆-product presents a noncommutative algebra – $(f \star g)(\xi) \neq (g \star f)(\xi)$ – and, for physical applications, the ⋆-product should be Hermitean: $\overline{f \star g} = \overline{g} \star \overline{f}$, where $\overline{f}$ denotes the complex conjugate of f. It is important to notice that this noncommutativity is due to the ⋆-product of the functions f and g at ξ, which involves not only the values of the functions f and g at this point, but also all higher derivatives of these functions at ξ. As a consequence, this noncommutativity disappears when $N \to \infty$ in Eq. (9), i.e., Eq. (9) reduces to the Eq. (1).

After that, we would like to explore the Moyal ⋆-product through the condition $\hbar/N << 1$, which reduces Eq. (9) to

$$f(\xi) \star g(\xi) = f(\xi)\left[1 + \frac{i\hbar}{N}\left(\frac{\overleftarrow{\partial}}{\partial Q_i}\,\frac{\overrightarrow{\partial}}{\partial P_i} - \frac{\overleftarrow{\partial}}{\partial P_i}\,\frac{\overrightarrow{\partial}}{\partial Q_i}\right)\right] g(\xi), \tag{10}$$

where the first term in the series is the scalar product, given in Eq. (1), while the power terms $\mathcal{O}[(\hbar/N)^n]$, for $n \geq 2$, are overlooked. From this point, we get the following relation

$$\begin{aligned} f(\xi) \star g(\xi) - g(\xi) \star f(\xi) &= \imath \frac{\hbar}{N} 2\{f(\xi), g(\xi)\}, \\ [f(\xi), g(\xi)]_\star &= \imath \frac{\hbar}{N} 2\{f(\xi), g(\xi)\}, \\ \frac{1}{\imath\hbar}[f(\xi), g(\xi)]_\star &= \frac{2}{N}\{f(\xi), g(\xi)\}, \end{aligned} \tag{11}$$

which reveals the connection between the classical and quantum behavior of the dynamical system. Now, we assume that f and g are, respectively, Q_i and P_j, consequently, Eq. (11) changes to

$$\frac{1}{\imath\hbar}[Q_i, P_j]_\star = \frac{2}{N}\{Q_i, P_j\}. \tag{12}$$

Since the quantum procedures, the canonical quantization, based on Eq. (6), and the ⋆-product, based on Eq. (12), must be consistent with each other, we must fix $N = 2$ and, consequently,

$$[Q_i, P_j]_\star \equiv \left[\widehat{Q}_i, \widehat{P}_j\right]. \tag{13}$$

In this sense, for instance, the time dependent Schrödinger equation changes to

$$\begin{aligned} \imath\frac{\partial\psi}{\partial t} &= H(\xi) \star \psi, \\ \imath\frac{\partial\psi}{\partial t} &= \left(\frac{1}{2}\mathbf{P} \star \mathbf{P} + V(\mathbf{Q}, t)\right) \star \psi, \\ \imath\frac{\partial\psi}{\partial t} &= H(\mathbf{Q}, \widetilde{\mathbf{P}})\psi, \end{aligned} \tag{14}$$

with

$$\widetilde{P}_i = P_i - \imath\frac{\hbar}{2}\frac{\partial}{\partial Q_i}. \tag{15}$$

At this point, $H(\mathbf{Q},\widetilde{\mathbf{P}}) \to \widehat{H}(\widehat{\mathbf{Q}},\widehat{\widetilde{\mathbf{P}}})$ and $\widetilde{P}_i \to \widehat{\widetilde{P}}_i$, then equation (14) reproduces equation (7).

4. Updating Planck's Radiation Law Through Noncommutativity

At this section, we present the motivation for revisiting Planck's radiation law, which describes the measured spectral density of black body electromagnetic radiation in thermal equilibrium at a given temperature T, in the scenario where there is no net flow of matter or energy between the body and its environment. Rayleigh and Jeans proposed a law which related the intensity of the radiation given off by a black body to the frequency at a specific temperature through classical arguments. This law present a problem for high frequencies, called ultraviolet catastrophe. The Fig. (1) [23] below makes the matter more clear. In order to fix the ultraviolet catastrophe problem presented in Fig. (1), Planck postulated that the energy of any harmonic oscillator system is discrete and multiple of a linear value at the frequency and, at the first time, the fundamental of Quantum Mechanics was introduced.

Now, we present a short discussion about the Planck's postulate. The Hamiltonian of a simple three-dimensional simple harmonic oscillator, written in the phase space $(\mathbf{Q},\mathbf{P})$, is

$$H(\mathbf{Q},\mathbf{P}) = \frac{1}{2}\left(\mathbf{P}^2 + \omega^2\mathbf{Q}^2\right), \quad (16)$$

with ω as a frequency. The energy is a physical quantity of the system at some time and, therefore, it is calculated by evaluating the Hamilton function at the point in phase space $\xi_0 = (\mathbf{Q}_0,\mathbf{P}_0)$ that characterizes the state of the system at this time, read as

$$W = \int H(\mathbf{Q},\mathbf{P})\delta^{(6)}(\mathbf{Q}-\mathbf{Q}_0,\mathbf{P}-\mathbf{P}_0)\; d\mathbf{Q}\; d\mathbf{P}, \quad (17)$$

where $\delta^{(6)}(\mathbf{Q}-\mathbf{Q}_0,\mathbf{P}-\mathbf{P}_0)$ is the six-dimensional Dirac delta function. Planck dealt with the relationship between energy, Eq. (17), and Hamiltonian, Eq. (16), of the harmonic oscillator by postulating that of the energy of harmonic oscillator is discrete, that is, a multiple of a linear value at the frequency, namely:

$$\begin{aligned} W &= \int H(\mathbf{Q},\mathbf{P})\delta^{(6)}(\mathbf{Q}-\mathbf{Q}_0,\mathbf{P}-\mathbf{P}_0)\; d\mathbf{Q}\; d\mathbf{P} \\ &\downarrow \\ W_n &= n\;\hbar\omega, \end{aligned} \quad (18)$$

with $H(\mathbf{Q},\mathbf{P})$ given by Eq. (16) and $n \in \mathbf{N}$. After that, Planck's investigated this subject in his book [8], entitled *The Theory of Heat Radiation*, where he got the ground state energy as being $\hbar\omega/2$, as well as, the quantized energy[1], a extension of Eq. (18), as being

$$W_n = (n - 1/2)\;\;\hbar\omega, \quad (19)$$

with $n \in \mathbf{N}^*$. Many years later, an equivalent expression to the equation (19), given by

$$W_n = (n + 1/2)\;\;\hbar\omega, \quad (20)$$

with $n \in \mathbf{N}$, it was obtained by the modern Quantum Mechanics, establish in 1925 [24–27, 29] and pages 361, 489 and 734 of ref. [28].

The usual Planck's radiation law for the black body radiation spectrum, proportional to the average energy $\langle\varepsilon(\nu,T)\rangle$, is presented in many textbooks as being

$$\rho(\omega,T) = \frac{\omega^2}{\pi^2c^3}\,\langle\varepsilon(\nu,T)\rangle\,, \quad (21)$$

where the average energy $\langle\varepsilon(\nu,T)\rangle$ is

$$\langle\varepsilon(\nu,T)\rangle = \frac{\displaystyle\sum_n W_n\, e^{-W_n/k\,T}}{\displaystyle\sum_n e^{-W_n/k\,T}}, \quad (22)$$

with k and T as being, respectively, the Boltzmann's constant and temperature. After that, Planck applies its postulate, given in Eq. (18), into Eq. (21) and get this following radiation law

$$\begin{aligned} \rho(\omega,T) &= \frac{\omega^2}{\pi^2c^3}\frac{\sum_{n=0}^{\infty} n\hbar\omega e^{-(n)\hbar\omega/kT}}{\sum_{n=0}^{\infty} e^{-(n)\hbar\omega/kT}}, \\ &= \frac{\omega^2}{\pi^2c^3}\left(\frac{\hbar\omega}{e^{\hbar\omega/kT}-1}\right), \end{aligned} \quad (23)$$

At the time Planck's article was published, where he introduced the quantum of the energy, he did not suspected that the ground state energy was not null. Some years later, Planck got the correct ground state energy, as we pointed out before, and corrected the result given in Eq. (23), as he demonstrated in his book [8].

At this work, we propose to get the non-null ground state energy by updating the Planck's quantize

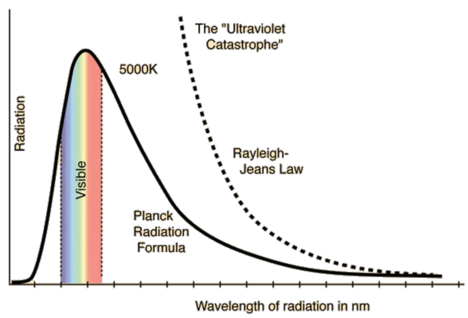

Figure 1: Comparison of the classical Rayleigh-Jeans Law and the quantum Planck radiation formula.

[1] See chapter III of ref. [8], page 141.

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energy for harmonic oscillator through noncommutativity [16, 17]. The noncommutativity [16, 17] will be introduced into the Rayleigh-Jeans theorem [2–6]. In order to get this, the scalar product among the phase space variables will be changed to the $\star$-product [16, 17], then the Hamiltonian of a harmonic oscillator in a noncommutative framework is

$$\begin{aligned} \widetilde{H}(\mathbf{Q},\mathbf{P}) &= \frac{1}{2}\left(\mathbf{P}\star\mathbf{P}+\omega^2\mathbf{Q}\star\mathbf{Q}\right), \\ &= \frac{(\mathbf{P}-\imath\omega\mathbf{Q})}{\sqrt{2}}\star\frac{(\mathbf{P}+\imath\omega\mathbf{Q})}{\sqrt{2}}, \\ &= \frac{1}{2}\left(\mathbf{P}^2+\omega^2\mathbf{Q}^2\right)+\frac{\hbar\omega}{N}, \\ &= H(\mathbf{Q},\mathbf{P})+\frac{\hbar\omega}{N}, \end{aligned} \tag{24}$$

where Eq. (10) was applied. Note that Eq. (24) is the usual Hamiltonian, given in Eq. (16), plus the NC term $\hbar\omega/N$. This allows us to update the result obtained from the application of the Planck's postulate, given in equation (18), to

$$\begin{aligned} W &= \int \widetilde{H}(\mathbf{Q},\mathbf{P})\delta^{(6)}(\mathbf{Q}-\mathbf{Q}_0,\mathbf{P}-\mathbf{P}_0)\,d\mathbf{Q}\,d\mathbf{P} \\ &= \int H(\mathbf{Q},\mathbf{P})\delta^{(6)}(\mathbf{Q}-\mathbf{Q}_0,\mathbf{P}-\mathbf{P}_0)\,d\mathbf{Q}\,d\mathbf{P}+\frac{1}{2}\hbar\omega \\ &\downarrow \\ W &= \left(n+\frac{1}{2}\right)\hbar\omega, \end{aligned} \tag{25}$$

with $\widetilde{H}(\mathbf{Q},\mathbf{P})$ given by Eq. (24), $N=2$ and $n\in\mathbf{N}$. In order to justify why $N=2$, we present a heuristic analysis of a vibrating string. Consider a string of length L with its extreme points fixed, which cannot move. The string vibrations modes are given by the Fig. (2). The Fig. (2) represents the equilibrium state, $n=0$, and the normal modes $n=1,2,3$. Note that for $n>0$, the expression $L=n\ \lambda/2$ is correct. As the vibration modes into the 3-dimensional black body cavity are analogous to what was briefly presented, the physical inferences obtained for string vibration modes are still valid when the 3-dimensional black body cavity is considered. At this context, Planck proposed that the energy is discrete and multiple of a linear value at the frequency ($n\hbar\omega$), as shown in the Fig. (2). However, the equilibrium state – ground state – of a vibrating system at frequency ω is the first vibration mode ($n=1$), which comprises the half wavelength ($\lambda/2$). Therefore, this state should comprise the half linear discrete energy at frequency ω – $\hbar\omega/N\to\hbar\omega/2$ –. To update Planck's original postulate, the term $\hbar\omega/2$ must be added *ad hoc* to your postulate, which results in the Eq. (25). Due to this, the quantum harmonic oscillator modes are represented in the Fig. (3) [30].

We would like to point out that the non-null ground state energy ($\hbar\omega/2$), as shown in Fig. (3), is also due to the noncommutativity and that the updated Planck quantum theory reproduces the result obtained by Planck in his book [8] and by the modern Quantum Mechanics [24–27, 29] and pages 361, 489 and 734 of ref. [28]. Recently, the quantization of simple harmonic oscillator through $\star$-product was obtained, in a quantum modern way, by Allen C. Hirshfeld and Peter Henselder in ref. [31].

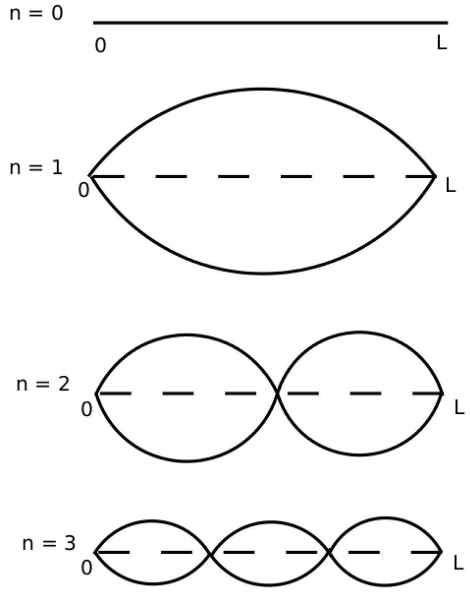

Figure 2: The string vibration modes.

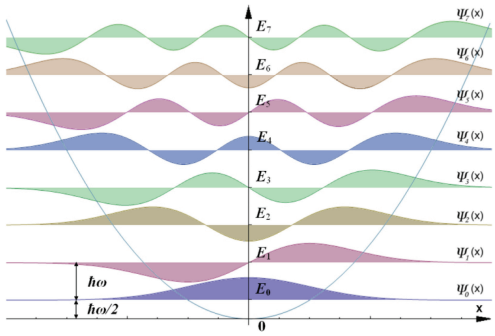

Figure 3: Wavefunction representations for the first eight bound eigenstates.

The black body electromagnetic energy W in a noncommutavive framework is given by

$$\begin{aligned} W &= \frac{1}{2}\int_V u(\nu,T)\,dV, \\ &= \frac{1}{2}\int_V\left(\varepsilon_0\,\vec{E}\star\vec{E}^*+\frac{1}{\mu_0}\,\vec{B}\star\vec{B}^*\right)dV, \end{aligned} \tag{26}$$

where $u(\nu,T)$, $\vec{E}$ and $\vec{B}$ are the electromagnetic energy density, the electric and the magnetic fields, respectively, and the usual scalar product is replaced by the $\star$-product. From the Maxwell's equations in vacuum, it is obtained the following differential equation for the potential vector

$$\vec{\nabla}^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0, \tag{27}$$

where the Lorentz's gauge was applied, namely, $\vec{\nabla}\cdot\vec{A} = 0$. The equation (27) is a wave equation, then the potential vector field $\vec{A}$ is a periodic vector function, with $\vec{A} = \vec{0}$ at the boundary of the black body cavity. After a straightforward calculation, equation (27) is solved and the periodic field vector is obtained as

$$\vec{A}(\vec{r},t) = \sum_n \sum_{\lambda=1}^{2} \left(\vec{q}\,(\vec{k}_n,\lambda)\, e^{i\,\vec{k}_n\cdot\vec{r}} + \vec{q}^{\,*}(\vec{k}_n,\lambda)\, e^{-i\,\vec{k}_n\cdot\vec{r}}\right), \tag{28}$$

with $\vec{k}_n$ and λ as being, respectively, the radiation propagation direction and the orthogonal directions of polarization. Further, $\vec{q}(\vec{k}_n,\lambda)$ are independent vectors and orthogonal to $\vec{k}_n$. As $\vec{E} = -\dfrac{\partial \vec{A}(\vec{r},t)}{\partial t}$ and $\vec{B} = \vec{\nabla}\times\vec{A}(\vec{r},t)$, equation (26) is rewritten as

$$W = 2\,V\,\varepsilon_0 \sum_n \sum_\lambda \left[\dot{\vec{q}}\star\dot{\vec{q}}^{\,*} + \omega_n^2\,(\vec{q}\star\vec{q}^{\,*})\right], \tag{29}$$

with $\omega_n^2 = k_n^2\,c^2$ and $c = \sqrt{1/\varepsilon_0\,\mu_0}$. Rewritten $\vec{q}$ and $\vec{q}^{\,*}$ in terms of the phase space coordinates $(\vec{Q},\vec{P})$, given by

$$\begin{aligned}
\vec{q} &= \frac{1}{2\sqrt{2\,V\,\varepsilon_0}}\left(\vec{Q} + i\,\frac{\vec{P}}{\omega_n}\right)\hat{e}(\vec{k}_n,\lambda)\,e^{-i\,\omega_n\,t},\\
\vec{q}^{\,*} &= \frac{1}{2\sqrt{2\,V\,\varepsilon_0}}\left(\vec{Q} - i\,\frac{\vec{P}}{\omega_n}\right)\hat{e}(\vec{k}_n,\lambda)\,e^{i\,\omega_n\,t},
\end{aligned} \tag{30}$$

the equation (29) reduces to

$$W = \sum_n \sum_\lambda \left[\frac{1}{2}\left(P^2 + \omega_n^2\,Q^2\right) + \frac{\hbar}{2}\,\omega_n\right], \tag{31}$$

which represents harmonic oscillators and $N = 2$. This is the noncommutative version of the Rayleigh-Jeans theorem for the radiation field.

Applying the Boltzmann distribution of energy among an infinite and enumerable number of oscillators confined in a cavity of volume V, considering the updated Planck's postulate, Eq. (25) and (31), one obtains, by analogy to the usual Planck development, the energy of radiation field as being

$$W_n = \left(n_\lambda(\omega) + \frac{1}{2}\right)\hbar\,\omega, \tag{32}$$

where $n_\lambda(\omega)$ is assumed to be the number of photons with frequency ω and polarization λ while, for $n_\lambda(\omega) = 0$, there is a non-null ground state energy equal to $\hbar\omega/2$. The energy of the radiation field is the sum of the eigenvalues W_n, given in Eq. (32), implies to the sum of $\hbar\omega/2$ term over all propagation direction, which leads to following divergent expression: $\sum_n \sum_\lambda \hbar\omega/2 \to \infty$.

The introduction of a noncommutativity, obtained through the replacement of the usual scalar product by the $\star$-product, induces a non-null ground state energy, as shown in Eq. (32), and, consequently, this approach changes the Planck's radiation law for the black body radiation spectrum, given by

$$\begin{aligned}
\rho(\omega,T) &= \frac{\omega^2}{\pi^2 c^3}\frac{\sum_{n=0}^{\infty}(n+1/2)\hbar\omega e^{-(n+1/2)\hbar\omega/kT}}{\sum_{n=0}^{\infty} e^{-(n+1/2)\hbar\omega/kT}},\\
&= \frac{\omega^2}{\pi^2 c^3}\left(\frac{\hbar\omega}{e^{\hbar\omega/kT}-1} + \frac{1}{2}\hbar\omega\right).
\end{aligned} \tag{33}$$

The equation (33) is exactly the Planck's radiation law[2] for the full black body energy spectrum, which was presented by Planck in his book [8]. Usually, this information is not presented in basic modern physics textbooks. In order to improve the education of physics students, this Planck's radiation law should be noticed in these textbooks.

It is important to notice that the molecules on the black body cavity boundary, a conducting surface connected to a thermal reservoir, oscillate and, consequently, emit electromagnetic waves that propagate through the interior of the cavity and then hits the internal surface of the cavity transferring energy to it, inducing the molecules to oscillate and, as a consequence, these molecules emit electromagnetic waves again; this process takes place successively. Those molecules behave as charged harmonic oscillators and they would still come to thermal equilibrium with the ambient thermal radiation [8]. Indeed, a classical charged harmonic oscillator acquires an average energy equal to the average energy per normal mode of the surround random classical radiation at the frequency ω_0 of the oscillator. Therefore, in the limit as the temperature decreases to absolute zero, the charged harmonic oscillators would be in equilibrium with the random radiation which exists at absolute zero temperature, i.e., there is a zero-point energy induced by particle motion. Since at the absolute zero temperature the quantum realm strongly prevails to the classical one, this random radiation is a quantun electromagnetic zero-point radiation. Another example of this is the Casimir effect [32], where the force between two uncharged parallel conducting plates depends upon all the radiation surrounding the plates. This force was experimentally measured [32–37] and it was shown that, at absolute zero, the Casimir force does not vanish, but this force was explained quantitatively by the existence of random radiation with an energy spectrum $\varepsilon_\omega = \hbar\omega/2$ per normal mode [9]. Due to this, we present an alternative way to fix $N = 2$ in equation (31) and, consequently, the correct ground state energy, whether for radiation fields or oscillating systems, is obtained.

[2] See chapter III of ref. [8], page 142.

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5. Conclusion

The $\star$-product among the phase space coordinates was applied to the process of determining the electromagnetic energy associated with the radiation emitted by the black body. With this, the correct relationship that determines the quantization of the harmonic oscillator was obtained, i.e., the non-null ground state energy was obtained and the original Planck's radiation law was extended in order to embrace the non-null ground state energy ($\hbar\omega/2$), as shown in ref. [8].

There is another point that we would like to speculate. Assume that the boundary of the black body cavity is at infinity so that the radiation field behaves like a free field, i.e., it seems that the radiation field propagates in a space which is not bounded by a cavity. In order to implement the transition from a scenario where the cavity edge is at infinity to one without a cavity, the first vibration mode ($n = 1$) in Fig. (2) tends to zeroth mode $n = 0$, consequently, the NC term ($\hbar\omega/N$) in Eq. (31) should be null, then $N \to \infty$. In this scenario, we argue that only vibration modes, integer multiples of the wavelength (λ), remain. Then, the quantum contribution due to the $\star$-product disappears, i.e., the $\star$-product reduces to the scalar one, as shown in section 3. Assuming this hypothesis as true, the NC contribution disappears and Eq. (32) reduces to

$$W_n = n_\lambda(\omega)\hbar\omega. \tag{34}$$

Now, the energy of the radiation field, which is the sum of the eigenvalues W_n, does not present a divergence. From what we presented using Moyal $\star$-product and edge, we can argue that the existence of non-zero ground state energy is due to the existence of a boundary.

In the papers [9–12] the Planck's law for black body energy spectrum and the ground state energy ($\hbar\omega/2$) where obtained by assuming the Lorentz-invariant electromagnetic radiation at the absolute zero of temperature, without the notion of energy discrete linear in frequency, or Dirac's classical-quantum analogy for the harmonic oscillator, or pure temperature ideas. On the other hand, we get the same result extending the Planck's postulate by using $\star$-product. Further, the deformation quantization method also reproduces the same previous result, even by the modern Quantum Mechanics. Due to this, there are, at least, five distinct descriptions of the Planck's law for black body energy spectrum and the ground state energy ($\hbar\omega/2$), equivalent when compared at experimental level. This shows that the realm for deciding which theory is correct is not only at the experimental level(e.g. black body radiation), but also at the theoretical level. Indeed, over time, modern Quantum Mechanics started at 1925 has shown to be the theory that best provides a description of the physical properties of nature at the scale of atoms and subatomic particles, far beyond the semi-classical limit. Despite that, deformation phase-space techniques are widely used, especially in quantum optics [38], chemistry [39] and application to quantum technologies [40].

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