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Angular invariant quantum mechanics in arbitrary dimension

(Mecânica quântica angularmente simétrica em dimensões arbitrárias)

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Alguns dos problemas de mecânica quântica unidimensional são generalizados em coordenadas esféricas e em dimensão arbitrária. São tratados os problemas do poço de potencial infinito, o oscilador harmônico, a partícula livre, o potencial da função delta de Dirac, o poço de potencial finito e a barreira de potencial finito. As soluções da equação de Schrödinger são escritas em termos das funções de Bessel e Whittaker e relacionadas a teorias físicas multi-dimensionais, como a teoria de cordas.
Palavras-chave: problemas quânticos em coordenadas esféricas; poço de potencial com simetria esférica.

One dimensional quantum mechanics problems, namely the infinite potential well, the harmonic oscillator, the free particle, the Dirac delta potential, the finite well and the finite barrier are generalized for finite arbitrary dimension in a radially symmetric, or angular invariant, manner. This generalization enables the Schrödinger equation solutions to be visualized for Bessel functions and Whittaker functions, and it also enables connections to multi-dimensional physics theories, like string theory.
Keywords: radially symmetric quantum problems; radially symmetric quantum well.

1. Introduction

An introductory quantum mechanics course deals with solutions to one-dimensional problems, as can be seen in the commonly used textbooks on the subject. Three-dimensional problems, like angular momentum, scattering and the hydrogen atom, are not generalizations of one-dimensional problems, and they normally require either a specific solution method to the Schrödinger equation or an additional symmetry input. As a multidimensional approach to one-dimensional problems does not necessarily lead to more relevant multidimensional models, it is merely regarded as a curiosity. However, the importance of multi-dimensional problems has increased in physics, since string theory has given rise to the possibility that there could be more than three dimensions of space. For example, the Schrödinger equation was semi-classically solved in various dimensions in order to quantize pulsating strings [1, 2].

On the other hand, the more dimensions a problem has, the more possibilities of motion, and the more symmetries it can have to restrict these possibilities. This means that a generalization can be a choice, depending of the symmetries of the $n-$dimensional solution of the problem. In this article we answer the question of what happens when the Schrödinger equation is solved using a formalism which generalizes one-dimensional problems into angular invariant $n-$dimensional cases. Accordingly, the Schrödinger equation is transformed into the Bessel equation, whose solutions appear in physics problems which either have cylindrical symmetry or spherical symmetry [3, 4].

The multidimensional approach provides a deeper understanding of the physics of the one-dimensional problems, as well as the capabilities and limitations of the mathematical apparatus. As some of these results may be scattered throughout literature, it is useful, for both students and researchers, to have them presented in a single place.

This article is organized as follows: in section two, the infinite $n-$dimensional cylindrical quantum well is solved. In the third section, the quantum harmonic oscillator is studied in various dimensions, and it is shown that this problem can be described in terms of Bessel or Whittaker functions. In section four, the free-particle is analyzed, requiring a particular Dirac delta function. Section five deals with the Dirac delta potential. The finite well and the finite barrier are dealt with in the sixth seventh sections, respectively. Finally, a brief conclusion rounds off this article.

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2. The infinite square well

The Schrödinger equation can be written schematically as an Eigenvalue equation

$$\big(\hat{\Pi}^2+V\big)\Psi=\mathcal{E}\Psi, \tag{1}$$

where Ψ is the wave-function, $\mathcal{E}$ is the energy, V is the potential and $\hat{\Pi}^2$ is the momentum operator. Considering an $(n+1)-$dimensional space, the squared momentum operator, expressed by means of spherical coordinates, depends on a Laplacian operator that has a radial term and an angular term so that

$$\hat{\Pi}^2=-\frac{\hbar^2}{2m}\Big(\hat{\nabla}_r^2+\frac{1}{r^2}\hat{\nabla}_\theta^2\Big), \tag{2}$$

where $\hat{\nabla}^2_{a=r,\,\theta}$ is the term of the Laplace operator for the radial coordinate and for the angular coordinates. For a radial-only dependent potential, the wave-function, expressed as $\Psi(r,\,\theta)=s(r)\,w(\theta)$, so that $w(\theta)$ has n angular variables, splits Eq. (1) into two equations, namely

$$\frac{1}{s}\hat{\nabla}_r^2 s+\epsilon-v-\frac{M}{r^2}=0, \tag{3}$$

$$\frac{1}{w}\hat{\nabla}_r^2 w=-M, \tag{4}$$

where $\epsilon=2m\mathcal{E}/\hbar^2$, $v=2mV/\hbar^2$ and M is a separation constant that is zero for $n=0$. Equation (4) can be solved in terms of $n-$dimensional spherical harmonics, which can be found elsewhere [5, 6]. In order to solve the radial equation, it has been set

$$s=\frac{u(r)}{r^\nu}, \tag{5}$$

and thus Eq. (3) becomes,

$$r^2u''+(n-2\nu)r\,u'+\big[(\epsilon-v)r^2+\nu(\nu-n+1)-M\big]u=0. \tag{6}$$

The solution for Eq. (6) depends on the particular potential v. For an infinite potential well, the potential is

$$v=\begin{cases}0 & \text{if} \quad r<R\\ \infty & \text{if} \quad r>R,\end{cases} \tag{7}$$

and for a radial dependent-only wave-function, $M=0$, the choice $\nu=\frac{n-1}{2}$ leads to the Bessel equation,

$$r^2\,u''+r\,u'+\big(\epsilon\,r^2-\nu^2\big)\,u=0. \tag{8}$$

Thus, the wave-function is expressed in terms of Bessel functions

$$\Psi_n(r)=\frac{1}{r^\nu}\Big(aJ_\nu\big(\sqrt{\epsilon}\,r\big)+b\,Y_\nu\big(\sqrt{\epsilon}\,r\big)\Big), \tag{9}$$

where a and b are integration constants. The choice of $\nu=\frac{n-2}{2}$ would lead to the spherical Bessel equation, whose spherical Bessel functions, j_ν and y_ν, are related to the usual Bessel functions as

$$j_\nu(x)=\sqrt{\frac{\pi}{2x}}J_{\nu+\frac{1}{2}} \quad \text{and} \quad y_\nu(x)=\sqrt{\frac{\pi}{2x}}Y_{\nu+\frac{1}{2}}, \tag{10}$$

which generates an identical wave-function, thus, Eq. (9) is indeed the most general solution to the problem. As $Y_\nu(x)$ is divergent in $x=0$ for $n>0$, then $b=0$ because otherwise the wave-function is not normalizable, as we will see in a moment. The potential is infinite at $r>R$, thus $\Psi(R)=0$. Defining $r=r_N^{(n)}$ as the $N-$th zero of $J_{\frac{\nu-1}{2}}(r)$, the quantized energy is obtained from $\sqrt{\epsilon}\,R=r_N^{(n)}$, and it is expressed as

$$\mathcal{E}_N^{(n)}=\frac{\hbar^2}{2m}\left(\frac{r_N^{(n)}}{R}\right)^2. \tag{11}$$

As R is a free parameter and n is fixed by the geometry, the more excited the level of the energy, the more zeros the wave-function in the interval $(0,R)$ has.

The wave-function is interpreted as a density of probability of finding a quantum particle in the space, and the sum of all probabilities is defined to be equal to one. The normalization is the condition that warrants the probability of finding the particle to be one, namely

$$\int_{\mathcal{V}} d\tau\,\Psi\,\Psi^*=1. \tag{12}$$

The integral is calculated over the entire space $\mathcal{V}$ using the complex conjugate Ψ^* and the volume element $d\tau$. For a three-dimensional space parameterized in spherical coordinates, the well-known formula $d\tau=r^2\sin^2\theta\,dr\,d\phi$ applies. Using the general wave-function $\Psi(x)=\mathcal{N}\Phi(x)$, the normalization constant $\mathcal{N}$ ajusts the value of Eq. (12) to one. Radial wave-functions, so that $\Psi=\Psi(r)$, permit to integrate the angular terms of $d\tau$ and absorb them in the normalization constant. Thus, an $(n+1)-$dimensional space in spherical coordinates has the effective volume element $d\tau=r^n\,dr$. Finally, we calculate the normalizalized wave-functions using the integral

$$\int dr\,r\Big(J_\nu\big(\sqrt{\epsilon}\,r\big)\Big)^2= \tag{13}$$
$$\frac{r^2}{2}\Big[\Big(J_\nu\big(\sqrt{\epsilon}\,r\big)\Big)^2-J_{\nu+1}\big(\sqrt{\epsilon}\,r\big)\,J_{\nu-1}\big(\sqrt{\epsilon}\,r\big)\Big],$$

so that $\int_0^\infty dr\,r^n|\Psi|^2=1$ implies the normalized wave-function

$$\Psi(r)=\frac{1}{R}\sqrt{\frac{2}{-J_{\frac{\nu+1}{2}}\big(\sqrt{\epsilon}\,r\big)\,J_{\frac{\nu-3}{2}}\big(\sqrt{\epsilon}\,r\big)}}\times$$
$$\frac{J_{\frac{\nu+1}{2}}\big(\sqrt{\epsilon}\,r\big)}{r^{\frac{n-1}{2}}}. \tag{14}$$

The wave-function is rotationally invariant. In this sense, the solution calculated above is not valid in the $n = 0$ case; the usual one-dimensional $n = 0$ solution has anti-symmetrical states due to the negative values that the argument of the wave-function has in this case, which are not included in a rotationally invariant $(n+1)-$dimensional wave-function.

3. The harmonic oscillator

The one-dimensional harmonic oscillator is solved analytically in terms of Hermite polynomials, as originally demonstrated by Schrödinger [7]. This method uses the asymptotic behavior of the wave-function to simplify the problem and to obtain the Hermite equation. However, using the variable $\rho = \mu r^2$ and the index $\nu = \frac{n+1}{2}$ in Eq. (6) it is obtained

$$u'' + \Big[-\frac{1}{4} + \frac{\epsilon}{4\,\mu\,\rho} - \frac{(n+1)(n-3)+4M}{16\rho^2} \Big]\, u = 0, \quad (15)$$

where $\mu = \frac{m\omega}{\hbar}$ and the prime means differentiation relative to ρ. Equation (15) with $M = 0$ is the Whittaker equation, whose general solution is

$$u(\rho) = a\, M_{\lambda,\,\eta}(\rho) + n\, W_{\lambda,\,\eta}(\rho), \quad (16)$$

where $\lambda = \frac{\epsilon}{4\mu} = \frac{\mathcal{E}}{2\hbar\omega}$, $\eta = \pm\frac{n-1}{4}$ and a and b are integration constants. The Whittaker functions $M_{\lambda,\,\eta}$ and $W_{\lambda,\,\eta}$ may be expressed as

$$M_{\lambda,\,\eta}(\rho) = e^{-\frac{\rho}{2}} \rho^{\frac{1}{2}+\eta}\, M\Big(\frac{1}{2} + \eta - \lambda,\, 1+2\eta,\, \rho\Big), \quad (17)$$

$$W_{\lambda,\,\eta}(\rho) = e^{-\frac{\rho}{2}} \rho^{\frac{1}{2}+\eta}\, U\Big(\frac{1}{2} + \eta - \lambda,\, 1+2\eta,\, \rho\Big). \quad (18)$$

$M(p, q, z)$ and $U(p, q, z)$ are confluent hyper-geometric functions known as Kummer functions. As always, wave-functions must be normalizable, and Kummer functions diverge if the first index p is not a negative integer. Thus, normalizable wave-functions are orthogonal polynomials of the order $N \in \mathbb{N}$, obtained through the following relations

$$M\Big(-N,\, \frac{1}{2},\, z^2\Big) = \frac{(-1)^N\, N!}{(2N)!}\, H_{2N}(z), \quad (19)$$

$$M\Big(-N,\, \frac{3}{2},\, z^2\Big) = \frac{(-1)^N\, N!}{(2N+1)!}\, \frac{H_{2N+1}(z)}{2z}, \quad (20)$$

$$U\Big(\frac{1-N}{2},\, \frac{3}{2},\, z^2\Big) = \frac{H_N(z)}{2^N z}, \quad (21)$$

$$U\big(-N,\, \alpha+1,\, z\big) = (-1)^N\, N!\, L_N^{(\alpha)}(z) = = (-1)^N\, (\alpha+1)_N\, M\big(-N,\, \alpha+1,\, z\big), \quad (22)$$

where $(\alpha+1)_N$ is a Pochhammer symbol, $H_N(z)$ are the Hermite polynomials, and $L_N^{(\alpha)}(z)$ are the generalized Laguerre polynomials. The one-dimensional solution for the harmonic oscillator is obtained by setting $n = 0$. In this situation, $b = 0$ in Eq. (16), according to two reasons: when N is even, Eq. (22) is divergent in $z = 0$ and so the wave-function is not normalizable, and when N is odd, the wave-function is normalizable but the energy integral [4]

$$\mathcal{E} = \int_{-\infty}^{\infty} dr\, r^n \big|\hat{\Pi}\Psi\big|^2 \quad (23)$$

is divergent, thus solutions involving Eq. (21) have to be discarded. The remaining conditions (19) and (20) imposed on Ψ in the $n = 0$ case, enables us to write the energy spectrum of the linear harmonic oscillator

$$\mathcal{E}_{2N} = \hbar\omega\Big(2N + \frac{1}{2}\Big), \quad (24)$$

$$\mathcal{E}_{2N+1} = \hbar\omega\Big(2N + \frac{3}{2}\Big). \quad (25)$$

Using the orthogonality condition for Hermite polynomials

$$\int_{-\infty}^{\infty} dx\, e^{-x^2}\, H_L(x)\, H_K(x) = 2^K\, K!\sqrt{\pi}\delta_{K,\,L}, \quad (26)$$

the normalized wave-function for the one-dimensional case is obtained with,

$$\Psi_K = \frac{1}{\sqrt{2^K\, K!}} \Big(\frac{\mu}{\pi}\Big)^{\frac{1}{4}} e^{-\frac{1}{2}\mu r^2} H_K\big(\sqrt{\mu}\, r\big) \quad (27)$$

where Eq. (27) is valid for $K = 2N$ and $K = 2N+1$. For the $n-$dimensional case, Eq. (22) indicates that the solution is given in terms of generalized Laguerre polynomials, $L_N^{(\alpha)}$ and thus $a = 0$ is established in Eq. (16), without loss of generality. When comparing Eqs. (18) and (22) we get $\alpha = \pm\frac{n1}{2}$. As $n > 0$ and $\alpha > 1$, the plus sign must be chosen. Also from Eqs. (18) and (22), we get the energy spectrum

$$\mathcal{E}_N = \hbar\omega\Big(2N + \frac{m+1}{2}\Big). \quad (28)$$

Using the orthogonality relation

$$\int_0^{\infty} dx\, x^\alpha e^{-x}\, L_M^{(\alpha)}(x)\, L_N^{(\alpha)}(x) = \Gamma(1+\alpha) \binom{N+\alpha}{N} \delta_{M,\,N} \quad (29)$$

the normalized wave-function is

$$\Psi_N(r) = \mu^{1/4}(-1)^N \sqrt{\frac{2\Gamma(N+1)}{\Gamma\Big(N + \frac{n+3}{2}\Big)}} \times e^{-\frac{1}{2}\mu r^2} L_N^{(\frac{n-1}{2})}(\mu r^2) \quad (30)$$

The result shows that the energy depends explicitly on the angular dimension n and that the wave-function has a rotational symmetry, as expected from the angular independence imposed by using $M = 0$ in the Schrödinger equation.

4. The free particle

The equation that describes a free particle is similar to the equation for the infinite well, as both have zero potential. The difference resides in the boundary conditions. In the $n = 0$ case, the solution is expressed in terms of complex exponentials, and in the arbitrary n case, it is expressed in terms of Hankel functions, which describe cylindrical travelling waves expressed in terms of Bessel functions, namely,

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z), \qquad \text{and} \tag{31}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z). \tag{32}$$

A general cylindrically symmetrical solution (5) of Eq. (3) for the free particle is,

$$\Psi_{\frac{n-1}{2}}(r) = \frac{1}{r^{\frac{n-1}{2}}}\Big(a\, H^{(1)}_{\frac{n-1}{2}}(\sqrt{\epsilon}r) + b\, H^{(2)}_{\frac{n-1}{2}}(\sqrt{\epsilon}r)\Big), \tag{33}$$

in which $H_\nu^{(1)}$ is a travelling mode towards $r = 0$, and $H_\nu^{(2)}$ is a travelling mode towards $r \to \infty$ and a and b are integration constants. As in the $n = 0$ case, the free-particle wave-function is not normalizable and it is understood as a wave packet which obeys

$$\int_0^\infty dr\, r^n \Psi^\dagger_\eta \Psi_\delta = \delta^{n+1}(\eta - \delta), \tag{34}$$

where $\delta^{n+1}(\eta - \delta)$ is an $(n+1)$dimensional Dirac delta function. By substituting two wave-functions like Eq. (33) with energies η and δ in Eq. (34), and considering

$$H_\nu^{(1)}(z) = \frac{1}{i\sin\nu\pi}\Big(J_{-\nu}(z) - e^{-i\nu\pi}J_\nu(z)\Big), \tag{35}$$

$$H_\nu^{(2)}(z) = \frac{1}{i\sin\nu\pi}\Big(-J_{-\nu}(z) + e^{i\nu\pi}J_\nu(z)\Big),$$

and $J_{-\nu} = (-1)^\nu J_\nu$, we find that

$$\delta^{n+1}(\epsilon - \eta) = \sqrt{\epsilon}\int_0^\infty dr\, r\, J_\nu(\sqrt{\epsilon}r)\, J_\nu(\sqrt{\eta}r), \tag{36}$$

which is the definition of a Dirac delta function in terms of Bessel functions, and thus the wave-function satisfies the mathematical requirements in order to describe a cylindrically symmetric free-particle. However, one physical aspect is missing: the behavior of the wave-function at $r = 0$. There is no external force or internal interaction, thus at this point the travelling wave must change direction and maintain intensity. This means that the integration constants, which give the wave amplitude of the incoming and outgoing waves, must have the same modulus in order to generate equal amplitudes for the wave-function at $r = 0$. As $Y_\nu(0)$ is divergent and cannot contribute to the solution, the wave-function is simply

$$\Psi_{\frac{n-1}{2}}(r) = \frac{1}{r^{\frac{n-1}{2}}}\, J_{\frac{n-1}{2}}(\sqrt{\epsilon}r). \tag{37}$$

Analogous to the $n = 0$ case, where wave-functions can be expressed in terms of a Fourier transform which allows the free particle to be interpreted as a wave packet, the same can be achieved here by expressing the free particle wave function as a Hankel transform [8]

$$\Psi(r) = \int_0^\infty d\sqrt{\epsilon}\,\phi(\sqrt{\epsilon})\, J_\nu(\sqrt{\epsilon}r)\,(\sqrt{\epsilon}r)^{1/2}, \tag{38}$$

for an appropriate function $\phi(\sqrt{\epsilon})$. Thus, the analogy between the $n = 0$ and the arbitrary n is complete. Physically the general $n-$dimensional case is more symmetrical because the wave-functions can only have one amplitude for both modes, something that does not constrain the $n = 0$ one-dimensional case.

5. The delta function potential

In this case there is a Dirac delta function potential,

$$V = \pm g\,\delta(r - R) \tag{39}$$

in which $g > 0$ is the coupling constant of the potential. A negative sign in Eq. (39) means a potential well and a positive sign means a potential barrier. Scattered states are possible for both signs of the potential, and a bound state occurs in the potential well for negative energies, which is discussed in the following subsection.

5.1. Bound state

The Schrödinger equation with negative energy $\mathcal{E} = -|\mathcal{E}|$ is expressed as

$$\nabla^2\Psi + \gamma\delta(r - R)\Psi = \epsilon\Psi, \tag{40}$$

where $\gamma = \frac{2m}{\hbar^2}g$ and $\epsilon = \frac{2m}{\hbar^2}|\mathcal{E}|$. The general solution to this problem is given in terms of modified Bessel functions I_ν and K_ν, integration constants a and b, and $\nu = \frac{n-1}{2}$, so that

$$\Psi(r) = \frac{1}{r^\nu}\Big(a\, I_\nu(\sqrt{\epsilon}\, r) + b\, K_\nu(\sqrt{\epsilon}\, r)\Big). \tag{41}$$

The modified Bessel function I_ν is divergent at $r \to \infty$ and at $r \to 0$, $K_\nu \to \infty$, thus the wave-function is

$$\Psi(r) = \begin{cases} \Psi_I = a\,\frac{I_\nu(\sqrt{\epsilon}\, r)}{r^\nu} & \text{if} \quad r < R, \\ \Psi_{II} = b\,\frac{K_\nu(\sqrt{\epsilon}\, r)}{r^\nu} & \text{if} \quad r > R. \end{cases} \tag{42}$$

At $r = R$, $\Psi_I = \Psi_{II}$ and one integration constant is eliminated

$$a = \frac{K_\nu(\sqrt{\epsilon}\, R)}{I_\nu(\sqrt{\epsilon}\, R)}\, b. \tag{43}$$

On the other hand, the first derivative of the wave-function is not continuous at $r = R$, as can be seen from integrating (40) in a $r = R$ neighborhood, which gives

$$\Delta\big(r^n \Psi'\big) = -\gamma \Psi(R). \tag{44}$$

The $\epsilon\Psi$ term is eliminated from Eq. (40) by integration, and this does not contribute to Eq. (44). This means that the energy sign of the energy is not important in order to determine whether the particle is bound or free; all information regarding this is in the potential sign. At $r = R$

$$\begin{aligned}\Delta\big(r^n \Psi'\big) &= R^n \big(\Psi'_{II}(R) - \Psi'_I(R)\big) = \\ &= -R^{n-\nu}\sqrt{\epsilon}\big(a\, I_\nu(\sqrt{\epsilon}\, R) - b\, K_\nu(\sqrt{\epsilon}\, R)\big). \end{aligned} \tag{45}$$

Using Eqs. (43), (44), (45), and the Wronskian [8],

$$K_\nu(x)\, I_{\nu+1}(x) + K_\nu(x)\, I_{\nu+1}(x) = \frac{1}{x},$$

$$I_\nu(\sqrt{\epsilon}\, R)\, K_\nu(\sqrt{\epsilon}\, R) = \frac{1}{\gamma\, R} \qquad \text{is obtained.}$$

Equation (46) is a transcendental equation and it enables us to determine the energy numerically or graphically for each n. However, some particular cases can be calculated. For

$$x \to \infty, \qquad K_\nu(x)\, I_\nu(x) \to \frac{1}{2x} \qquad \text{and thus}$$

$$\epsilon = \frac{\gamma^2}{4} \qquad \text{or} \qquad |\mathcal{E}| = \frac{mg^2}{2\hbar^2}, \tag{46}$$

which is the unique bound state of this regime. On the other hand if $x \ll 1$, then

$$K_\nu(x)\, I_\nu(x) \to \frac{1}{2\nu} - \frac{x^2}{2\nu(\nu^2-1)}, \tag{47}$$

where $\nu > 1$ and the energy for this regime is

$$\epsilon = 2\frac{\nu^2-1}{R}\Big(1 - \frac{2\nu}{\gamma\, R}\Big), \tag{48}$$

which is also a one−state solution only, as has been observed in the well-known one-dimensional case.

5.2. Scattering state

In this problem, the particle comes from infinity towards $r = 0$ and is scattered by a Dirac delta well at $r = R$. In fact, the transmitted wave is totally reflected at $r = 0$, thus in the $r < R$ region the waves have the same intensity in directions; accordingly, the wave-function in

$$\Psi(r) = \begin{cases} \Psi_I = a\,\frac{J_\nu(\sqrt{\epsilon}\, r)}{r^\nu} & \text{if} \quad r < R \\ \Psi_{II} = \frac{1}{r^\nu}\Big(b\, H_\nu^{(1)}(\sqrt{\epsilon}\, r) + H_\nu^{(1)}(\sqrt{\epsilon}\, r)\Big) & \text{if } r > R \end{cases}$$

so that $\nu = \frac{n-1}{2}$ and $H_\nu^{(2)}$ describes the incident wave. From the continuity of the wave function and the integration of the Schrödinger equation, we obtain

$$\begin{aligned} &a\, J_\nu - b\, H_\nu^{(1)} = H_\nu^{(2)} \\ &a\, J_{\nu+1} + b\Big(\frac{\gamma}{\sqrt{\epsilon}} H_\nu^{(1)} + H_{\nu+1}^{(1)}\Big) = H_{\nu+1}^{(2)} - \frac{\gamma}{\sqrt{\epsilon}} H_\nu^{(2)}. \end{aligned} \tag{49}$$

All Bessel functions are evaluated at $\sqrt{\epsilon}\, R$. Using a Wronskian for Hankel and Bessel functions, the above system can be solved for a and b, whose modulus give us the reflection rate and the transmission rate, namely

$$R = |a|^2 = 1, \qquad \text{and} \tag{50}$$

$$T = |b|^2 = \frac{16}{\big(\pi\gamma R J_\nu Y_\nu\big)^2 + \big(\pi\gamma R J_\nu^2 - 2\big)^2}.$$

A reflection rate equal to one is understandable considering the fact that at $r = 0$ the wave is totally reflected, and as the wave-function describe stationary states, everything coming from infinite will be reflected. On the other hand, the transmission rate is something altogether more subtle. It may be greater than one, and if $R = 1$, it would be expected that $T = 0$. However, there is a reflection of the wave inside the region $r \leq R$ at $r = R$, and thus it is understandable that, within this region, the intensity of the wave will be greater than outside the region; the incoming wave is not immediately reflected to infinity and in fact a stationary wave-function is generated by totally reflection at the origin $r = 0$ and a partial transmission at $r = R$. Thus, T cannot be interpreted as a transmission of the incoming wave, but as a relative intensity of the beams in the confined region and open region. Inside each region, the incoming beam and the outgoing beam have equal intensities. The intensity of the stationary wave drops to zero if the position of the potential $R \to \infty$, but in the zeros of the Bessel functions, it is four times greater than the incoming wave, irrespective of how far the zero is from the origin of the coordinate system.

The relative intensity of the wave-functions also enables energy quantization according to the value of T, which is an oscillating function. For each particular value of T, there is an infinite spectrum of energy where the transmission has this particular value. Thus, it can be said that the energy is quantized for this system, because only particular values of the energy are permitted.

One last comment about this case must be made about the wave-function. In the region $r > R$ there are incoming and outgoing waves represented by Hankel functions. The intensity of these waves is equal, so $|b|^2 = 1$, although the coefficients are not necessarily equal, thus, by ansatz, $b \neq 1$. In the $r < R$ region, the situation is different, and the intensity and the Hankel functions coefficients functions must be equal in both of the directions because $Y_\nu \to \infty$ at $r \to 0$, something which does not occur in the $r > R$ region,thus the greater generality of the wave-function there.

5.3. Dirac delta function barrier

The case of the Dirac delta function is totally analogous to the scattering of the Dirac delta function well tackled above, the only difference being that the sign of γ sign in the potential term is flipped from plus to minus. This change, however, does not alter any of the results, which depend only on γ^2, so the well and the barrier are physically indistinguishable.

A physical analogy of these models can be executed with a laser beam, which is produced from an oscillating light-wave inside a partially reflecting chamber. A light wave is produced inside the device and when the intensity of the wave inside it is high enough, the coherent light escapes through one of the sides of the chamber, which acts as a barrier, in a situation similar to the Delta scattering. To be more realistic, the model would, of course, need a source at $r = 0$.

6. The finite potential well

In this case the potential is

$$v = \begin{cases} v_I = -v_0 = -\frac{2m}{\hbar^2}\mathcal{V}_0 & \text{if} \quad r < R, \\ v_{II} = 0 & \text{if} \quad r > R, \end{cases} \tag{51}$$

where $\mathcal{V}_0 > 0$ and the potential describes a cylindrical well whose center is located at $r = 0$. There are two possible solutions: a bounded-state solution with negative energy and a scattering-state solution with positive energy.

6.1. bound states

This problem has negative energy $\mathcal{E} = -|\mathcal{E}|$ and $\nu = \frac{n-1}{2}$ and $M = 0$ were chosen in Eq. (6), thus obtaining

$$r^2\, u'' + r\, u' + \big(\mathcal{Q}_a\, r^2 - \nu^2\big) u = 0, \tag{52}$$

so that,

$$\mathcal{Q}_a = \begin{cases} \mathcal{Q}_I = v_0 - |\epsilon| & \text{if} \quad r < R; \\ \mathcal{Q}_{II} = -|\epsilon| & \text{if} \quad r > R. \end{cases} \tag{53}$$

The general solution to both regions is

$$u_I = a\, J_\nu\big(\sqrt{\mathcal{Q}_I}\, r\big) + b\, Y_\nu\big(\sqrt{\mathcal{Q}_I}\, r\big), \quad \text{and} \tag{54}$$

$$u_{II} = c\, I_\nu\big(\sqrt{\epsilon}\, r\big) + d\, K_\nu\big(\sqrt{\epsilon}\, r\big). \tag{55}$$

The wave-function must be finite at $r = 0$ and at $r \to \infty$, thus $b = c = 0$. The continuity of the wave-function and its first derivative at $r = R$ generates

$$\frac{K_\nu\big(\sqrt{\epsilon}\, R\big)}{K_{\nu+1}\big(\sqrt{\epsilon}\, R\big)}\frac{J_{\nu+1}\big(\sqrt{\epsilon}\, R\big)}{J_\nu\big(\sqrt{\epsilon}\, R\big)} = \sqrt{\frac{|\mathcal{E}|}{\mathcal{V}_0 - |\mathcal{E}|}}. \tag{56}$$

This transcendental equation is solved numerically for each n, and the intersection points of the graphs of both sides give us the quantized energy. Quantized energy may be obtained for specific cases. If $\mathcal{E} \approx \mathcal{V}_0$, then $J_\nu \to 0$, thus the quantized energy comes from, $\sqrt{\mathcal{Q}_I} R = x_N^{(n)}$

$$\mathcal{E}_N = \mathcal{V}_0 - \frac{2m}{\hbar^2}\left(\frac{x_N^{(\nu)}}{R}\right)^2, \tag{57}$$

where $x_N^{(\nu)}$ is the Nth zero of J_ν. In this situation $R \gg x_N^{(\nu)}$. If $n = 0$, Bessel functions turn to trigonometric functions and the known regular spacing among these zeros for the one dimensional well appears. Other quantizing possibilities come from the $\mathcal{V}_0 \gg \mathcal{E}$ regime. In this case the quantized energy comes from the zero of $J_{\nu+1}$, namely $\sqrt{\epsilon}\, R = x_N^{(\nu)}$. This case can be understood as a deep well or tiny energy.

6.2. Scattering states

In this case the state has positive energy and the potential is the same used in the bounded states (51). As discussed in the case of the free particle, the intensity of the wave-function is maintained at $r = 0$, and so the solution is simply,

$$\Psi(r) = \begin{cases} \Psi_I \;= b\,\frac{J_\nu\left(\sqrt{\mathcal{P}}\, r\right)}{r^\nu} & \text{if} \quad r < R; \\ \Psi_{II} = a\,\frac{H_\nu^{(1)}\left(\sqrt{\epsilon}\, r\right)}{r^\nu} + \frac{H_\nu^{(2)}\left(\sqrt{\epsilon}\, r\right)}{r^\nu} & \text{if} \quad r > R, \end{cases}$$

so that $\mathcal{P} = \epsilon + v_0$. From the continuity of the wave-function and its first derivative, we obtain $|b|^2 = 1$ and,

$$T = |a|^2 = \frac{16/\big(\pi\,\epsilon\,R^2\big)}{\Big(\tilde{J}_\nu\,J_{\nu+1} - \mu\,J_\nu\,\tilde{J}_{\nu+1}\Big)^2 + \Big(\tilde{J}_\nu\,Y_{\nu+1} - \mu\,Y_\nu\,\tilde{J}_{\nu+1}\Big)^2}$$

where $\mu = \sqrt{\frac{\mathcal{V}_0}{\mathcal{E}} + 1}$, $\tilde{J}_\nu = J_\nu(\sqrt{\mathcal{P}}\,R)$, $J_\nu = J_\nu(\sqrt{\epsilon}\,R)$ and $Y_\nu = Y_\nu(\sqrt{\epsilon}\,R)$. The result is compatible with the situation for scattered states observed in the delta function model, where the incoming wave is totally reflected at $r = R$ because it is totally reflected at $r = 0$, and the intensity of the wave-function is greater in the $r < R$ region because the outgoing wave is partially transmitted at $r = R$. For each particular value of T, which is an oscillating function, there is an infinite energy spectrum that gives us this value, and then the energy is quantized according to this particular value. This result has also already been obtained for the scattering in the Dirac potential.

7. Conclusion

In this article several one-dimensional quantum mechanics problems have been generalized in an angular invariant manner. The results confirm expectations such as a contribution to the zero-point energy of the harmonic oscillator due to the angular dimension, and several results are not so obvious: the equality of the wave-function intensities to free particles, the existence of various quantized states in the Dirac delta potential and the quantum scattering states for the finite well. It is hoped that this set of results will be useful in understanding quantum-mechanics problems that link angular invariance and multi-dimensionality.

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