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A derivation of the Stokes theorem

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Motivated by electrodynamics we discuss a derivation of the Stokes theorem which is based on the variations of the fluxes and line integrals of vector fields. We show how this procedure can be extended to higher rank tensors in the context of the explicitly Lorentz invariant equations of electromagnetism and finally we present a general derivation of the Stokes theorem for differential forms.

Keywords: Stokes theorem, Differential forms, Lorentz Invariant Integral Maxwell equations.

1. Introduction

The laws of electrodynamics as they were first formulated are based on the Faraday's idea of lines of field and their fluxes through spatial surfaces and circuitations along curves. The understanding of these laws in terms of vector fields in $\mathbb{R}^3$, namely, the electric and magnetic fields, was later introduced by Maxwell, as well as the complete set of linear partial differential equations describing a huge variety of electromagnetic phenomena.

The Stokes theorem provides the natural bridge between the integral equations and the Maxwell equations; in fact, given the integral equations one can see the corresponding differential equations as local consistency conditions on the fields required by the Stokes' theorem, which is a mathematical identity.

In many occasions, the process of derivation of a given mathematical relation can itself bring more insight into its understanding. In physics text-books [1, 2] the discussion of the Stokes theorem is based on the projection of $\mathbb{R}^3$ vector fields on plaquettes and pill-boxes. In basic calculus text-books [3] the Stokes theorem is generally verified directly and discussed through examples and in more advanced texts in mathematics [4], the approach is probably too dense for undergraduate students to appreciate its content.

The main idea of this paper is to present a simple way to derive the Stokes theorem based on an interesting approach using the concept that is intrinsic to the so called integral equations of electromagnetism: that the behaviour or the eletric and magnetic fields is given by the changes of their fluxes. Thus, we used the context of electrodynamics as a motivation since this is where generally this theorem and its applications first appear for physics students.

The discussion of the derivation of the Stokes theorem can be understood in three different levels: starting in section 3, we derive the curl and divergence theorems for vector fields in $\mathbb{R}^3$ in a way that is reachable for students with vector calculus knowledge; then, in section 4 we apply the scheme of derivation for tensors in Minkowski space-time and present the argument using the Lorentz invariant integral equations of electromagnetism. Finally, in section 5 we give a more general derivation of the Stokes theorem for differential forms and some final considerations are given in section 6.

2. Stokes: The Connection Between Faraday and Maxwell

The Maxwell equations of electromagnetism are a set of linear partial differential equations which define the dynamics of the electric and magnetic vector fields[1] $\mathbf{E}$ and $\mathbf{B}$. These equations are in fact equations for the curl and divergence of these fields and in the Gaussian system of units they read

$$\boldsymbol{\partial} \cdot \mathbf{E} = 4\pi\rho \tag{1}$$

$$\boldsymbol{\partial} \times \mathbf{B} - \frac{1}{c}\partial_t \mathbf{E} = \frac{4\pi}{c}\mathbf{j} \tag{2}$$

and

$$\boldsymbol{\partial} \cdot \mathbf{B} = 0 \tag{3}$$

$$\boldsymbol{\partial} \times \mathbf{E} + \frac{1}{c}\partial_t \mathbf{B} = 0 \tag{4}$$

where ρ and $\mathbf{j}$ are respectively the electric charge and current densities and $\boldsymbol{\partial}$ and ∂_t stand for the gradient operator and the partial derivative in time respectively.

The Faraday idea of lines of fields was crucial in the development of the mathematical formalism known as field theories, on which is based the construction of the fundamental theories of Nature and several models describing physical phenomena. With the concept of

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[1] In free 3-dimensional Euclidean space $\mathbb{R}^3$.

field, the problem of action-at-a-distance in electrodynamics, namely, the fact that electrically charged bodies will attract or repel each other when separated in space by a finite distance can be eliminated as the interaction happens from one point to another, infinitesimally near, obeying the differential Maxwell equations.

On the other hand, originally the laws of electrodynamics were given in their integral form, namely, as equations for the flux and circulation of the electric and magnetic fields, obtained from empirical investigations.

The flux of a vector field $\mathbf{v}$ in $\mathbb{R}^3$ is defined by its projection on a 2-dimensional orientable surface Σ as

$$\Phi(\mathbf{v},\Sigma) \equiv \int_\Sigma \mathbf{v}\cdot d\mathbf{S}, \tag{5}$$

with $d\mathbf{S}$ standing for the surface's area element; the line integral of this vector field is defined by its projection along a 1-dimensional orientable curve γ as

$$f(\mathbf{v},\gamma) \equiv \int_\gamma \mathbf{v}\cdot d\boldsymbol{\ell}, \tag{6}$$

with $d\boldsymbol{\ell}$ giving the curve's line element.

With these definitions, the integral equations of electromagnetism read

$$\Phi(\mathbf{E},\partial\Omega) \;=\; 4\pi Q, \tag{7}$$

$$f(\mathbf{B},\partial\Sigma) - \frac{1}{c}\frac{d\Phi(\mathbf{E},\Sigma)}{dt} \;=\; \frac{4\pi}{c}\Phi(\mathbf{j},\Sigma) \tag{8}$$

and

$$\Phi(\mathbf{B},\partial\Omega) \;=\; 0, \tag{9}$$

$$f(\mathbf{E},\partial\Sigma) + \frac{1}{c}\frac{d\Phi(\mathbf{B},\Sigma)}{dt} \;=\; 0. \tag{10}$$

with Ω and Σ a 3-dimensional spatial volume and a 2-dimensional spatial surface respectively and

$$Q \equiv \int_\Omega \rho\, dV, \tag{11}$$

the total electric charge in the volume Ω.

While the differential Maxwell equations define local relations between the fields, the integral equations deal with globally defined quantities. Moreover, these integral equations establish relations between fields on the border of volumes and surfaces with fields inside them. It is through the divergence and curl theorems, which here are referred to as the Stokes theorem, that one can move from the integral laws to the differential Maxwell equations since this theorem defines relations between fields on borders with fields inside volumes and surfaces.

3. A Derivation of the Curl and Divergence Theorems

In the context of the integral equations of electromagnetism we deal with the flux and circulation of the electric and magnetic fields and how these quantities change due to the presence of electrically charged matter sources. Let us take, for instance, the Ampère-Maxwell law (8): it relates the circulation of the magnetic field around the border of the surface Σ with the flux of electric current and the variation in time of the flux of electric field inside this surface. So, in order to establish a local equation between these fields it is necessary to find how the circulation of the magnetic field can be written in terms of something evaluated also in the surface Σ, for instance. Let us now see how this can be done.

Consider the line integral of the field $\mathbf{v}$ (generically the electric or magnetic fields) along the path γ_0 with coordinates x^i, $i = 1,2,3$, which we will parameterize with σ conveniently chosen to vary from 0 to 2π corresponding to the initial and final points of the path, $x_0 \equiv x^\mu(\sigma = 0)$ and $x_{2\pi} \equiv x^\mu(\sigma = 2\pi)$ respectively[2]:

$$f(\mathbf{v},\gamma) \equiv \int_0^{2\pi} v_i \frac{dx^i}{d\sigma} d\sigma. \tag{12}$$

Let us then consider a smooth infinitesimal transformation of γ_0 given by $x^i \to x^i + \delta x^i$, such that its endpoints are kept unchanged, i.e., $\delta x^\mu|_{x_0,x_{2\pi}} = 0$.

As we vary the path so does the line integral of the vector field change and its infinitesimal variation can be calculated as follows

$$\begin{aligned}\delta f \;=&\; \int_0^{2\pi} \delta v_i \frac{dx^i}{d\sigma} d\sigma + \int_0^{2\pi} v_i \frac{d\delta x^i}{d\sigma} d\sigma \\ =&\; \int_0^{2\pi} \partial_j v_i \delta x^j \frac{dx^i}{d\sigma} d\sigma \\ &- \int_0^{2\pi} \partial_j v_i \frac{dx^j}{d\sigma} \delta x^i d\sigma + \left(v_i \delta x^i\right)\Big|_0^{2\pi} \\ =&\; \int_0^{2\pi} \left(\partial_j v_i - \partial_i v_j\right) \frac{dx^i}{d\sigma} \delta x^j d\sigma,\end{aligned}$$

where we have used that at first order, $\delta v_i \equiv v_i(x+\delta x) - v_i(x)$ with $v_i(x+\delta x) = v_i(x) + \partial_j v_i(x)\delta x^j$ and $\partial_i \equiv \frac{\partial}{\partial x^i}$.

Defining $C_{ij} \equiv \partial_i v_j - \partial_j v_i$ we have

$$\delta f = -\int_0^{2\pi} C_{ij} \frac{dx^i}{d\sigma} \delta x^j \; d\sigma. \tag{13}$$

The above relation defines how the line integral changes infinitesimally under a respective infinitesimal change of the curve. There are two independent ways of deforming the curve γ_0 at a given point: along its tangent direction and perpendicular to it.

The variations along the tangent direction will simply define a reparameterization of the curve and consequently will not contribute to any change of the line integral. So, we consider only variations along the perpendicular direction at each point of γ and we conveniently parameterize them by $\tau \in [0, 2\pi]$ such

[2] We adopt Einstein's summation convention for indices.

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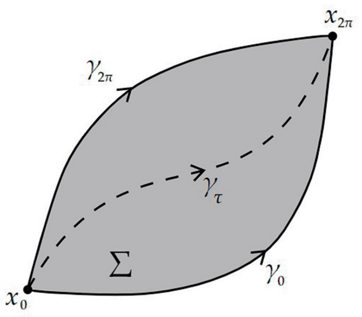

Figure 1: The variation of the path γ_0 along its normal direction will be parameterized by $\tau \in [0, 2\pi]$ and the path $\gamma_0^{-1} \circ \gamma_{2\pi}$ will define the border of the surface Σ.

that $\tau = 0$ will label the points of the curve γ_0 while $\tau = 2\pi$, those of a curve $\gamma_{2\pi}$, obtained from γ_0, sharing its borders x_0 and $x_{2\pi}$ (see Fig. 1). Then, we have that $\delta x^i = \frac{\partial x^i}{\partial \tau}\delta\tau$ and $\delta f = \frac{df}{d\tau}\delta\tau$ so the relation (13) will define a differential equation for the line integral with respect to the deformations of the curve:

$$\frac{df}{d\tau} = -\int_0^{2\pi} C_{ij}\frac{\partial x^i}{\partial \sigma}\frac{\partial x^j}{\partial \tau}\, d\sigma. \tag{14}$$

For a finite transformation taking γ_0 to $\gamma_{2\pi}$ we can obtain the variation of the flux from one curve to another by directly integrating this equation in τ which gives

$$\begin{aligned}\Delta f \;&\equiv\; f(\mathbf{v}, \gamma_{2\pi}) - f(\mathbf{v}, \gamma_0) \\ &= -\int_0^{2\pi}\int_0^{2\pi} C_{ij}\frac{\partial x^i}{\partial \sigma}\frac{\partial x^j}{\partial \tau}\, d\sigma\, d\tau.\end{aligned} \tag{15}$$

Rewriting the l.h.s using the definition of the line integral given in (12) we get

$$\Delta f = \int_{\gamma_{2\pi}} \mathbf{v}\cdot d\boldsymbol{\ell} - \int_{\gamma_0} \mathbf{v}\cdot d\boldsymbol{\ell} \tag{16}$$

and reversing the direction of the normal vector to the curve γ_0 it becomes

$$\Delta f = \oint_{\gamma} \mathbf{v}\cdot d\boldsymbol{\ell}, \tag{17}$$

where $\gamma \equiv \gamma_0^{-1} \circ \gamma_{2\pi}$.

Next, for the r.h.s we can use the identity $\partial_i v_j - \partial_j v_i \equiv \epsilon_{ijk}\epsilon_{klm}\partial_l v_m = \epsilon_{ijk}\left(\boldsymbol{\partial}\times\mathbf{v}\right)_k$ and therefore

$$\int_0^{2\pi}\int_0^{2\pi} C_{ij}\frac{\partial x^i}{\partial \sigma}\frac{\partial x^j}{\partial \tau}\, d\sigma\, d\tau = \int_\Sigma \boldsymbol{\partial}\times\mathbf{v}\cdot d\mathbf{S}, \tag{18}$$

where the area element is given by $dS_k = \epsilon_{ijk}\frac{\partial x^i}{\partial \sigma}\frac{\partial x^j}{\partial \tau}\, d\sigma\, d\tau$.

Finally, the relation given in (15) can be written as

$$\oint_{\partial\Sigma} \mathbf{v}\cdot d\boldsymbol{\ell} = -\int_\Sigma \boldsymbol{\partial}\times\mathbf{v}\cdot d\mathbf{S}, \tag{19}$$

which is the Stokes theorem or the curl theorem, relating the evaluation of the line integral of the vector field $\mathbf{v}$ along the (closed) border of the 2-dimensional surface Σ with its curl inside it.

The sign on the r.h.s. expresses the convention on the basis orientation. While the vector $\frac{d\mathbf{x}}{d\sigma}$ is along the tangent direction of the path, the vector $\frac{d\mathbf{x}}{d\tau}$ points along its normal direction. The orientation of these vectors agrees with the choice where the tangent direction is defined counter-clockwise along the curve and the normal direction is opposite to the direction defined by the path variation. Thereby, a positively oriented surface is obtained when its normal is given by $\frac{d\mathbf{x}}{d\sigma}\times\frac{d\mathbf{x}}{d\tau}$. On the other hand, in the case of Σ on r.h.s. of equation (19), the normal direction of the surface follows this basis convention but its border $\partial\Sigma$ has the opposite direction because its orientation is clockwise.

Now the Ampère-Maxwell law and also the Faraday induction law (10) can be rearranged as

$$\Phi(\boldsymbol{\partial}\times\mathbf{B} - \frac{1}{c}\partial_t\mathbf{E}, \Sigma) = \frac{4\pi}{c}\Phi\left(\mathbf{j}, \Sigma\right) \tag{20}$$

and

$$\Phi(\boldsymbol{\partial}\times\mathbf{E} + \frac{1}{c}\partial_t\mathbf{B}, \Sigma) = 0, \tag{21}$$

from where the local differential Maxwell equations follow directly.

Now, the Gauss law for the electric and magnetic fields (7) and (9) define relations between the flux of these fields through the border of a volume and the charges inside this volume (which vanishes for the magnetic case). In order to connect the flux of these vector fields with something inside the volume so that this can be associated locally with the charge density we use the relation established by the Stokes theorem which is now derived.

We start by considering the flux of a generic vector field $\mathbf{v}$ through the 2-dimensional surface Σ_0 whose coordinates x^i are conveniently parameterized by $\sigma \in [0, 2\pi]$ and $\tau \in [0, 2\pi]$:

$$\begin{aligned}\Phi(\mathbf{v}, \Sigma_0) \;&\equiv\; \int_{\Sigma_0} \mathbf{v}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{2\pi} v_k\epsilon_{kij}\frac{\partial x^i}{\partial \sigma}\frac{\partial x^j}{\partial \tau}\, d\sigma\, d\tau \\ &\equiv \int_{\Sigma_0} H_{ij}\frac{\partial x^i}{\partial \sigma}\frac{\partial x^j}{\partial \tau}\, d\sigma\, d\tau,\end{aligned} \tag{22}$$

where we have defined the antisymmetric tensor $H_{ij} = \epsilon_{ijk}v_k$ for simplicity.

Our aim is to understand how this flux on the border of a 3-dimensional volume can be related to something inside it, so, we construct such a closed surface corresponding to this volume from the given surface Σ_0 by continuously deforming it.

At each point of Σ_0 we take the deformation $x^\mu \to x^\mu + \delta x^\mu$ changing this surface everywhere except for its border $\partial\Sigma_0$, which is kept fixed. Because tangent variations to the surface will simply reparameterize the integral which defines the flux in (22), we shall only regard variations along the normal direction at each point of Σ_0. Then, the infinitesimal change of the flux can be determined by considering the flux as a functional in parameters σ and τ as follows:

$$\begin{aligned}
\delta\Phi \; = & \int_{\Sigma_0} \delta H_{ij} \frac{\partial x^i}{\partial\sigma}\frac{\partial x^j}{\partial\tau}\, d\sigma d\tau + \int_{\Sigma_0} H_{ij}\frac{\partial \delta x^i}{\partial\sigma}\frac{\partial x^j}{\partial\tau}\, d\sigma d\tau \\
& + \int_{\Sigma_0} H_{ij}\frac{\partial x^i}{\partial\sigma}\frac{\partial \delta x^j}{\partial\tau}\, d\sigma d\tau \\
= & \int_{\Sigma_0} \partial_k H_{ij}\frac{\partial x^i}{\partial\sigma}\frac{\partial x^j}{\partial\tau}\delta x^k\, d\sigma d\tau \\
& - \int_{\Sigma_0} \frac{d}{d\sigma}\left(H_{ij}\frac{\partial x^j}{\partial\tau}\right)\delta x^i\, d\sigma d\tau \\
& - \int_{\Sigma_0} \frac{d}{d\tau}\left(H_{ij}\frac{\partial x^i}{\partial\sigma}\right)\delta x^j\, d\sigma d\tau \\
= & \int_{\Sigma_0} \partial_k H_{ij}\frac{\partial x^i}{\partial\sigma}\frac{\partial x^j}{\partial\tau}\delta x^k\, d\sigma d\tau \\
& - \int_{\Sigma_0} \partial_k H_{ij}\frac{\partial x^k}{\partial\sigma}\frac{\partial x^j}{\partial\tau}\delta x^i\, d\sigma d\tau \\
& - \int_{\Sigma_0} \partial_k H_{ij}\frac{\partial x^i}{\partial\sigma}\frac{\partial x^k}{\partial\tau}\delta x^j\, d\sigma d\tau,
\end{aligned}$$

where in the second line we have integrated by parts the second and third terms and we have dropped all the terms which are evaluated on the border since they will vanish. Finally, using the antisymmetry of H_{ij} we can rewrite the above expression as

$$\begin{aligned}
\delta\Phi \; = & \int_0^{2\pi}\int_0^{2\pi} (\partial_i H_{jk} + \partial_j H_{ki} + \partial_k H_{ij}) \\
& \cdot \frac{\partial x^i}{\partial\sigma}\frac{\partial x^j}{\partial\tau}\delta x^k\, d\sigma d\tau. \qquad (23)
\end{aligned}$$

The equation above gives the infinitesimal variation of the flux of $\mathbf{v}$ through a 2-dimensional surface as this surface is deformed while its border remains fixed.

We may then parameterize these deformations with $\zeta \in [0, 2\pi]$ such that the points on the surface Σ_0 are labeled by $\zeta = 0$ and those on $\Sigma_{2\pi}$ by $\zeta = 2\pi$ (see Fig. 2). in this case, equation (23) becomes a differential equation for the change of the flux due to smooth changes of the surface:

$$\begin{aligned}
\frac{d\Phi}{d\zeta} \; = & \int_0^{2\pi}\int_0^{2\pi} (\partial_i H_{jk} + \partial_j H_{ki} + \partial_k H_{ij}) \\
& \cdot \frac{\partial x^i}{\partial\sigma}\frac{\partial x^j}{\partial\tau}\frac{\partial x^k}{\partial\zeta}\, d\sigma d\tau. \qquad (24)
\end{aligned}$$

Now, for a finite variation of the surface, the change of the flux can be obtained from the integration of this differential equation in ζ, giving

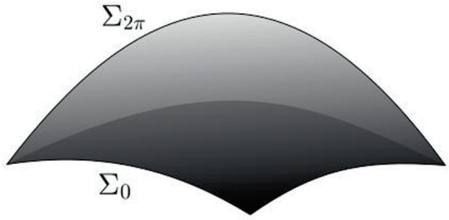

Figure 2: The variation of the surface Σ_0 along its normal direction will result in a change of the flux of the vector field $\mathbf{v}$ through it. This change will be related to the divergence of this field inside the volume bounded by $\Sigma_0^{-1} \cup \Sigma_{2\pi}$.

$$\begin{aligned}
\Delta\Phi \; \equiv & \; \Phi(\mathbf{v}, \Sigma_{2\pi}) - \Phi(\mathbf{v}, \Sigma_0) \\
= & \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} (\partial_i H_{jk} + \partial_j H_{ki} + \partial_k H_{ij}) \\
& \cdot \frac{\partial x^i}{\partial\sigma}\frac{\partial x^j}{\partial\tau}\frac{\partial x^k}{\partial\zeta}\, d\sigma d\tau d\zeta. \qquad (25)
\end{aligned}$$

Using the definition of the flux to rewrite the l.h.s of the equation above we have

$$\Delta\Phi = \int_{\Sigma_{2\pi}} \mathbf{v}\cdot d\mathbf{S} - \int_{\Sigma_0} \mathbf{v}\cdot d\mathbf{S} \qquad (26)$$

and reversing the orientation of Σ_0 it becomes

$$\Delta\Phi = \int_{\Sigma} \mathbf{v}\cdot d\mathbf{S}, \qquad (27)$$

where $\Sigma = \Sigma_0^{-1} \cup \Sigma_{2\pi}$.

For the r.h.s of (25) we can use

$$\begin{aligned}
& \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} (\partial_i H_{jk} + \partial_j H_{ki} + \partial_k H_{ij}) \\
& \quad \cdot \frac{\partial x^i}{\partial\sigma}\frac{\partial x^j}{\partial\tau}\frac{\partial x^k}{\partial\zeta} d\sigma d\tau d\zeta \\
& = \frac{1}{3!}\int_{\Omega} (\partial_i H_{jk} + \partial_j H_{ki} + \partial_k H_{ij})\, \epsilon_{ijk}\, d^3x \\
& = \int_{\Omega} \partial_i v_i\, d^3x = \int_{\Omega} \boldsymbol{\partial}\cdot\mathbf{v}\, dV,
\end{aligned}$$

where Ω is the 3-dimensional volume enclosed by Σ. Finally, we arrive at the relation

$$\oint_{\partial\Omega} \mathbf{v}\cdot d\mathbf{S} = \int_{\Omega} \boldsymbol{\partial}\cdot\mathbf{v}\, dV, \qquad (28)$$

which is more commonly known as the divergence theorem relating the flux of a vector field across a 2-dimensional closed surface with its divergence inside it[3].

With this identity it is direct to obtain the differential Gauss law for the electric and magnetic fields from their

[3] The Gauss' theorem in 2 dimensions is given in the appendix.

integral equations by replacing the flux of these fields by their corresponding divergences integrated over the spatial volume:

$$\Phi(\mathbf{E},\partial\Omega) = \int_\Omega \boldsymbol{\partial}\cdot\mathbf{E}\, dV = 4\pi\int_\Omega \rho\, dV \tag{29}$$

and

$$\Phi(\mathbf{B},\partial\Omega) = \int_\Omega \boldsymbol{\partial}\cdot\mathbf{B}\, dV = 0. \tag{30}$$

4. The Lorentz Invariant integral Equations of Electromagnetism: The Stokes Theorem in Minkowski Space-Time

The reconciliation of the Maxwell equations of electromagnetism with the principle of relativity has in its core the observation that the electric and magnetic fields in $\mathbb{R}^3$ are not vectors (i.e., they do not transform covariantly) under Lorentz transformations. Instead, these quantities appear as components of a more fundamental field, the electromagnetic field: an antisymmetric rank-2 Lorentz covariant tensor $F_{\mu\nu}$, $\mu,\nu = 0,1,2,3$ in Minkowski 4-dimensional space-time with metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. So, what we read as the electric or magnetic fields $\mathbf{E}$ and $\mathbf{B}$ depends on our choice of inertial frame of reference since the components $F_{0i} = E_i$ and $F_{ij} = -\epsilon_{ijk}B_k$ can be mixed from one frame to another after a Lorentz transformation.

If, on one hand, the direct substitution of the electric and magnetic fields in terms of the electromagnetic field components into Maxwell equations leads us to an explicit Lorentz covariant form of these equations, on the other hand, it is not that direct to obtain the Lorentz invariant integral equations in this way, the reason being that these integral equations are defined in terms of fluxes and circulation of fields over specific surfaces and curves, namely, with a specific splitting of space-time into space and time.

Nevertheless, let us discuss how the straightforward application of the Stokes theorem derivation as proposed in the previous section defines the differential covariant Maxwell equations from a proposed set of integral equations for the electromagnetic field, which can therefore be recognized as the desired Lorentz invariant integral equations of electromagnetism [6, 7].

In 4-dimensional Minkowski space-time, one can naturally define a flux over a 2-dimensional surface Σ given in terms of the electromagnetic field and its Hodge dual [5]

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{31}$$

as

$$\Phi(F_{\mu\nu},\Sigma) \equiv \int_\Sigma F_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}d\sigma\, d\tau \qquad \text{and}$$

$$\Phi(\widetilde{F}_{\mu\nu},\Sigma) \equiv \int_\Sigma \widetilde{F}_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}d\sigma\, d\tau. \tag{32}$$

Then, we postulate the integral equations of electromagnetism to be given by the following relations[4]

$$\Phi(F_{\mu\nu},\partial\Omega) = 0, \tag{33}$$

$$\Phi(\widetilde{F}_{\mu\nu},\partial\Omega) = -\frac{4\pi}{c}\mathcal{Q}, \tag{34}$$

where Ω is a 3-dimensional volume in the 4-dimensional space-time and

$$\mathcal{Q} \equiv \int_\Omega J^\mu dV_\mu, \tag{35}$$

with $dV_\mu = \epsilon_{\mu\nu\lambda\gamma}\frac{\partial x^\nu}{\partial\sigma}\frac{\partial x^\lambda}{\partial\tau}\frac{\partial x^\gamma}{\partial\zeta}d\sigma d\tau d\zeta$ the 3-dimensional volume element and $J^\mu = (c\rho,\mathbf{j})$ the electric 4-current thus defining $\mathcal{Q}$ as the electric charge and/or current.

The differential Maxwell equations can then be obtained as local relations for the fields which give the consistency of the above integral equations with the Stokes theorem.

In order to see this we take the flux of the field strength and of its Hodge dual over a 2-dimensional surface Σ_0 parameterized by $\sigma\in[0,2\pi]$ and $\tau\in[0,2\pi]$ as given above and then consider continuous deformations of this surface keeping its border fixed, as done before. Writing $B_{\mu\nu}$ for either $F_{\mu\nu}$ or $\widetilde{F}_{\mu\nu}$, a completely analogous calculation to the case where we derived the divergence theorem leads us to

$$\begin{aligned}
\delta\Phi = &\int_\Sigma \delta B_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\, d\sigma d\tau + \int_\Sigma B_{\mu\nu}\frac{\partial\delta x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\, d\sigma d\tau \\
&+ \int_\Sigma B_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial\delta x^\nu}{\partial\tau}\, d\sigma d\tau \\
= &\int_\Sigma \partial_\lambda B_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\delta x^\lambda\, d\sigma d\tau \\
&- \int_\Sigma \frac{d}{d\sigma}\left(B_{\mu\nu}\frac{\partial x^\nu}{\partial\tau}\right)\delta x^\mu\, d\sigma d\tau \\
&- \int_\Sigma \frac{d}{d\tau}\left(B_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\right)\delta x^\nu\, d\sigma d\tau \\
= &\int_\Sigma \partial_\lambda B_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\delta x^\lambda\, d\sigma d\tau \\
&- \int_\Sigma \partial_\lambda B_{\mu\nu}\frac{\partial x^\lambda}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\delta x^\mu\, d\sigma d\tau \\
&- \int_\Sigma \partial_\lambda B_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\lambda}{\partial\tau}\delta x^\nu\, d\sigma d\tau
\end{aligned}$$

where in the second line we have integrated by parts the second and third terms and we have dropped all the

[4] The generalization of these integral equations to electrodynamics in $2+1$ and $1+1$ dimensional space-times is straightforward: while the equation of the field strength states that the flux of a rank-2 tensor over a closed 2-dimensional surface vanishes, the equation for the Hodge dual field defines that the flux of a $D-1$ rank tensor over a closed $(D-1)$-dimensional hypersurface in a $(D+1)$-dimensional space-time equals to the electric charge inside its hypervolume.

terms which are evaluated on the border since they will vanish. Then we can rewrite the above expression as

$$\delta\Phi = \int \left(\partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}\right) \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\delta x^\lambda \, d\sigma d\tau. \tag{36}$$

Next, parameterizing the variations of the surface with $\zeta \in [0, 2\pi]$ such that the points on the surface Σ_0 are labeled by $\zeta = 0$ and those on $\Sigma_{2\pi}$ by $\zeta = 2\pi$, the above relation becomes a differential equation for the change of the flux due to smooth changes of the surface:

$$\frac{d\Phi}{d\zeta} = \int \left(\partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}\right) \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\lambda}{\partial\zeta} \, d\sigma d\tau. \tag{37}$$

So finally, integrating over ζ we get the change of the flux from the initial to the final surface:

$$\begin{aligned} \Delta\Phi &\equiv \Phi(B, \Sigma_{2\pi}) - \Phi(B, \Sigma_0) \\ &= \int_\Omega \left(\partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}\right) \\ &\quad \cdot \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\lambda}{\partial\zeta} \, d\sigma d\tau d\zeta. \end{aligned} \tag{38}$$

Using the definition of the flux to rewrite the l.h.s and reversing the orientation of Σ_0 the above equation becomes

$$\begin{aligned} \oint_{\partial\Omega} B_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau} d\sigma d\tau \;=\; &\int_\Omega \left(\partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}\right) \\ &\cdot \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\lambda}{\partial\zeta} \, d\sigma d\tau d\zeta. \end{aligned} \tag{39}$$

where Ω is the 3-dimensional volume enclosed by the closed surface $\partial\Omega = \Sigma_0^{-1} \cup \Sigma_{2\pi}$.

So, we have now that the integral equations can be rewritten as

$$\begin{aligned} &\int_\Omega \left(\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}\right) \\ &\qquad \cdot \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\lambda}{\partial\zeta} \, d\sigma d\tau d\zeta = 0 \end{aligned} \tag{40}$$

$$\begin{aligned} &\int_\Omega \left(\partial_\lambda \widetilde{F}_{\mu\nu} + \partial_\mu \widetilde{F}_{\nu\lambda} + \partial_\nu \widetilde{F}_{\lambda\mu}\right) \\ &\qquad \cdot \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\lambda}{\partial\zeta} \, d\sigma d\tau d\zeta \\ &\quad = \frac{4\pi}{c}\int_\Omega \epsilon_{\mu\nu\lambda\gamma} J^\gamma \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\lambda}{\partial\zeta} d\sigma \, d\tau \, d\zeta, \end{aligned} \tag{41}$$

from where we derive the local conditions. The first equation gives immediately the Bianchi identity, corresponding to the Faraday law of induction and the Gauss law for the magnetic field:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0. \tag{42}$$

For the second equation we have

$$\partial_\lambda \widetilde{F}_{\mu\nu} + \partial_\mu \widetilde{F}_{\nu\lambda} + \partial_\nu \widetilde{F}_{\lambda\mu} = \frac{4\pi}{c}\epsilon_{\mu\nu\lambda\gamma} J^\gamma, \tag{43}$$

from where taking the contraction with $\epsilon^{\alpha\lambda\nu\mu}$ and using[5] $\epsilon^{\mu\nu\alpha\lambda}\epsilon_{\mu\nu\rho\sigma} = -2(\delta^\alpha_\rho\delta^\lambda_\sigma - \delta^\lambda_\rho\delta^\alpha_\sigma)$ and $\epsilon^{\mu\nu\lambda\alpha}\epsilon_{\mu\nu\lambda\gamma} = -6\delta^\alpha_\gamma$ we get

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \tag{44}$$

corresponding to the Gauss law for the electric field and the Ampère-Maxwell law.

5. A general Proof of Stokes Theorem

One of the consequences of the principle of equivalence is that physical equations must be tensorial equations so that each of the terms transforms covariantly under the respective symmetry group. Besides tensors, the differential forms [5, 8] are a very elegant language to describe physical laws and in particular, for gauge theories such as electrodynamics, it becomes a powerful tool.

A differential form [9, 10] may be seen as something which is naturally integrated over a curve or a surface or a volume or anything else with more dimensions. We shall refer to these geometrical structures in a generic way as hyper-surfaces. So, a 0-form is nothing but a function $f(x)$ which can be evaluated at each point of space-time[6]. A 1-form can be defined as something which is ready to be integrated over a 1-dimensional hyper-surface: $\omega = \omega_\mu dx^\mu$. Next, a 2-form will be naturally integrated over a 2-dimensional hyper-surface, $\omega = \frac{1}{2}\omega_{\mu\nu}dx^\mu \wedge dx^\nu$, where $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$ and therefore the components of the 2-form, the tensor $\omega_{\mu\nu}$ is antisymmetric. Generally speaking, a p-form is defined by

$$\omega = \frac{1}{p!}\omega_{\mu_1\dots\mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \tag{45}$$

which is something that is integrated over a p-dimensional hyper-surface.

In particular, the electromagnetic field is defined by the components of a 2-form [11], $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ and therefore, its flux, and analogously the flux of its Hodge dual is nothing but the integration of a 2-form over a 2-dimensional hyper-surface; the most natural thing one can think of doing with a 2-form.

The scheme we have used to obtain the relation known as the Stokes theorem for the case of a 2-form can be generalized in a straightforward manner to a p-form and that is what we are going to show in what follows.

We consider an orientable hyper-surface Σ of dimension p to be immersed in space-time $\mathcal{M}$ of dimension $D > p$, with local coordinates $x^{\mu_1}, \dots, x^{\mu_D}$. The flux of the p-form ω is defined as the integration of that form over Σ:

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[5] We use $\epsilon^{0123} = 1$.

[6] In particular, the Stokes theorem for a 0-form, which is the Fundamental Theorem of Calculus, is derived explicitly in the appendix.

$$\Phi = \int_\Sigma \frac{1}{p!}\omega_{\mu_1\ldots\mu_p}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} = \int_\Sigma \omega_{\mu_1\ldots\mu_p}\frac{\partial x^{\mu_1}}{\partial\sigma_1}\ldots\frac{\partial x^{\mu_p}}{\partial\sigma_p}d\sigma_1\ldots d\sigma_p. \tag{46}$$

Under an infinitesimal "orthogonal deformation" of this hyper-surface, $x\to x+\delta x$, keeping its border fixed, the flux will change by

$$\begin{aligned}\delta\Phi \;=\; & \int_\Sigma \partial_\lambda\omega_{\mu_1\ldots\mu_p}\delta x^\lambda\frac{\partial x^{\mu_1}}{\partial\sigma_1}\ldots\frac{\partial x^{\mu_p}}{\partial\sigma_p}d\sigma_1\ldots d\sigma_p\\ & -\int_\Sigma \frac{d}{d\sigma_1}\left(\omega_{\mu_1\ldots\mu_p}\frac{\partial x^{\mu_2}}{\partial\sigma_2}\ldots\frac{\partial x^{\mu_p}}{\partial\sigma_p}\right)\delta x^{\mu_1}d\sigma_1\ldots d\sigma_p\;+\ldots+\\ & -\int_\Sigma \frac{d}{d\sigma_n}\left(\omega_{\mu_1\ldots\mu_p}\frac{\partial x^{\mu_1}}{\partial\sigma_1}\ldots\frac{\partial x^{\mu_{n-1}}}{\partial\sigma_{n-1}}\frac{\partial x^{\mu_{n+1}}}{\partial\sigma_{n+1}}\ldots\frac{\partial x^{\mu_p}}{\partial\sigma_p}\right)\delta x^{\mu_n}d\sigma_1\ldots d\sigma_p\;+\ldots+\\ & -\int_\Sigma \frac{d}{d\sigma_p}\left(\omega_{\mu_1\ldots\mu_p}\frac{\partial x^{\mu_2}}{\partial\sigma_2}\ldots\frac{\partial x^{\mu_{p-1}}}{\partial\sigma_{p-1}}\right)\delta x^{\mu_p}d\sigma_1\ldots d\sigma_p\end{aligned} \tag{47}$$

where we have integrated by parts the derivatives of the hyper-volume element, throwing away the terms which vanish at the border, where $\delta x = 0$.

Now the derivatives of the remaining terms will be as follows

$$\begin{aligned}&\int_\Sigma \frac{d}{d\sigma_n}\left(\omega_{\mu_1\ldots\mu_p}\frac{\partial x^{\mu_1}}{\partial\sigma_1}\ldots\frac{\partial x^{\mu_{n-1}}}{\partial\sigma_{n-1}}\frac{\partial x^{\mu_{n+1}}}{\partial\sigma_{n+1}}\ldots\frac{\partial x^{\mu_p}}{\partial\sigma_p}\right)\delta x^{\mu_n}d\sigma_1\ldots d\sigma_p\\ &\quad=\int_\Sigma \partial_\lambda\omega_{\mu_1\ldots\mu_p}\left(\frac{\partial x^{\mu_1}}{\partial\sigma_1}\ldots\frac{\partial x^{\mu_{n-1}}}{\partial\sigma_{n-1}}\frac{\partial x^\lambda}{\partial\sigma_n}\frac{\partial x^{\mu_{n+1}}}{\partial\sigma_{n+1}}\ldots\frac{\partial x^{\mu_p}}{\partial\sigma_p}\right)\delta x^{\mu_n}d\sigma_1\ldots d\sigma_p\\ &\qquad+\int_\Sigma\left(\omega_{\mu_1\ldots\mu_p}\frac{\partial^2 x^{\mu_1}}{\partial\sigma_n\partial\sigma_1}\ldots\frac{\partial x^{\mu_{n-1}}}{\partial\sigma_{n-1}}\frac{\partial x^{\mu_{n+1}}}{\partial\sigma_{n+1}}\ldots\frac{\partial x^{\mu_p}}{\partial\sigma_p}\right)\delta x^{\mu_n}d\sigma_1\ldots d\sigma_p\;+\ldots\\ &\qquad+\int_\Sigma\left(\omega_{\mu_1\ldots\mu_p}\frac{\partial x^{\mu_1}}{\partial\sigma_1}\ldots\frac{\partial^2 x^{\mu_{n-1}}}{\partial\sigma_n\sigma_{n-1}}\frac{\partial x^{\mu_{n+1}}}{\partial\sigma_{n+1}}\ldots\frac{\partial x^{\mu_p}}{\partial\sigma_p}\right)\delta x^{\mu_n}d\sigma_1\ldots d\sigma_p\;+\ldots\\ &\qquad+\int_\Sigma\left(\omega_{\mu_1\ldots\mu_p}\frac{\partial x^{\mu_1}}{\partial\sigma_1}\ldots\frac{\partial x^{\mu_{n-1}}}{\partial\sigma_{n-1}}\frac{\partial^2 x^{\mu_{n+1}}}{\partial\sigma_n\partial\sigma_{n+1}}\ldots\frac{\partial x^{\mu_p}}{\partial\sigma_p}\right)\delta x^{\mu_n}d\sigma_1\ldots d\sigma_p\;+\ldots\\ &\qquad+\int_\Sigma\left(\omega_{\mu_1\ldots\mu_p}\frac{\partial x^{\mu_1}}{\partial\sigma_1}\ldots\frac{\partial x^{\mu_{n-1}}}{\partial\sigma_{n-1}}\frac{\partial x^{\mu_{n+1}}}{\partial\sigma_{n+1}}\ldots\frac{\partial^2 x^{\mu_p}}{\partial\sigma_n\partial\sigma_p}\right)\delta x^{\mu_n}d\sigma_1\ldots d\sigma_p.\end{aligned}$$

So that relabelling all the indices and using the anti-symmetry of the tensor $\omega_{\mu_1\ldots\mu_p}$, all terms involving second derivatives in the r.h.s of (47) will cancel and what remains is

$$\delta\Phi = (-1)^p\int_\Sigma\left(\partial_\lambda\omega_{\mu_1\ldots\mu_p}+\ldots\right)\frac{\partial x^{\mu_1}}{\partial\sigma_1}\ldots\frac{\partial x^{\mu_p}}{\partial\sigma_p}\delta x^\lambda d\sigma_1\ldots d\sigma_p. \tag{48}$$

We notice that the rearrangement of the indices involves permutations leading to an anti-symmetric tensor $\partial_\lambda\omega_{\mu_1\ldots\mu_p}+\ldots$ which will be later recognized as the components of a $(p+1)$-form in a given basis.

This variation can be parameterized by $s\in[0,2\pi]$ so that we obtain a differential equation for the flux as

$$\frac{d\Phi}{ds} \;=\; (-1)^p\int_\Sigma\left(\partial_\lambda\omega_{\mu_1\ldots\mu_p}\ldots\right)\frac{\partial x^{\mu_1}}{\partial\sigma_1}\ldots\frac{\partial x^{\mu_p}}{\partial\sigma_p}\frac{\partial x^\lambda}{\partial s}d\sigma_1\ldots d\sigma_p \tag{49}$$

$$=\;\int_\Sigma\frac{(-1)^p}{(p+1)!}\left(\partial_\lambda\omega_{\mu_1\ldots\mu_p}\ldots\right)dx^\lambda\wedge dx^{\mu_1}\wedge\cdots\wedge dx^p, \tag{50}$$

which can be directly integrated and the l.h.s will give the difference of the flux calculated at the hyper-surface at $s=0$ and at $s=2\pi$.

Changing the orientation of the hyper-surface at $s=0$, the border shared by these two hyper-surfaces will disappear as they can be merged as a closed oriented hyper-surface $\partial\Omega$, enclosing the hyper-volume Ω of dimension $p+1$. Then, we get the Stokes theorem for a p-form:

$$\oint_{\partial\Omega}\omega = (-1)^p\int_\Omega d\omega. \tag{51}$$

The minus sign appearing for the odd differential forms can be changed by a redefinition of the orientation of the closed hyper-surface $\partial\Omega$.

6. Conclusions

The development of the ideas of physics depends upon and, at the same time, boosts the construction of new mathematical knowledge and techniques. The Stokes theorem finds in the study of electrodynamics a perfect "pedagogical match" concerning its applications: in many important physical situations for which symmetry arguments can be used, this theorem makes the task of finding solutions to the Maxwell equations much simpler than solving them directly. In the present discussion we have explored the natural insights one can obtain by looking at the fact that electrodynamics deals exactly with fluxes and circuitations in order to propose a derivations of the Stokes' theorem which defines mathematical identities exactly for the variations of fluxes and circuitations. Thus, we have found a quite simple way to determine these identities by analysing how the fluxes and circuitations change when we change the corresponding surfaces and curves where they are defined. We believe that this method of deriving Stokes theorem fills an important gap for undergraduate and graduate students concerning their learning of this important theorem.

Supplementary material

The following online material is available for this article:
Appendix

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