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A small oscillation model for the "water dancing ball"

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Inspired by the "water dancing ball" problem proposed in the International Physicists' Tournament 2019, we study the small-angle oscillations of a cylinder on a plane subject to a thin sheet of water. The interaction between the water flow and the cylinder is modeled using basic physics principles such as Newton's laws, momentum, and energy conservation. We found that the flux of water around the cylinder applies a force which is responsible for the oscillations of the cylinder. From the motion equation of the cylinder, the period of oscillation is calculated in the harmonic approximation. Our results are contrasted with the period measurements presented recently by Pagaud and Delance, with a good agreement found, showing an error of only 10%.

Keywords: Classical Mechanics, Oscillations, Coandă effect.

1. Introduction

The International Physicists' Tournament (IPT) is an annual competition among teams of physics students from all around the world. Every year, a list of 17 problems is created for the IPT. The solutions to the problems are presented in "Physics Fights" where three teams compete with each other. In each fight, the teams have to play the roles of Reporter, Opponent and Reviewer. Unlike the typical physics exam, the problems used in the IPT allow multiple approaches (theoretical and/or experimentally), and the solutions must also be presented, reviewed, and challenged by the other participants. The performance of the teams in each role is judged by an experienced jury, usually consisting of professors from different Universities. Detailed information about the IPT can be found on the official web page of the tournament. [1].

The original "water dancing ball" problem was presented in the IPT 2019 and consisted of a ball subjected to a vertical water jet. Because of the interaction with the jet, the ball starts to oscillate instead of being ejected outward from the jet as we might intuitively think. This phenomenon is closely related to the Coandă effect which basically states that "the tendency of a jet of fluid emerging from an orifice is to follow an adjacent flat or curved surface and to entrain fluid from the surroundings so that a region of lower pressure develops" [2–5]. The main objective of the problem is to estimate the period of oscillation and its dependency on the system parameters. This is a problem of fluid dynamics with complicated time-dependent boundary conditions. There are several approaches to address this problem depending on the approximations made to simplify it. For instance, a very simplistic model could assume that the impact of the jet is an elastic collision where the horizontal component of the momentum is preserved. In this simple model, the oscillation of the ball is given by the torque due to the vertical force applied by the jet. A more elaborate model could use the static pressure distribution due to the jet around a ball [6, 7]. From the pressure distribution, it is possible to calculate the force on the ball and determine the subsequent movement. Recently, a theoretical and experimental study about the "water dancing ball" developed by former IPT participants was presented in Ref. [8]. In this work, the ball is replaced by a cylinder and some experimental measurements of the oscillation period are contrasted with the results given by a simple model based on an empirical force and torque which depends on two fit parameters. In this paper, we also present a simple approach to describe the motion of the cylinder based on basic physics tools such as Newton's laws, momentum, and energy conservation [9–12]. However, our model does not involve any fit parameter. In this way, the solution presented can be understood by young physics or engineering students without knowledge of advanced tools such as Lagrangian mechanics, Navier-Stokes equations, etc. We consider that the proposed solution can be used as an example in basic physics courses to show students how the use of different physical concepts can describe the behavior of non-trivial mechanical systems.

2. Description of the Model

Consider a cylinder of finite length and radius R on a flat horizontal rigid surface subjected to a vertical water jet with constant flux. The jet, a thin vertical

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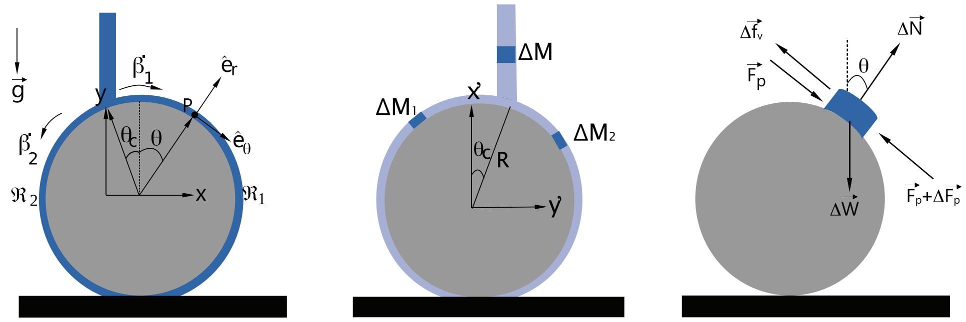

Figure 1: (a) The position of the center of the cylinder is denoted by x, and θ_c represents the point where the jet hits the cylinder. (b) After the collision, an incoming mass differential ΔM splits into two parts, ΔM_1 and ΔM_2, which move in opposite directions on the cylinder. (c) The diagram depicts the forces acting on an arbitrary mass differential of water.

sheet of water with thickness ϵ, falls on the cylinder parallel to its axis. The mass, length, and radius of the cylinder are M_b, L, and R, respectively. Experience shows that due to the action of the jet, the cylinder starts to oscillate (see the supplementary material of Ref. [8]). As mentioned before, for the sake of simplicity it is necessary to use several assumptions in order to simplify this complex problem avoiding the use of Navier-Stokes equations with moving boundary conditions. The first approximation we use in this paper assumes that the cylinder performs planar motion for a significant duration. The flow is considered laminar and incompressible, there is no water mass loss during the impact and the cylinder rolls without sliding on the flat rigid horizontal surface. The water only flows over the circular face of the cylinder as shown in Fig. 1. In this way the cross section of the water jet is $A = \epsilon, L$. These assumptions are expected to hold approximately for small enough flow rates. The density of the water is ρ and its velocity just before the impact with the cylinder is v. Thus, the incoming mass flux of water on the cylinder is $\rho A v$. The position of the center of mass of the cylinder is x, and the center of the jet strikes the cylinder at an angle θ_c (see Fig. 1 (a)). In the chosen coordinate system, the vertical axis, y, coincides with the water jet while the horizontal axis, x, is at a height R above the surface. We assume that the friction force between the cylinder and the flat surface is given by the Coulomb friction law and that the water that flows off the cylinder does not affect how the cylinder rolls. The angular position θ_c and the position of the cylinder's center are related by $x = -R\sin(\theta_c)$, and both are functions of time t whose forms are yet to be determined. As usual, the gravity acceleration is $\vec{g}$.

3. Stationary Approximation for the Flow Distribution

In this revised model, the water jet that makes contact with the cylinder bifurcates into two sections, which move in diametrically opposing directions along the cylinder, as depicted in Fig. 1 (b). We make the assumption that the water layer's angular velocity adheres to $\dot{\beta}(\theta) = \dot{\beta}_1$ for $\pi \geq \theta \geq \theta_c$ ($\Re_1$) and $\dot{\beta}(\theta) = -\dot{\beta}_2$ for $2\pi + \theta_c \geq \theta \geq \pi$ ($\Re_2$). For simplicity's sake, we posit that the flow entirely encapsulates the cylinder, with $\dot{\beta}_1$ and $\dot{\beta}_2$ remaining constant. This presumption overlooks the augmentation in kinetic energy of $\Delta M_{1,2}$ due to the alteration in gravitational potential energy as it navigates around the cylinder; in essence, it presumes that $v^2 \gg 2gR$. The position of $\Delta M_{1,2}$ can be expressed as the summation of the center of the cylinder's position and the vector $\vec{r}_i = R\sin(\theta)\hat{i} + R\cos(\theta), \hat{j}$. It should be noted that $\vec{r}_i$ signifies the position of ΔM_i in relation to the center of the cylinder. The equation (1) below illustrates the velocity of a water mass differential at the position θ on the cylinder:

$$\vec{v}_i = \left(\dot{x} + R\dot{\beta}(\theta)\cos(\theta)\right)\hat{i} - R\dot{\beta}(\theta)\sin(\theta)\hat{j}, \qquad (1)$$

where $i = 1$, $\dot{\beta}(\theta) = \dot{\beta}_1$ corresponds to $\Re_1$ and $i = 2$, $\dot{\beta}(\theta) = \dot{\beta}_2$ aligns with $\Re_2$. To evaluate the flux in the $\Re_1$ and $\Re_2$ regions, we conjecture that during the collision, mass, tangential momentum, and energy remain conserved. ΔM denotes the incoming water mass flux over a time interval of Δt. As displayed in Fig. 1(b), following the collision, ΔM partitions into ΔM_1 and ΔM_2. Thus, mass conservation is demonstrated by equation (2) as

$$\Delta M = \Delta M_1 + \Delta M_2, \qquad (2)$$

while equations (3) and (4) respectively encapsulate the conditions of momentum and energy conservation as

$$\Delta M_1 v_1^t - \Delta M_2 v_2^t = \Delta M v \sin(\theta_c) \qquad (3)$$

and

$$\frac{\Delta M_1 (v_1^t)^2}{2} + \frac{\Delta M_2 (v_2^t)^2}{2} = \frac{\Delta M v^2}{2}, \qquad (4)$$

In the equations (3) and (4), v_i^t represents the tangential components of the velocities of ΔM_i. Radial momentum is not conserved owing to the application of the normal force by the cylinder's surface. In effect,

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the normal force transforms radial momentum into tangential momentum. Moreover, the tangential velocities of ΔM_1 and ΔM_2 are given by $v^t_{1,2} = \vec{v}_{1,2} \cdot \hat{e}_\theta$, with $\hat{e}_\theta = \cos(\theta)\hat{\imath} - \sin(\theta)\hat{\jmath}$, as shown in Fig. 1 (a). In this scheme, $\Delta M_{1,2} = \rho, A_{1,2}, v^t_{1,2}, \Delta t$ whereby $A_{1,2}$ refers to the cross-sectional area of the flux in the $\Re_{1,2}$ regions. Moving forward, we will assume that the transverse area in the $\Re_1$ and $\Re_2$ regions are equal, or in other words, $A_1 = A_2$. Solving equations (2) to (4) for $A_{1,2}$ and $\beta_{1,2}$ (as shown in Appendix A), we discover the following:

$$A_{1,2} = \frac{A}{\sqrt{4 - 3\sin^2(\theta_c)}}, \tag{5}$$

and

$$\dot{\beta}_{1,2} = -\frac{\dot{x}}{R}\cos(\theta_c) + v\frac{\pm\sin(\theta_c) + \sqrt{4 - 3\sin(\theta_c)^2}}{2R}, \tag{6}$$

Here, the plus and minus signs correspond respectively to the indices 1 and 2. A significant limitation of equation (6) is the dependency of $\dot{\beta}_{1,2}$ on the cylinder velocity $\dot{x}$. This results in a nonlinear differential equation that necessitates numerical solution. However, for instances where $v \gg \dot{x}$, it is plausible to disregard the first term in equation (6). This "stationary approximation" assumes that the time taken for mass ΔM_i to move on the cylinder from the point of impact until it reaches the ground is minimal enough such that the cylinder essentially remains stationary in that time interval. The stationary approximation fails if the oscillation period compares to the time taken by the fluid to reach the ground, i.e., $\pi R/v \sim T$, where T signifies the oscillation period. Hence, sufficiently large fluxes are required so that $\pi R/v \ll T$. The parameters considered in this paper satisfy this condition. The behavior of the mass flux $\dot{M}_{1,2} = \lim \Delta t \to 0 \Delta M_{1,2}/\Delta t$ in the stationary approximation is depicted in Fig. (2). As θ transitions from $-\pi/2$ to $\pi/2$, $\dot{M}_1$ rises from zero to Av/R while $\dot{M}_2$ descends from Av/R to zero. Additionally, it's worth noting that for small values of θ, the flux behaves linearly. Moving forward, we will utilize equations (5) and (6) to estimate the mass flux $\dot{M}_{1,2} = \rho A_{1,2} R \dot{\beta}_{1,2}$ in the $\Re_1$ and $\Re_2$ regions.

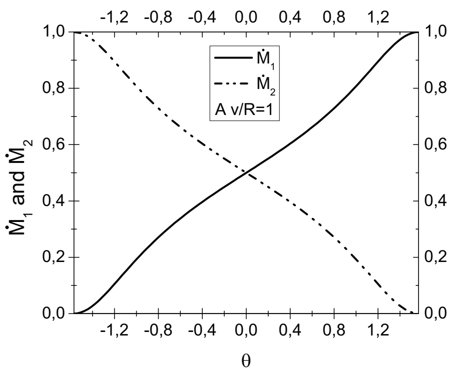

Figure 2: Mass flux in the regions $\Re_1$ and $\Re_2$ as functions of θ_c. A linear behavior is found for a wide region of the domain of the function.

4. Equations of Motion

To derive the equations of motion for the cylinder, we must ascertain the forces acting upon it. Essentially, we account for the interaction between the water and the cylinder using two forces. The first one is the impact force exerted by the water jet on the cylinder, applied at position θ_c. The second force stems from the water flowing around the cylinder and thus is applied across the entirety of the curved surface of the cylinder.

4.1. Impact force

In our model, the force arising from the impact of the jet on the cylinder, denoted as $\vec{F}_c$, is determined by the relationship between the net force and the momentum change of an incoming mass differential, ΔM. We operate under the assumption that the tangential component of the velocity of ΔM is preserved during the collision. This is a sensible assumption for fluids with low viscosity where the exerted force by the fluid is nearly perpendicular to the surface. Consequently, $\vec{F}_c$ is a radial force. Thus, we determine

$$F_c - \Delta M g\cos(\theta_c) = \frac{\Delta M_1 v^r_1 + \Delta M_2 v^r_2 + \Delta M v\cos(\theta_c)}{\Delta t}, \tag{7}$$

where the radial component of the velocity is given by $v^r_{1,2} = \vec{v}_{1,2} \cdot \hat{e}_r$. Using the Eqs. (1) and (7), it can be shown that the force applied during the collision by the cylinder on the mass differential ΔM is given by

$$F_c = \rho A v\left(\dot{x}\sin(\theta_c) + v\cos(\theta_c)\right), \tag{8}$$

where it is assumed that the impact is instantaneous, thus, ΔM, $\Delta t \to 0$, with $\Delta M/\Delta t \to \rho A v$. The force given by Eq. (8) is proportional to the incoming mass flow and not only depends on v but also on the speed of the cylinder.

4.2. Force due to the flowing water around the cylinder

The water circulating around the cylinder exerts an additional force on the cylinder, which can be split into normal and tangential components, N and f_v, respectively. To compute this force, we consider a mass differential, ΔM, at the position θ on the cylinder. The forces affecting ΔM are illustrated in Fig. 1(c), where the radial (ΔN) and tangential (Δf_v) forces exerted on ΔM are depicted. The force related to the pressure is represented as $\vec{F}_p$. The motion equations for ΔM corresponding to the vertical and horizontal directions are

$$\begin{aligned}\Delta N\cos(\theta) + \Delta f_v\sin(\theta) - \Delta M g + \Delta F_p\sin(\theta) \\ = -\Delta M R\dot{\beta}(\theta)^2\cos(\theta).\end{aligned} \tag{9}$$

and

$$\Delta N\sin(\theta)-\Delta f_v\cos(\theta)-\Delta F_p\cos(\theta) \\ =\Delta M\left(\ddot{x}-R\dot{\beta}(\theta)^2\sin(\theta)\right), \qquad (10)$$

respectively. Therefore, the total horizontal component of the normal force applied on the cylinder $N_x = -\int d\theta\Delta N\sin(\theta)$ can be written as

$$N_x = -\int_0^{2\pi} d\theta RA\rho\big(-R\dot{\beta}(\theta)^2+g\cos(\theta) \\ +\ddot{x}\sin(\theta)\big)\sin(\theta), \qquad (11)$$

where Eqs. (9) and (10) are used to eliminate ΔF_p and the dependence on ΔM is encoded in the term $RA\rho d\theta$, which accounts for the distributed mass in a differential sector $d\theta$ of the cylinder. The additional negative sign stems from Newton's third law, providing the correct force direction since in Eq. (10) we have the force on the water mass differential exerted by the cylinder. The integration can be readily performed to find

$$N_x=\rho AR^2\left(\dot{\beta}_1^2-\dot{\beta}_2^2\right)(1+\cos(\theta_c))-\rho A\pi R\ddot{x}. \qquad (12)$$

The first term in Eq. (12) includes the effect of the relative movement between the water mass on the cylinder and the cylinder itself. According to Eq. (6), this term is zero for the equilibrium position $\theta_c = 0$, but for arbitrary values of θ_c, its value depends on the competition between the mass fluxes in regions $\Re_1$ and $\Re_2$. The second term in Eq. (12) is the force required to move the mass of water around the cylinder with a horizontal acceleration of $\ddot{x}$.

The tangential component $\vec{f}_v$ due to the friction between the cylinder and the water is proposed to be proportional to the relative velocity between the cylinder and the mass ΔM [13]. In this way, the horizontal component of $\vec{f}_v$, f_v^x, can be written as

$$f_v^x = cR\left[\int_{\theta_c}^{\pi} d\theta\left|\dot{\theta}-\dot{\beta}_1\right|\cos(\theta) - \int_{\pi}^{2\pi+\theta_c} d\theta\left|\dot{\theta}+\dot{\beta}_2\right|\cos(\theta)\right] \\ = cx\left(\left|\dot{\theta}-\dot{\beta}_1\right|+\left|\dot{\theta}+\dot{\beta}_2\right|\right), \qquad (13)$$

with c a constant which depends on the viscosity of the fluid. Similarly, the torque due to f_v around the center of mass, τ_v, is given by

$$\tau_v=cR^2\left|\dot{\theta}-\dot{\beta}_1\right|(\pi-\theta_c)-cR^2\left|\dot{\theta}+\dot{\beta}_2\right|(\pi+\theta_c). \qquad (14)$$

As expected, both f_v^x and τ_v are zero at the equilibrium position for $\dot{\theta}=0$.

4.3. Resulting equations

Using Eqs. (8), (13) and (14) the equation of motion for the center of mass of the cylinder can be written as

$$N_x+f_v^x+f_r-F_c\sin(\theta_c)=M_b\ddot{x}, \qquad (15)$$

where f_r represents the friction force between the cylinder and the surface. The minus sign before F_c is due to Newton's third law. To determine f_r we consider the rotational motion around the center of mass which is given by

$$-f_rR+\tau_v=\frac{I_0}{R}\ddot{x}. \qquad (16)$$

Note that to write Eq. (16) we have used the "rolling without slipping" condition which completely determines the friction force, as stated before we assume that this condition holds despite the thin layer of water between the surface an the cylinder. Thus, using Eqs. (15) and (16) to eliminate $\ddot{\theta}$ we find

$$N_x+f_v^x+\frac{\tau_v}{R}-F_c\sin(\theta_c)=\left(M_b+\frac{I_0}{R^2}\right)\ddot{x}, \qquad (17)$$

where F_c, N_x, f_v^x and τ_v, are given by Eqs. (8), (12), (13) and (14), respectively. Equation (17) does not involve any fit parameter and can be solved numerically by standard methods such as Euler or Runge-Kutta methods. It is worth noting that to arrive to Eq. (17) we not only use conservation laws, but also, we have assumed that the motion of the water around the cylinder is given by Eq. (1). This last assumption is used to calculate the force applied by the cylinder on the flowing water. The third Newton's law does the rest of the work.

5. Results

For arbitrary values of the parameters Eq. (17) must be solved numerically. However, in the limit of small oscillations and zero viscosity ($c = 0$) Eq. (17) takes a simple form. In this case $\theta_c \approx -x/R$, $\beta_{1,2} \approx v/r \pm \theta_c v/(2r)$ and $A_{1,2} \approx A/2$. Similarly, the impact force F_c given by Eq. (7) can be written as

$$F_c \approx A\rho v\left(v-\frac{x\dot{x}}{R}\right), \qquad (18)$$

while the horizontal component of the normal force given by Eq. (12) takes the form

$$N_x=-\frac{1}{2}A\rho\pi\ddot{x}R-2A\rho v^2\frac{x}{R}. \qquad (19)$$

Using these results, it is easy to find that, for small oscillations, Eq. (17) reduces to

$$M_{eff}\,\ddot{x}=-\frac{A\,\rho\,v^2}{R}\,x, \qquad (20)$$

where we have defined the effective mass according to $M_{eff}=\pi\,A\,\rho\,R/2+M_b+I_0/R^2$. Therefore, in this limit the oscillation period is given by

$$T=2\pi\sqrt{\frac{R\,M_{eff}}{A\,\rho\,v^2}}. \qquad (21)$$

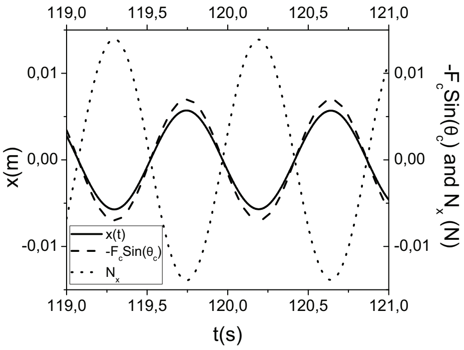

Figure 3: Behavior of the position of the center of the cylinder. The horizontal component of the force applied by the water around the cylinder is twice larger than the horizontal component of the impact force F_c.

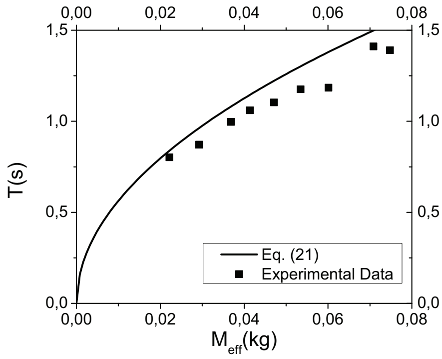

Figure 4: Behavior of the oscillation period as function of the effective mass.

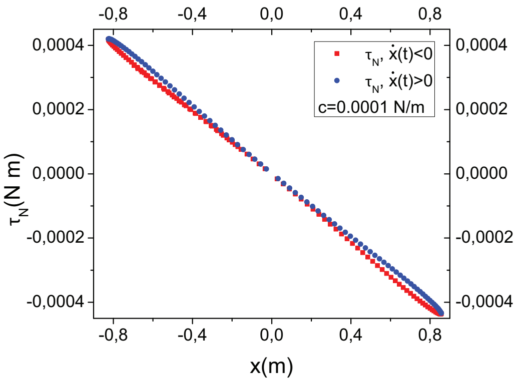

Figure 5: Behavior of the net torque τ_N as a function of $x(t)$.

In order to check the validity of the model, we solved numerically Eq. (17) taking $c = 0$ and using the parameters reported in Ref. [8], $A = 1.22 \times 10^{-5}\, m^2$, $\rho = 1000\, kg/m^3$, $v = 1.72\, m/s$, $M_b = 0.0124\, kg$, $R = 0.029\, m$ and $g = 9.81\, m/s^2$. The solution for long times is shown in Fig. 3, the position of the center of the cylinder is represented by the solid line. The forces N_x and F_c are included, dotted and dashed lines respectively. Note that N_x acts as a restoring force while the impact force tries to move away the cylinder from the equilibrium position. However, the maximum value of N_x is close to $0.014N$ which is almost twice the maximum value of the impact force. In fact, in the small oscillation limit, from Eqs. (18) and (19) it can be shown that $-F_c \sin(\theta_c) \approx A\rho v^2 x/R$ and $N_x \approx -2A\rho v^2 x/R$, i.e., the horizontal component of the impact force is the half of N_x. Our results show that the origin of the oscillations is due to the difference between the forces applied by the water in regions $\Re_1$ and $\Re_2$. For the parameters mentioned before, the authors of Ref. [8] experimentally found that the oscillation period is approximately $T \approx 0.8\, s$ which is close to that from Eq. (21), $T \approx 0.89\, s$ and in agreement with Fig. 3 which shows the numerical solution of Eq. (17).

The behavior of T as function of the M_{eff} is shown in Fig. 4, the experimental data reported in Ref. [8] were also included. Our model describes qualitatively and quantitatively the oscillations and the period T of the cylinder and, in contrast with the previous analytical model presented in [8], ours does not involve fit parameters.

Finally, Fig. 5 shows the behavior of the net torque $\tau_N = I_0\, \ddot{x}/r$ as a function of the position of the center of mass of the cylinder. We use the same parameters as before but now taking $c = 0.0001\, kg/s$. The plotted data were taken in the interval $2s > t > 1s$. The blue and red points correspond to the part of the trajectory where the cylinder is moving to the right and left, respectively. The friction force generates the “hysteresis” phenomena reported in Ref. [8] which is related with the energy loss and the subsequent decrease in the amplitude of oscillation.

However, for larger values of the flux, we can expect that the impact force is large enough to generate the opposite effect, i.e., increase the amplitude of oscillation [8]. Due to the assumptions used, especially the mass conservation during collision, this behavior is not reproduced by our model where for small oscillations the condition $2\,|F_c|\sin(\theta_c) \approx N_x$ guarantees the stability of the oscillations regardless the value of the parameters. This relation strongly depends on the assumption of mass conservation of water during the collision. If the speed before the collision of the incoming water is large enough the loss of water is not negligible and F_c can be larger than N_x throughout the entire trajectory. In this case there are not oscillations and the cylinder moves away from the water jet.

6. Conclusions

The model presented here, although simple and involving certain assumptions, provides both a qualitative and quantitative description of the oscillations experienced by a cylinder subjected to a vertical water jet. Our calculations are based on foundational mechanics concepts; hence, we believe that this system, as presented, can serve as an illustrative example in engineering dynamics courses. Despite the simplifications made, the predicted oscillation period in the small oscillations limit closely matches that found experimentally by Pagaud and Delance, with an error near 10%. In our model, the oscillations are a result of the force applied by the water flow around the cylinder. We determined that the period is proportional to the square root of the effective mass ($M_{eff}^{1/2}$), which aligns with the results from Pagaud's experiments. The effective mass incorporates three distinct contributions: the mass of the cylinder, the inertia associated with rotation, and the mass of the water around the cylinder. The latter contribution is included empirically in the model proposed in Pagaud's research but appears naturally in our model. The period also depends on the cylinder's radius, the incoming water flux, and the speed of this flux. Lastly, we would like to emphasize that the approach utilized in this study does not involve any fitting parameters.

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Supplementary material

The following online material is available for this article: Appendix A.

References

[1] https://iptnet.info/the-principles-of-ipt/
[2] T. Panitz and D. Wasan, AIChE J. **18**, 51 (1972).
[3] I. Reba, Scientific American **214**, 84 (1966).
[4] D.J. Tritton, *Physical Fluid Dynamics* (Springer, Holanda, 1977).
[5] I. Manabu and J.I. Olusegun, *The Coanda Effect* (Springer, New York, 2011).
[6] M. Kaushik, in *Governing Equations of Fluid Flows* (Springer, Singapore, 2019).
[7] M. Kaushik, *Potential Flow Theory* (Springer, Singapore, 2019).
[8] F. Pagaud and L. Delance, Emergent Scientist **5**, 1 (2021).
[9] D. Kleppner and R.K. Kolenkow, *An Introduction to Mechanics* (Cambridge University Press, Cambridge, 2010), 2 ed.
[10] A.J. French and M.G. Ebison, *Introduction to Classical Mechanics* (Springer, Holanda, 1986).
[11] R.P. Feynmann, *The Feynman Lectures on Physics* (Addison-Wesley, Boston, 2005), v. 1, 2 ed.
[12] J.M. Finn, *Classical Mechanics* (Jones & Bartlett Publishers, Burlington, 2009).
[13] Y. Cengel and J. Cimbala, *Fluid Mechanics: Fundamentals and Applications* (McGraw Hill, New York, 2006).