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Moments of inertia by summation

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This study delves into moment of inertia in rigid body mechanics, a crucial concept in physics. While it's traditionally taught through integration, we propose an alternative method using discrete mass segments. This approach simplifies calculations, particularly for symmetric bodies, without complex integration procedures. Our goal is to enhance students' understanding of fundamental physics principles, making the subject more accessible. We also provide visual aids and online animations to aid comprehension.

Keywords: Rigid Body, Moment of Inertia, Summation.

1. Introduction

In the initial exploration of rigid body mechanics, there arises a necessity to elucidate an abstract notion pertaining to the mass distribution of objects in rotational motion about a specified axis. This physical quantity is denoted as the moment of inertia. Typically, it finds comprehensive treatment towards the culmination of the introductory university physics course, attended by engineering, exact science, and various educational program students.

However, within the majority of university curricula, these same students concurrently engage in the study of differential and integral calculus. Through this mathematical endeavor, they acquire a set of new concepts and analytical tools that can be skillfully employed in the computation of the moment of inertia for rigid bodies exhibiting a specific class of symmetry, such as axial, radial, or spherical symmetry.

Although the definition of the moment of inertia [1, 2] originates from the discrete distribution of mass through a *summation* of massive points with a constant separation between them and positioned at a certain distance from the axis of rotation, lets us to extend this notion to the *integration* of contributions from infinitesimal elements of mass of a continuous body spatially distributed along the axis of rotation:

$$I = \sum_j m_j r_j^2, \qquad \text{(discrete)}, \tag{1a}$$

$$I = \int dm\, r^2, \qquad \text{(continuous)}. \tag{1b}$$

However, the principal objective of the physics discipline is not centered on instructing functional integration techniques. Instead, its core mission is to instill a profound understanding of the diverse concepts governing the characterization of both translational and rotational motion within rigid bodies. This necessitates a shift in pedagogical approach, which in turn enhances students' comprehension of novel physical principles.

To this end, this study introduces an alternative methodology for determining the moment of inertia in the case of commonly encountered symmetric rigid bodies, a subject widely documented in the literature. This approach circumvents the need for explicit integration procedures as previously indicated [3–5], as well as the reliance on theorems pertaining to parallel and perpendicular axes. Instead, our approach involves the partitioning of the body into a set of N discrete mass segments, each possessing its unique moment of inertia. The cumulative effect of these individual inertial components converges towards the moment of inertia of the entire rigid body as the collection size approaches infinity, symbolized as $N \to \infty$.

Accordingly, within the forthcoming exposition that delineates the methodology for computing the moment of inertia for certain rigid bodies, there arises a requisite engagement with the ensuing outcomes. These outcomes pertain to the summation of the initial N integer values raised to varying exponents, encompassing linear, quadratic, cubic, and quartic orders. It is noteworthy that these outcomes are amenable for direct application without necessitating a formal demonstration of their

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derivation [6, 7],

$$
\begin{aligned}
\sum_{j=1}^{N} j &= 1+2+3+\cdots+N = \frac{N(N+1)}{2},\\
\sum_{j=1}^{N} j^2 &= \frac{N(N+1)(2N+1)}{6} = \tfrac{1}{3}N^3+\tfrac{1}{2}N^2+\tfrac{1}{6}N,\\
\sum_{j=1}^{N} j^3 &= \frac{N^2(N+1)^2}{4} = \tfrac{1}{4}N^4+\tfrac{1}{2}N^3+\tfrac{1}{4}N^2,\\
\sum_{j=1}^{N} j^4 &= \frac{N(N+1)(2N+1)(3N^2+3N-1)}{30}.
\end{aligned} \tag{2}
$$

On the other hand, to facilitate a deeper comprehension of the outlined procedure for calculating the moment of inertia of different rigid bodies, we will provide visual analyses and freely accessible animation videos available online.

2. Solid Bar: Rod

To begin the physical calculation of the moment of inertia of a rigid rod with mass M and length L, which can freely rotate about an axis perpendicular to its length at a distance s from one of its ends, we can conceptualize the rod as an ensemble of N mass points, each with an individual mass $m = M/N$, uniformly distributed along the length L. The positions of these mass points are defined by $x_j = L(j-1)/(N-1)$. Regarding an axis of rotation located at a distance s within the interval $[0, L]$ from the boundary, the moment of inertia for each individual mass point can be expressed as follows,

$$I_j = \frac{M}{N}\left(s-x_j\right)^2. \tag{3}$$

Where j is the index denoting the individual mass point. The total moment of inertia is given by eq.(1a) as,

$$
\begin{aligned}
I &= \sum_{j=1}^{N} \frac{M}{N}\left(s^2 + x_j^2 - 2sx_j\right),\\
&= \sum_{j=1}^{N} \frac{M}{N}\left[s^2 + \frac{L^2(j^2+1-2j)}{(N-1)^2} - \frac{2sL(j-1)}{(N-1)}\right],\\
&= Ms^2 + \frac{2MsL}{(N-1)} + \frac{ML^2}{(N-1)^2} + \frac{ML^2}{N(N-1)^2}\sum_{j=1}^{N} j^2\\
&\quad - \frac{2ML^2}{N(N-1)^2}\sum_{j=1}^{N} j - \frac{2MsL}{N(N-1)}\sum_{j=1}^{N} j.
\end{aligned} \tag{4}
$$

When we substitute the summations into their corresponding expression in eq.(2) and subsequently rearrange the terms through a series of simplifications, we get,

$$I = Ms^2 - MsL + \frac{ML^2N}{(N-1)^2} + \frac{ML^2(N+1)(2N+1)}{6(N-1)(N-1)}. \tag{5}$$

When the amount of partitions goes to infinity ($N\to\infty$), physically we have a transition from a discrete distribution of mass to a continuous rigid body,

$$I = Ms^2 - MsL + \cancelto{0}{\frac{ML^2N}{(N-1)^2}} + \frac{ML^2}{6}\cancelto{1}{\frac{(N+1)}{(N-1)}}\cancelto{2}{\frac{(2N+1)}{(N-1)}}. \tag{6}$$

We subsequently obtained the moment of inertia for a rigid rod rotating about an axis significantly distant from its end by a distance denoted as s,

$$I = Ms^2 - MsL + \tfrac{1}{3}ML^2. \tag{7}$$

It is noteworthy to mention that the moment of inertia demonstrates a quadratic relationship with respect to the variable s, displaying a positively concave behavior. The maximal magnitudes of the moment of inertia coincide with the endpoints of the parameter s, wherein $I_0 = I_L$ holds. This signifies that at the midpoint $s = L/2$, the moment of inertia attains its minimum value due to the inherent symmetry of mass distributed around the central axis. Thus, the rod manifests its least moment of inertia as the rotational axis passes through its center of mass,

$$I_0 = \tfrac{1}{3}ML^2, \qquad I_{\mathrm{CM}} = \tfrac{1}{12}ML^2. \tag{8}$$

The reason why the moment of inertia of a rod, relative to its center of mass, is four times less than when it rotates about its end is due to the fact that one-half of its length represents the distance between these two rotating axes. According to the definition of moment of inertia, it exhibits a quadratic dependence on distance. Therefore, the one-half factor reduces to one-quarter, which explains why the moment of inertia at the center of the rod is smaller.

Finally, in Figure 1 we observe the variation in moment of inertia as the rod transitions from discrete to continuous, encompassing an increasing number of closely spaced point masses. We encourage you to view the accompanying video by clicking here [8], which provides a comprehensive visualization of the progression from a finite number of point masses to an infinite quantity, depicting the continuous limit.

3. Ring

Calculating the moment of inertia for a ring rotating about an axis perpendicular to its plane can be somewhat intuitive. In this scenario, we will discretize the ring into a series of N equidistant point masses, each having a mass of $m = M/N$. Consequently, the moment of inertia for each individual point mass j is expressed as $I_j = MR^2/N$. Consequently, the total moment of inertia for this assembly of point masses can then be computed through the summation of their individual contributions,

$$I_{\mathrm{ring}} = \sum_{j=1}^{N} \frac{MR^2}{N} = \frac{MR^2}{N}N = MR^2. \tag{9}$$

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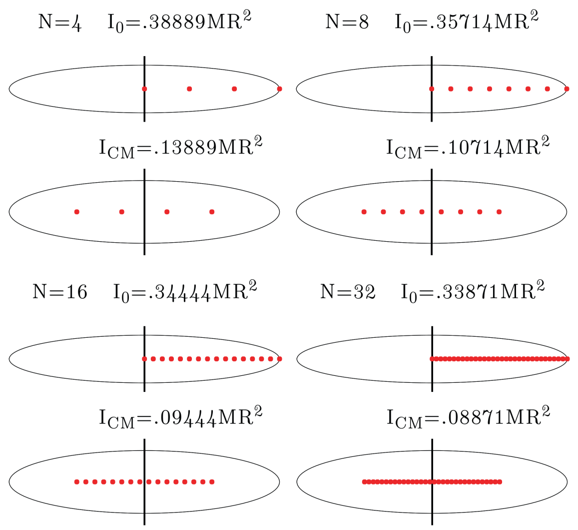

Figure 1: Solid rod as a collection of point masses. In each instance, the situation depicted above illustrates a rod undergoing rotational motion around its end ($s=0$), whereas the situation below exemplifies a rod rotating about its center of mass ($s=L/2$).

This process underscores the noteworthy observation that the outcome remains unaltered regardless of the degree of discretization necessitated to compose the rigid ring. However, it is worthwhile to acknowledge that this approach aids students in grasping the fundamental notion of spatial mass distribution within a rigid body, all without the requirement of intricate integration procedures.

4. Disc

In connection to the preceding outcome, we will consider a solid disc of mass M and radius R as a collection of N thin rings closely spaced, every one with a mass $m=M/N$ and radius $r_j=jR/N$. If the mass is uniformly distributed throughout the entire circumference of the collection of thin rings, represented by the summation $\sum_{j=1}^{N} 2\pi jR/N$, the concept of linear mass density can be introduced as follows,

$$\lambda = \frac{\text{Mass}}{\text{Length}} = \frac{M}{\pi R(N+1)}. \tag{10}$$

Thus, each ring has a moment of inertia denoted as $I_j = 2\pi\lambda(jR/N)^3$, where j is the index denoting the individual ring. The summation of these individual inertias yields the total moment of inertia,

$$\begin{aligned} I &= 2\pi\lambda\frac{R^3}{N^3}\sum_{j=1}^{N} j^3, \\ &= 2\pi\frac{M}{\pi R(N+1)}\frac{R^3}{N^3}\frac{N^2(N+1)^2}{4}, \\ &= \tfrac{1}{2}MR^2\left(\frac{N+1}{N}\right). \end{aligned} \tag{11}$$

As the number of rings tends towards infinity ($N\to\infty$), the expression within the parentheses approaches unity. Ultimately, this yields the moment of inertia of a disc with respect to an axis of rotation passing through its center of mass and perpendicular to its plane,

$$I_{\text{disc}} = \tfrac{1}{2}MR^2. \tag{12}$$

In contrast to the moment of inertia exhibited by a ring, our recent findings reveal that the moment of inertia of a solid disc is precisely half that of a ring. This phenomenon can be elucidated by partitioning the entire mass of the disc into two radial components with the same mass: a central disc with a radius of $R/\sqrt{2}$ and a wide annular ring with an inner radius of $R/\sqrt{2}$ and an outer radius of R. This physical approach demonstrates that the entire mass of the solid disc, in terms of its moment of inertia about its center, can be equivalently represented as that of a thin ring with identical mass but a radius of $R/\sqrt{2}$. However, it is essential to emphasize that in the case of the solid disc, mass distribution is uniform across its entire surface, encompassing even the inner regions.

Finally, in Figure 2 we can observe the variation in the moment of inertia of a solid disc discretized into an assembly of thin rings. As N increases, it becomes evident that the moment of inertia undergoes a reduction from unity to a value approximating one-half. However, as we progressively augment the number of subdivisions, it becomes apparent that an infinite aggregation of these thin rings converges to form a solid disc characterized by a distinct moment of inertia. We encourage you to view the accompanying video by clicking here [9]. This video offers an elaborate visualization of the transition from a finite number of rings to an infinite continuum.

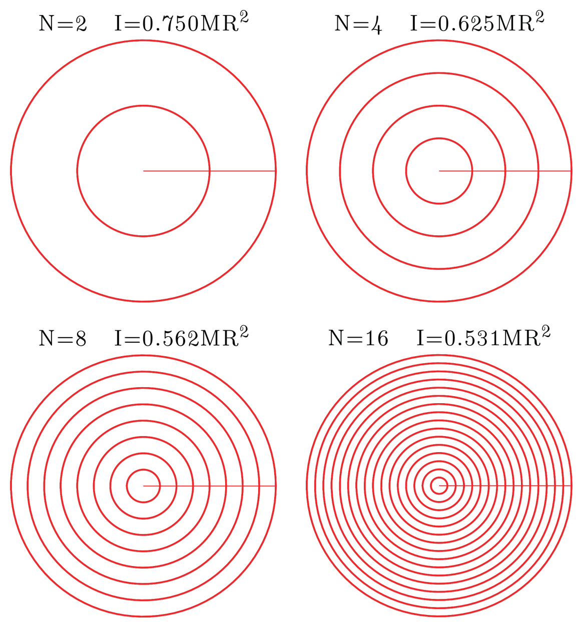

Figure 2: Rigid disc as a collection of thin rings.

5. Spherical Shell

For this particular rigid body characterized by three-dimensional symmetry (spherical symmetry), a spherical shell of mass M and radius R rotates about its center. We can analyze it as an assemblage of thin rings akin to geographical lines of latitude encircling our planet. Considering each ring with mass $m = M/N$ and radius $r_j = jR/N$, where j is the index denoting the individual ring. These rings are situated at a vertical distance from the equatorial plane given by $z_j = \pm\sqrt{R^2 - r_j^2}$, with the $\pm$ sign indicating their position on the upper or lower hemisphere. The moment of inertia of each thin ring is given by the expression,

$$I_j = \frac{M}{N}\left(\frac{jR}{N}\right)^2 = \frac{MR^2}{N^3} j^2. \tag{13}$$

The total moment of inertia is obtained by accounting for the contributions from the rings on both hemispheres (twice). This cumulative moment of inertia can be expressed as,

$$\begin{aligned} I &= 2\sum_{j=1}^{N} I_j = 2\frac{MR^2}{N^3}\left[\tfrac{1}{3}N^3 + \tfrac{1}{2}N^2 + \tfrac{1}{6}N\right], \\ &= MR^2\left[\tfrac{2}{3} + \frac{1}{N} + \frac{1}{3N^2}\right]. \end{aligned} \tag{14}$$

As the number of rings approaches infinity ($N \to \infty$), certain terms become negligible, leading to the physical consideration of an infinitely sized collection of thin rings join together to compound a spherical surface of mass uniformly distributed on its area. The moment of inertia for this configuration is then given by,

$$I_{\text{Shell}} = \tfrac{2}{3}MR^2. \tag{15}$$

In contrast to the obtained result, we can observe that the spherical shell exhibits a greater moment of inertia than that of a solid disc with the same mass and radius. To understand the significance of the factor $2/3$, we must analyze the spherical surface and compare it to that of a disc with the same radius. For the former, its surface area is four times greater than that of the disc, which is πR^2. In light of this, we determine the radius of a concentric cap about the axis of rotation that has an equivalent area to that of a disc with radius R, we can use the expression for the area of a spherical cap, which is $2\pi Rh$, where h is its height. Thus, the radius of the cap should be $\sqrt{3}R/2$ to match the area of the disc.

Conversely, for a solid disc to match the moment of inertia of the spherical shell, its radius should be larger than the sphere's radius by a factor of $2/\sqrt{3}$. In essence, the factor $2/3$ arise from equating the moment of inertia of the spherical shell to that of a disc with a radius $2/\sqrt{3}$ times larger than the sphere's radius.

Finally, in Figure 3, we can observe the variation in the moment of inertia of a spherical shell comprised of an assemblage of N rings. As the number of rings increases, it becomes evident that the moment of inertia experiences a reduction, transitioning from two to a value that approximates two-thirds. Therefore, as the number of rings approaches infinity, convergence occurs, resulting in the formation of a rigid spherical shell distinguished by a unique moment of inertia. We recommend viewing the accompanying video for a comprehensive visual representation of the entire process, encompassing the evolution from a finite number of rings to an infinite continuum. To access the video, please click here [10].

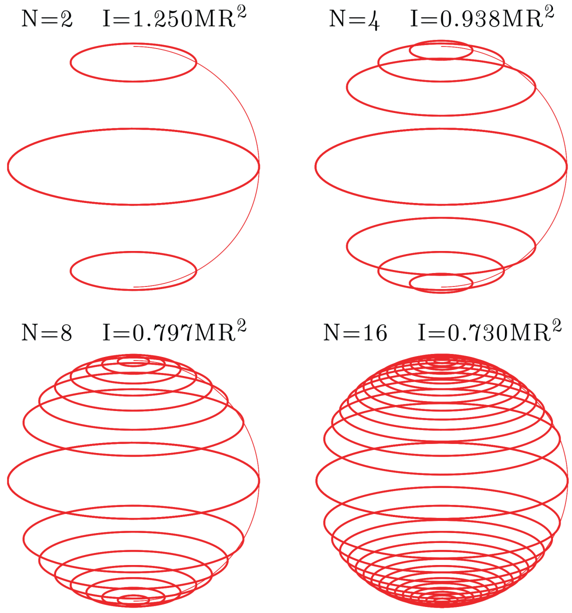

Figure 3: Spherical shell as a collection of thin rings.

6. Solid Sphere

In this concluding analytical scenario, we opt to assemble a rigid solid sphere characterized by a mass M and a radius R using a collection of N concentric shells with radii $r_j = jR/N$, where j signifies the index indicating each individual shell. Assuming a uniform distribution of the total mass M across the surfaces of these shells,

$$\begin{aligned} \text{Area} &= \sum_{j=1}^{N} 4\pi r_j^2 = \frac{4\pi R^2}{N^2}\sum_{j=1}^{N} j^2, \\ &= \frac{4\pi R^2}{N^2}\frac{N(N+1)(2N+1)}{6}, \\ &= \tfrac{2}{3}\pi R^2 \frac{(N+1)(2N+1)}{N}. \end{aligned} \tag{16}$$

Subsequently, introducing the notion of surface mass density, we find that,

$$\sigma = \frac{\text{Mass}}{\text{Area}} = \frac{M}{\frac{2}{3}\pi R^2 \frac{(N+1)(2N+1)}{N}}. \tag{17}$$

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The moment of inertia of the j-th shell is given by,

$$\begin{aligned} I_j &= \tfrac{2}{3} m_j r_j^2, \\ &= \tfrac{8}{3}\pi\sigma \frac{j^4 R^4}{N^4}. \end{aligned} \tag{18}$$

Therefore, summing these individual moments of inertia yields the collective moment of inertia for this discretized solid sphere, assembled through N concentric shells with respect to its center,

$$\begin{aligned} I &= \tfrac{8}{3}\pi \frac{M}{\frac{2}{3}\pi R^2 \frac{(N+1)(2N+1)}{N}} \frac{R^4}{N^4} \sum_{j=1}^{N} j^4, \\ &= \frac{4MR^2}{(N+1)(2N+1)} \frac{N(N+1)(2N+1)(3N^2+3N-1)}{30N^3}, \\ &= \tfrac{4}{30} MR^2 \left(3 + \frac{3}{N} - \frac{1}{N^2}\right). \end{aligned} \tag{19}$$

As the quantity of shells tends towards infinity ($N \to \infty$), specific terms diminish in significance, culminating in the physical concept of an aggregate of infinitesimally sized spherical shells fusing to form a solid rigid sphere. In the limiting scenario, the moment of inertia for this arrangement is expressed as follows,

$$I_{\text{sphere}} = \tfrac{2}{5} MR^2. \tag{20}$$

The factor 2/5 has physical significance, indicating that the mass within a solid sphere is evenly distributed throughout its entire volume. Consequently, the moment of inertia for a solid sphere is 60 percent less than that of a spherical shell with equivalent mass and radius. This reduction arises due to the greater concentration of mass toward the central region within the solid sphere.

Finally, in Figure 4 we can observe the variation in the moment of inertia of a rigid sphere comprised of an assemblage of N spherical shells. With an increasing number of shells, a discernible reduction in the moment of inertia occurs, transitioning from a two-thirds ratio to an approximation of two-fifths. Consequently, in the limit as the number of shells approaches infinity, a state of convergence emerges, resulting in the formation of a solid rigid sphere characterized by a distinctive moment of inertia. A comprehensive visual representation of this progression, spanning from a finite number of shells to an infinite continuum, is provided in the accompanying video. To access the video, please click here [11].

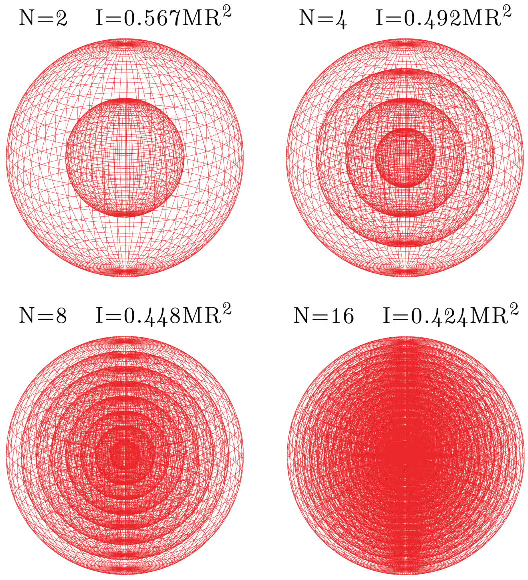

Figure 4: Solid sphere as a collection of spherical shells.

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