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The change in the motions of the Earth and spacecraft launching - a college physics level analysis

(As mudanças no movimento da terra e o lançamento de veículos espaciais - uma análise no nível da graduação em física)

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Both the translational velocity and the angular velocity of the Earth change during a spacecraft launching process, in which a spacecraft is accelerated from the ground and eventually sent into space. This article presents a systematic study of the role played by the changes in the translation and rotation of the Earth in spacecraft launching. Neglecting these changes, which inevitably arise in the interaction between the Earth and the spacecraft, there is an obvious conflict with the conservation laws of momentum and angular momentum. Nevertheless, this flaw in principle is not accompanied by any technically erroneous answers when college students solve the often-encountered exercise problems, thanks to the special reference frames students use. It is pointed out that the technical validity of the Earth-in-constant-motion approximation cannot be generalized to arbitrary reference frames. For example, the correct values of the second and third cosmic velocities cannot be found in an arbitrary reference frame if the velocity of the Earth is treated as a constant. In an arbitrary reference frame, the increase in the translational kinetic energy of the Earth, which is caused by the work done by the gravitational pull by the spacecraft, is not negligible if compared with the increase in the kinetic energy of the spacecraft. It is also demonstrated that the disparity in the energy consumed in launching a spacecraft from the ground along different directions cannot be well interpreted if the angular velocity of the Earth is treated as a constant. When the spacecraft is launched eastwards, the increase in its kinetic energy is partly gained, either directly or indirectly, at the expense of a decrease in the rotational kinetic energy of the Earth.
Keywords: second cosmic velocity, third cosmic velocity, Earth motion, conservation of momentum, conservation of angular momentum, kinetic energy of the Earth.

Tanto a velocidade de translação quanto a velocidade angular de rotação da Terra mudam durante o processo de um lançamento espacial, em que o veículo é acelerada a partir do solo e, finalmente, enviado para o espaço. Este artigo apresenta um estudo sistemático da influência das mudanças na translação e na rotação da Terra sobre o processo de lançamento de um veículo espacial. Negligenciando essas mudanças, há um conflito evidente com as leis de conservação de momento e momento angular. No entanto, esta falha, em princípio, não é acompanhada por respostas tecnicamente erradas quando os estudantes universitários resolvem esse tipo comum de exercício, devido à utilização de sistemas de referência muito particulares. É importante observar que a validade técnica da aproximação da "Terra em movimento constante" não pode ser generalizada para sistemas de referência arbitrários. Por exemplo, os valores corretos da segunda e da terceira velocidades cósmicas não podem ser encontrados em um sistema de referência arbitrário se a velocidade da Terra for tratada como uma constante. Em um sistema de referência arbitrário o aumento da energia cinética de translação da Terra, causado pelo trabalho realizado pela força gravitacional da veículo espacial, não é insignificante quando comparado com o aumento da energia cinética do próprio veículo. Também fica demonstrado que a disparidade na energia consumida no lançamento de um veículo espacial a partir do solo ao longo de direções diferentes não pode ser explicada se a velocidade angular da Terra for tratada como uma constante. Quando o veículo é lançado para leste, o aumento da sua energia cinética é parcialmente ganho, quer direta ou indiretamente, às custas de uma diminuição da energia cinética de rotação da Terra.
Palavras-chave: segunda velocidade cósmica, terceira velocidade cósmica, movimento da Terra, conservação do momento, conservação do momento angular, energia cinética da Terra.

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1. Introduction

As is well known, neglecting the motion of the Earth in studying the motions of particles in the vicinity of the Earth leads to contradictions in principle. The most cited example seems to be that the momentum of the particle-Earth system is not conserved when the particle is supposed to fall to a "motionless" ground [1]. Generally, however, this conceptual deficiency is not accompanied by technically detectable mistakes. This article systematically re-examines this topic from the perspective of spacecraft launching. The changes in the motion of the Earth are studied during both the accelerating process and escaping process of a spacecraft. To maintain a rigorous application of Newton's second law, inertial reference frames, in which the center of mass of the spacecraft-Earth system is either at rest or in a uniform rectilinear motion, are employed. Changes in the motion of the Earth play arguably an indispensable role in the process of spacecraft launching. Omitting these changes could lead to erroneous results.

For clarity, the well-known definitions of the second and third cosmic velocities are repeated here. The minimum velocity (or more unambiguously, speed) required for a spacecraft at the surface of the Earth to escape the gravitational pull of the Earth is defined as the second cosmic velocity (or the "escape velocity"). The minimum velocity for a spacecraft at the surface of the Earth to escape the gravitational pull of the Sun is defined as the third cosmic velocity. In these definitions, it has to be emphasized, the two cosmic velocities are both measured with respect to the Earth center.

2. Some ambiguities in determining the second cosmic velocity and their clarifications

2.1. Possible ambiguities

The mass of the spacecraft, the mass and radius of the Earth are denoted by m, M_e and R_e, respectively. As is usually taught in textbooks, the second cosmic velocity (denoted here by v_2) can be found by invoking the theorem of kinetic energy:

$$0 - \frac{1}{2}mv_2^2 = -\frac{GM_em}{R_e}. \quad (1)$$

Despite the extreme simplicity of this problem, several ambiguities could arise, including: (i) the reference frame in which the problem is treated; (ii) the precise meaning of the term "$-\frac{GM_em}{R_e}$"; and (iii) the reasonableness in neglecting the kinetic energy of the Earth.

2.2. The appropriate reference frames

Newton's second law is valid only in an inertial reference frame. With all the external forces, e.g. those from the Sun and the Moon, omitted, a reference frame in which the center of mass of the spacecraft-Earth system is either at rest or in uniform rectilinear motion should be rigorously speaking an inertial reference frame. Hereafter, a frame in which the center of mass of the spacecraft-Earth system is at rest is termed the "S frame" and a frame in which the center of mass undergoes uniform rectilinear motion is termed the "S' frame".

Of course, one can also opt to use a reference frame in which the center of the Earth is either at rest or in uniform rectilinear motion. In this case, the reference frame is non-inertial because of the gravitational pull received by the Earth from the spacecraft and inertia forces should be introduced to guarantee conceptual rigor [2]. The discussion in this article is restricted to inertial frames only.

2.3. The exact meaning of the term "$-\frac{GM_em}{R_e}$"

In some textbooks, it is explicitly stated and even emphasized that the gravitational potential energy is associated with the particle-Earth system instead of the particle alone [3]. Nonetheless, a student might still be dimly impressed that a particle alone has a gravitational potential energy near the Earth. For example, according to Eq. (1), during the escape of the spacecraft from the Earth, the gravitational potential energy increases from $-\frac{GM_em}{R_e}$ to zero at the expense of a loss in the kinetic energy of the spacecraft alone. That is, the increase in the kinetic energy of the Earth does not appear to play any role in the increase in the potential energy. (Hereafter, the word "increase" means the difference between the kinetic energies at two states, with the kinetic energy of the final state as the minuend and that of the initial state as the subtrahend. An "increase" can be either positive or negative.) Therefore, it is quite natural for a student to relate the potential energy $-\frac{GM_em}{R_e}$ solely to the spacecraft.

As common knowledge, the introduction of potential energy arises from the work done by conservative forces. It should be emphasized that work is actually done by a pair of forces in the above example. That is, the gravitational pull exerted on the Earth by the spacecraft also plays an indispensable role. The following discussion is restricted to the assumption that the force and counter force between two arbitrary particles lie along the straight line that joins the two particles. The work done by such a pair of force and counter force between two particles is

$$dW = \mathbf{F}_{12} \cdot d\mathbf{r}_1 + \mathbf{F}_{21} \cdot d\mathbf{r}_2 = \mathbf{F}_{12} \cdot d(\mathbf{r}_1 - \mathbf{r}_2), \quad (2)$$

where $d\mathbf{r}_1$ and $d\mathbf{r}_2$ are respectively the elementary displacements of particles 1 and 2 in an arbitrary reference frame, $\mathbf{F}_{12}$ is the force exerted on particle 1 by particle 2, and $\mathbf{F}_{21}$ that on particle 2 by particle 1.

Let $F_{12} = |\mathbf{F}_{12}|$, $\boldsymbol{\rho} = \mathbf{r_1} - \mathbf{r_2}$, and $\rho = |\boldsymbol{\rho}|$. Then [4]

$$dW = \pm F_{12} d\rho. \tag{3}$$

Importantly, $d\rho$ is actually the increase in the distance between particles 1 and 2 and is independent of the choice of reference frame in Newtonian mechanics. Therefore, the work done by this force pair is independent of reference frame and consequently, can be calculated in any reference frame. Generally, the most convenient frame for the calculation is the frame in which one of the two particles, *e.g.* particle 2, is at rest. In this case, $d\mathbf{r}_2 = 0$ and

$$dW = \mathbf{F}_{12} \cdot d\mathbf{r}_1 + \mathbf{F}_{21} \cdot d\mathbf{r}_2 = \mathbf{F}_{12} \cdot d\mathbf{r}_1. \tag{4}$$

That is, calculating the work done by the force on particle 1 only in this particle-2-at-rest frame is equivalent to calculating the work done by the force pair in an arbitrary reference frame.

As commonly practiced in most textbooks, in the reference frame in which the Earth's center is at rest, the work done by the Earth's gravitational pull on the spacecraft is easily calculated to be "$-\frac{GM_e m}{R_e}$". From the above discussion, it is readily known that in the S frame the sum of the work done by the gravitational pull exerted on the spacecraft and that on the Earth is also "$-\frac{GM_e m}{R_e}$". Equivalently, the term "$-\frac{GM_e m}{R_e}$" can be understood as the potential energy of the spacecraft-Earth system instead of that of the spacecraft alone.

Consequently, in the S frame, the term $-\frac{GM_e m}{R_e}$ should be equal to the increase in the kinetic energy of the spacecraft-Earth system instead of the kinetic energy of the spacecraft alone. According to König's theorem, the kinetic energy of the Earth, which is treated as a rigid body here, is the sum of two parts, namely, its "translational kinetic energy" and its "rotational kinetic energy". During the escape of the spacecraft, the rotational kinetic energy of the Earth remains unchanged and is not taken into consideration when the theorem of kinetic energy is used.

The escape of the spacecraft from the Earth is reached if its relative velocity with respect to the Earth is at least zero at infinity. Because the center of mass of the spacecraft-Earth system is motionless in the S frame, after escape both the spacecraft and the Earth have (at least) zero velocity in the S frame. Thus Eq. (1) should be replaced by

$$-\frac{GM_e m}{R_e} = 0 - \left[\frac{1}{2} m \left(\mathbf{V}_e + \mathbf{v}_2\right)^2 + \frac{1}{2} M_e V_e^2\right], \tag{5}$$

where $\mathbf{V}_e$ denotes the initial velocity of the Earth in the S frame when escape has just commenced. (It might be necessary to remind possible freshman readers again that both the second and the third cosmic velocities are measured with respect to the Earth center. Therefore, when measured in the S frame, the velocity of a spacecraft with the second cosmic velocity is $\mathbf{V}_e + \mathbf{v}_2$, not $\mathbf{v}_2$.)

Moreover, another equation derived from the conservation of momentum should be added

$$m \left(\mathbf{V}_e + \mathbf{v}_2\right) + M_e \mathbf{V}_e = 0. \tag{6}$$

The process involving a spacecraft escaping the Earth, as observed in the S frame, is illustrated in Fig. 1. Two cases are described. In Fig. 1(a), both the initial velocity of the spacecraft and that of the Earth are along the straight line that joins the spacecraft and the Earth's center. In Fig. 1(b), the spacecraft is launched along a tangent to the Earth's surface. Fig. 1(a) provides the most direct visualization of this problem whereas Fig. 1(b) illustrates the use of the Earth's rotation in the actual launch, as will be discussed in section 4.

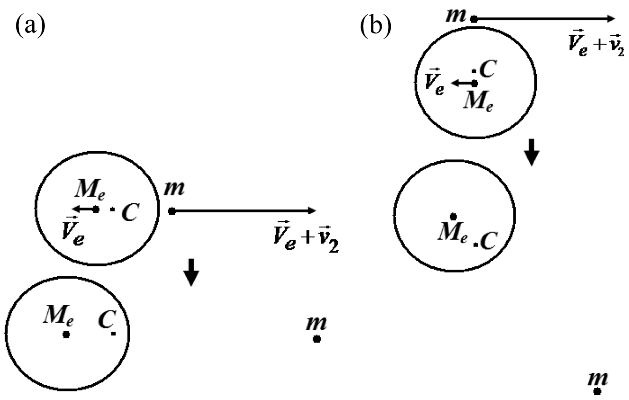

Figure 1 - Launching a spacecraft with the second cosmic velocity as observed in the S frame. (The initial velocity of the Earth and the distance from the center of mass of the spacecraft-Earth system to the Earth's center are both greatly exaggerated.) (a) The spacecraft is launched along the straight line that joins the spacecraft and the Earth's center; (b) The spacecraft is launched along a tangent to the Earth's surface.

2.4. On the technical validity of omitting the increase in the Earth's translational kinetic energy

Using Eqs. (5) and (6), the second cosmic velocity can be found

$$v_2 = \sqrt{\frac{2G\left(M_e + m\right)}{R_e}}, \tag{7}$$

from which the often-used approximate result can be comfortably obtained after dropping m

$$v_2 = \sqrt{\frac{2GM_e}{R_e}}. \tag{8}$$

As common practice, Eq. (8) is usually obtained directly from Eq. (1), thus it should be interesting to study the connection between Eqs. (5), (6) and Eq. (1).

In fact, after eliminating $\mathbf{V}_e$ using Eq. (6), Eq. (5) is rewritten as

$$-\frac{GM_e m}{R_e} = \left[0 - \frac{1}{2}m\left(\frac{M_e}{m+M_e}\mathbf{v}_2\right)^2\right] + \left[0 - \frac{1}{2}M_e\left(-\frac{m\mathbf{v}_2}{m+M_e}\right)^2\right]. \quad (9)$$

Now, Eq. (1) can be obtained from Eqs. (5) and (6) using two approximations. First, the initial speed of the spacecraft in the S frame, $\frac{M_e v_2}{m+M_e}$, is approximated by v_2. Second, and more importantly, the increase in the Earth's kinetic energy, $0 - \frac{1}{2}M_e\left(-\frac{m\mathbf{v}_2}{m+M_e}\right)^2$, is omitted directly. Both approximations can be well justified when the condition $m << M_e$ is taken into consideration.

However, the situation is different when a reference frame in which the spacecraft-Earth system is in motion is used. Let $\mathbf{u}_c$ denote the constant non-zero velocity of the center of mass of this system in the S′ frame. The velocity of the Earth at the beginning of the escape in the S' frame is denoted by $\mathbf{U}_e$. Apparently, the definition of v_2 demands both the velocity of the spacecraft and that of the Earth should be $\mathbf{u}_c$ after the escape of the spacecraft. The process of escape is illustrated in Fig. 2. Accordingly, Eqs. (5) and (6) should be revised to state

$$-\frac{GM_e m}{R_e} = \frac{1}{2}(m+M_e)u_c^2 - \left[\frac{1}{2}m(\mathbf{U}_e+\mathbf{v}_2)^2 + \frac{1}{2}M_e U_e^2\right], \quad (10)$$

and

$$m(\mathbf{U}_e+\mathbf{v}_2) + M_e\mathbf{U}_e = (m+M_e)\mathbf{u}_c. \quad (11)$$

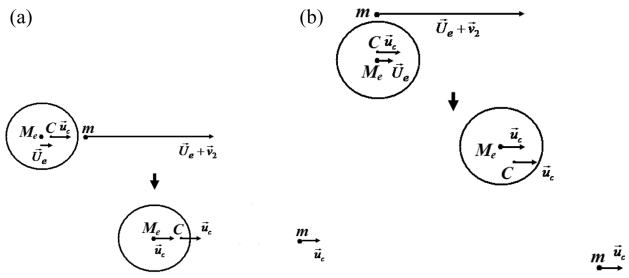

Figure 2 - Launching a spacecraft with the second cosmic velocity as observed in the S' frame. (For simplicity, $\mathbf{u}_c$ and $\mathbf{U}_e$ are assumed to be along the same direction.) (a) The spacecraft is launched along the straight line joining the spacecraft and the Earth's center; (b) The spacecraft is launched along a special tangent to the Earth's surface.

Equation (7), and thus Eq. (8), can still be obtained from Eqs. (10) and (11). That $\mathbf{u}_c$ does not appear in the final result is expected because v_2 is the relative speed of the spacecraft with respect to the Earth and is independent of the reference frame.

The right side of Eq. (10) separates into two parts: the increase in the kinetic energy of the spacecraft, denoted by ΔE_{km}, and that of the Earth, denoted by ΔE_{ke}. Taking into consideration $m << M_e$, their expressions reduce to

$$\Delta E_{km} = \frac{1}{2}m\left[u_c^2 - (\mathbf{U}_e+\mathbf{v}_2)^2\right] = -\frac{1}{2}m\left[2\mathbf{u}_c\cdot\mathbf{v}_2 + v_2^2\right], \quad (12)$$

and

$$\Delta E_{ke} = \frac{1}{2}M_e\left(u_c^2 - U_e^2\right) = m\mathbf{u}_c\cdot\mathbf{v}_2. \quad (13)$$

Apparently, ΔE_{ke} is generally not negligible in comparison with ΔE_{km} unless $|\mathbf{u}_c\cdot\mathbf{v}_2| << v_2^2$. Consequently, when solving the problem in the S' frame, neglecting the increase in the Earth's kinetic energy during the escape of the spacecraft, i.e. treating the velocity of the Earth as constant, generally leads to a wrong result.

The physical origin of this disparity in treating the Earth's kinetic energy in the two frames can be readily understood by comparing the power applied to the Earth by the gravitational field with that applied to the spacecraft, here denoted by P_e and P_m, respectively. The two gravitational forces here are identical in magnitude but opposite in direction. Furthermore, the velocity of the Earth in the S frame at a certain moment is denoted by $\mathbf{V}$ and that of the spacecraft by $\mathbf{v}$. Obviously, $V = |\mathbf{V}| << v = |\mathbf{v}|$ and

$$\left|\frac{P_e}{P_m}\right| = \frac{V}{v} << 1. \quad (14)$$

Therefore, the increase of the kinetic energy of the Earth is negligible. However, when one moves to the S' frame, the corresponding ratio becomes:

$$\left|\frac{P_e'}{P_m'}\right| = \gamma\frac{|\mathbf{u}_c+\mathbf{V}|}{|\mathbf{u}_c+\mathbf{v}|}, \quad (15)$$

where γ is a factor determined by the velocity directions of the spacecraft and the Earth. Although $\left|\frac{V}{v}\right| << 1$, $\frac{|\mathbf{u}_c+\mathbf{V}|}{|\mathbf{u}_c+\mathbf{v}|}$ is not necessarily a small quantity. That is, the increase in the kinetic energy of the Earth is not generally negligible in comparison with the increase in the kinetic energy of the spacecraft.

Of course, when trying to determine the second cosmic velocity, few students (and teachers) would use the S' frame, because using such a frame only makes the issue unnecessarily more troublesome. However, as shown in the next section, in order to determine the third cosmic velocity, the S' frame could be used, perhaps unconsciously. In this case, the correct answer is not obtainable if the increase in the Earth's kinetic energy is erroneously omitted.

3. Calculating the third cosmic velocity directly in the Solar System

The velocity and speed of the Earth in the reference frame fixed to the Sun are:

$$U_e = |\mathbf{U}_e| = \sqrt{\frac{GM_s}{R_{se}}}, \quad (16)$$

where M_s and R_{se} are the mass of the Sun and the radius of the orbit of the Earth around the Sun, respectively. (The orbit is approximated by a circle.)

The third cosmic velocity is denoted by v_3 here. One ostensibly plausible approach to determine v_3 directly in the fixed-to-the-Sun reference frame is to resort to the theorem of kinetic energy for the spacecraft

$$-\frac{GM_e m}{R_e} - \frac{GM_s m}{R_{se}} = 0 - \frac{1}{2}m\left(\mathbf{U}_e + \mathbf{v}_3\right)^2, \quad (17)$$

from which the wrong result

$$v_3 = 1.38 \times 10^4 \text{ m/s} \quad (18)$$

is obtained.

The introduction of a critical speed, $\xi = |\boldsymbol{\xi}|$, defined as the minimal speed required for the spacecraft in the Earth's orbit to escape the Solar System, can establish a connection between Eq. (17) and the discussion in the previous section: given that

$$\frac{1}{2}m\xi^2 - \frac{GM_s m}{R_{se}} = 0, \quad (19)$$

Eq. (17) can be re-written as

$$-\frac{GM_e m}{R_e} = \frac{1}{2}m\xi^2 - \frac{1}{2}m\left(\mathbf{U}_e + \mathbf{v}_3\right)^2. \quad (20)$$

In this treatment, the left side of Eq. (20) is understood as the work done by the Earth's gravitation field on the spacecraft during the escape process. The right side is the increase in the kinetic energy of the spacecraft during this process. Right after the spacecraft escapes from the Earth, i.e., when it is sufficiently far away from the Earth but still near the Earth's orbit around the Sun, the residual speed needs to be at least "ξ" for the spacecraft to further escape the Sun's gravitational pull. From the discussion in the preceding section, the reason behind the mistake can be readily seen: $-\frac{GM_e m}{R_e}$ should be understood as the total work done by the gravitational pull between the Earth and the spacecraft and is equal to the increase in the kinetic energy of the spacecraft-Earth system instead of that of the spacecraft alone. That is, the erroneous omission of the increase in the Earth's kinetic energy leads to the wrong answer [5].

Denoting the velocity of the Earth after the escape of the spacecraft by $\mathbf{U}_{e1}$, Eq. (20) is revised as

$$-\frac{GM_e m}{R_e} = \left(\frac{1}{2}m\boldsymbol{\xi}^2 + \frac{1}{2}M_e\mathbf{U}_{e1}^2\right) - \left[\frac{1}{2}m\left(\mathbf{U}_e + \mathbf{v}_3\right)^2 + \frac{1}{2}M_e\mathbf{U}_e^2\right]. \quad (21)$$

Assuming the spacecraft escape time from the Earth is very short, the impulse due to the gravitational pull from the Sun during this time is negligible. Therefore, the momentum of the spacecraft-Earth system is approximately conserved

$$M_e\mathbf{U}_{e1} + m\boldsymbol{\xi} = M_e\mathbf{U}_e + m\left(\mathbf{U}_e + \mathbf{v}_3\right). \quad (22)$$

Taking $m << M_e$ and using Eqs. (16), (19), (21), and (22), v_3 is found to be

$$v_3 = \sqrt{\xi^2 + U_e^2 - 2U_e\xi + \frac{2GM_e}{R_e}} = \sqrt{\left(3 - 2\sqrt{2}\right)\frac{GM_s}{R_{se}} + \frac{2GM_e}{R_e}} = 1.66 \times 10^4 \text{ m/s}. \quad (23)$$

4. The change in Earth's rotation in accelerating a spacecraft

So far, the S frame has proved to be an appropriate reference frame in studying the spacecraft escape process from the Earth *after* attaining the necessary high speed, e.g. v_2, because the omission of the increase in the Earth's translational kinetic energy in this reference frame makes no technical difference to the final result. However, if the S frame is further used in determining the energy required to accelerate an initially grounded spacecraft to the same high speed, caution must be taken again not to neglect the change in the motion of the Earth.

As is well known, in terms of energy saving, a spacecraft should be launched eastwards, so that the speed of the Earth surface can be used [6]. However, to escape the Earth, irrespective of launch direction, the final speed of the spacecraft after the acceleration stage has finished is always (almost) v_2 as observed in the S frame. If the increase in the kinetic energy of the spacecraft alone is considered, the energy expended would be independent of launch direction. That is, there is no difference in the energy consumed in launching eastwards or westwards. As is discussed in this section, when studied in the S frame, this difference is actually incorporated into the change in the Earth's rotation. The argument is demonstrated in two scenarios. The first is directly perceivable but unrealistic; the second has a more factual basis in modern astronautics.

4.1. Verne's gun

Verne envisaged a powerful gun that could send a spacecraft to the Moon. Impractical as this approach is, it is nonetheless interesting to study the process where a spacecraft is accelerated to the second cosmic velocity by such a gun on the surface of the Earth in the S frame. For simplicity, it is assumed that the gun is in rigid joint with the Earth. That is, the relative motion between the gun and the Earth caused by the spacecraft launching is neglected. Instead, it is imagined that the Earth and the gun jointly receives the recoil.

Let $\mathbf{v}_i = v_i\hat{e}$ ($v_i = |\mathbf{v}_i|$) and $\mathbf{V}_i = -V_i\hat{e}$ ($V_i = |\mathbf{V}_i|$), where $\hat{e}$ is the eastward unit vector, denote the respective velocities of the spacecraft grounded at point "A" on the Earth surface and the center of the Earth "O" in the S frame before the launching. In the S frame, the center of mass of the spacecraft-Earth system "C" is at rest, thus

$$m\mathbf{v}_i + M_e\mathbf{V}_i = 0. \qquad (24)$$

Before launching, as depicted in Fig. 3, the spacecraft is motionless with respect to the surface of the Earth and its velocity in the S frame is purely the "velocity of following". For simplicity, the spacecraft is assumed to be on the equator.

$$\mathbf{v}_i = \mathbf{V}_i + \mathbf{\Omega}_i \times \mathbf{OA}, \quad v_i = -V_i + \Omega_i R_e, \qquad (25)$$

where $\mathbf{\Omega}_i$ is the angular velocity of the Earth before launch, and $\Omega_i = |\mathbf{\Omega}_i|$. Thus,

$$V_i = \frac{m\Omega_i R_e}{M_e + m} \text{ and } v_i = \frac{M_e\Omega_i R_e}{M_e + m}. \qquad (26)$$

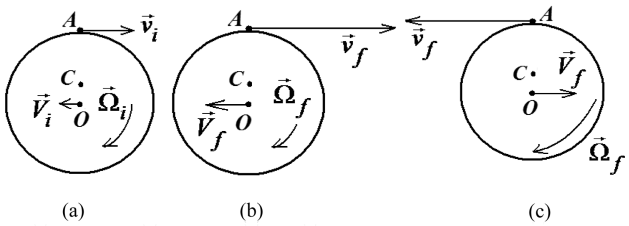

Figure 3 - Launch of an initially grounded spacecraft using Verne's gun, as observed in the S frame. (a) Before launch; (b) after an eastward launch; (c) after a westward launch.

When the accelerating process concludes, the spacecraft gains a relative velocity $\mathbf{v}_2$ with respect to the Earth's center. That is,

$$\mathbf{v}_f - \mathbf{V}_f = \mathbf{v}_2 = \pm v_2\hat{e}, \qquad (27)$$

where the upper sign corresponds to an eastward launch and the lower sign to a westward launch. Again, using momentum conservation, the final velocities of the spacecraft and the Earth's center can be calculated

$$\mathbf{v}_f = \pm\frac{M_e v_2}{M_e + m}\,\hat{e} \text{ and } \mathbf{V}_f = \mp\frac{m v_2}{M_e + m}\,\hat{e}. \qquad (28)$$

For both eastward and westward launches, the increase in the spacecraft's kinetic energy is

$$\Delta E_{km} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \frac{1}{2}m\left(\frac{M_e}{M_e + m}\right)^2\left(v_2^2 - \Omega_i^2 R_e^2\right). \qquad (29)$$

Similarly, in both eastward and westward launchings, the increase in the Earth's translational kinetic energy is

$$\Delta E_{kt} = \frac{1}{2}M_e V_f^2 - \frac{1}{2}M_e V_i^2 = \frac{1}{2}M_e\left(\frac{m}{M_e + m}\right)^2\left(v_2^2 - \Omega_i^2 R_e^2\right) << \Delta E_{km}. \qquad (30)$$

Furthermore, the recoil of the powerful gun inevitably changes not only the velocity of the Earth's center, but also, more importantly, the angular velocity of the Earth. Neglecting external forces, the angular momentum of the spacecraft-Earth system is conserved. Also, the period of acceleration is approximated as being very short, so that the change in the spacecraft's position during acceleration can be neglected. The most straightforward reference point (not necessarily the most convenient one) for calculating angular momenta is the center of mass, i.e. point C. The axis that passes the Earth's center O and is perpendicular to the equator plane is approximated as a principal axis of inertia and the corresponding moment of inertia is denoted by I. Then,

$$\mathbf{CA} \times m\mathbf{v}_i + \mathbf{CO} \times M_e\mathbf{V}_i + I\mathbf{\Omega}_i = \mathbf{CA} \times m\mathbf{v}_f + \mathbf{CO} \times M_e\mathbf{V}_f + I\mathbf{\Omega}_f, \qquad (31)$$

where $\mathbf{\Omega}_f$ is the angular velocity of the Earth after the accelerating process.

Using conservation of momentum, Eq. (31) can be rewritten as

$$R_e \cdot mv_i + I\Omega_i = \pm R_e \cdot mv_f + I\Omega_f. \qquad (32)$$

Equation (32) demonstrates that the most convenient reference point to calculate the angular momentum of the spacecraft-Earth system is actually the spatial point in the S frame that momentarily coincides with the Earth's center O during the accelerating of the spacecraft. Rearrangement of Eq. (32) yields

$$\Omega_f = \Omega_i + \frac{R_e m}{I}\left(v_i \mp v_f\right). \qquad (33)$$

Now the increase in the rotational kinetic energy of the Earth can be calculated

$$\Delta E_{kr} = \frac{1}{2} I\Omega_f^2 - \frac{1}{2} I\Omega_i^2 =$$
$$R_e m (v_i \mp v_f) \cdot \left[\frac{R_e m}{2I} (v_i \mp v_f) + \Omega_i \right]. \quad (34)$$

As a good approximation, it is assumed that $I = kM_e R_e^2$. Obviously, in Eq. (34), $\left| \frac{R_e m}{2I} (v_i \mp v_f) \right| = \left| \frac{m}{2kM_e} \frac{(v_i \mp v_f)}{R_e} \right| << \Omega_i$. Hence, by using Eqs. (26), (28), (29), (30), (34) and approximating $\frac{M_e}{M_e+m}$ and $\frac{m}{M_e+m}$ by 1 and 0, respectively, the increase in the kinetic energy of the spacecraft-Earth system is calculated to be

$$\Delta E_k = \Delta E_{km} + \Delta E_{kt} + \Delta E_{kr} =$$
$$\Delta E_{km} + \Delta E_{kr} = \frac{1}{2} m (v_2 \mp \Omega_i R_e)^2. \quad (35)$$

This result is in agreement with that obtained in the reference frame fixed to the ground. Irrespective of whether observed in the S frame or the frame fixed to the ground, the conclusion is the same: launching the spacecraft eastwards consumes less energy than launching westwards. Nonetheless, the interpretations in the two frames differ. In the frame fixed to the ground, the initial spacecraft speed is zero. The final spacecraft speed in an eastwards launch, $v_2 - \Omega_i R_e$, is smaller than that in a westwards launch, $v_2 + \Omega_i R_e$. In the S frame, when the spacecraft is launched eastwards, the gun's recoil slows the rotation of the Earth and $\Delta E_{kr} < 0$. That is, a certain amount of the Earth's rotational kinetic energy is converted into kinetic energy of the spacecraft; thus less chemical energy from the gunpowder is consumed. When the spacecraft is launched westwards, the recoil pushes the Earth to rotate faster and $\Delta E_{kr} > 0$. In this case, the expended chemical energy from the gunpowder is used to increase both the kinetic energy of the spacecraft and the rotational kinetic energy of the Earth.

4.2. Tsiolkovsky's rocket

The derivation of Tsiolkovsky's ideal rocket equation can be routinely found in college textbooks. It is repeated here with all velocities written in vector form.

The total mass of the rocket before launching is m_0. The mass of the spacecraft, which is the payload of the rocket, is m. The initial velocity of the grounded rocket observed in the S frame is $\mathbf{v}_i = v_i \hat{e}$, where $\hat{e}$ is still the eastward-pointing unit vector, and the final velocity of the spacecraft after depletion of all fuel is $\mathbf{v}_f = \pm v_f \hat{e}$. The effective exhaust velocity of the fuel is $\boldsymbol{\eta} = \mp \eta \hat{e}$. Here $v_f = |\mathbf{v}_f| > 0$ and $\eta = |\boldsymbol{\eta}| > 0$. Again, upper signs refer to eastward launchings and lower signs to westward launchings. The velocity of the rocket at a given moment is $\mathbf{v}$. The mass of the rocket at this moment is denoted also by m. The increase in the rocket's mass due to the expended fuel over an infinitesimal time interval is $dm < 0$. Correspondingly, the increase in the velocity of the rocket is $d\mathbf{v}$.

The conservation of the momentum of the rocket system requires:

$$m\mathbf{v} = (m + dm)(\mathbf{v} + d\mathbf{v}) + (-dm)(\mathbf{v} + d\mathbf{v} \mp \eta \hat{e}). \quad (36)$$

Therefore,

$$\int_{v_i \hat{e}}^{\pm v_f \hat{e}} d\mathbf{v} = \mp \eta \hat{e} \int_{m_0}^{m} \frac{dm}{m}, \quad (37)$$

from which the relation between the mass of the spacecraft and the initial mass of the rocket can be established:

$$m_0 = m e^{\frac{v_f \mp v_i}{\eta}}. \quad (38)$$

That is, for identical final speed $v_f (> v_i)$, a rocket can have a smaller mass when launched eastwards than launched westwards. Unlike Verne's gun, here the advantage in launching eastwards is apparent without the need to consider the increase in the rotational kinetic energy of the Earth. The reason is very simple. In obtaining the ideal rocket equation, the rocket, including payload and fuel, is approximated as an isolated system. For Verne's gun, the accelerating of the spacecraft requires accelerating the ground in the opposite direction. For Tsiolkovsky's rocket, the accelerating of the spacecraft is negated by expelling spent fuel in the opposite direction.

However, as illustrated in Fig. 4, because of the interaction between the spent fuel and the Earth, launching of a spacecraft still inevitably changes the motion of the Earth. To obtain analytical expressions appropriate for college-level physics, a number of approximations, some of which could not be very well justified in the actual situation, have to be employed. For example, the accelerating process is assumed to take place over a short duration, so that the displacements of the spacecraft, fuel, and the Earth from their original positions can be neglected. (For clarity, in Fig. 4, spacecraft and fuel are drawn as being separated by a large distance after launching. In the calculation, this separation is neglected.) Furthermore, Earth's atmosphere is disregarded. In reality, the spent fuel leads directly to changes in the motion of the surrounding atmosphere and eventually changes the Earth's motion. In this article, for simplicity, it is assumed that spent fuel falls to the ground and adheres to the Earth in a perfectly inelastic collision.

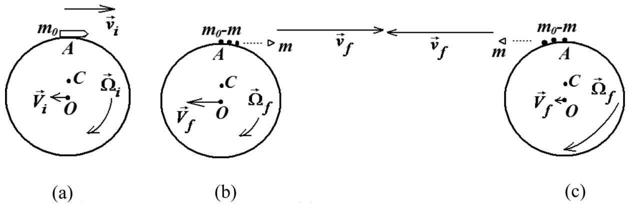

Figure 4 - Launching of an initially grounded spacecraft using Tsiolkovsky's rocket as observed in the S frame. (a) Before the launching; (b) after an eastward launching; (c) after a westward launching.

For an eastward launching, the rocket's velocity before launching is $\mathbf{v}_i = (-V_i + R_e\Omega_i)\,\hat{e}$, and the velocity of the fuel after the launching, which is first expelled from the rocket and eventually falls to ground, is $[-(V_i + \Delta V) + R_e(\Omega_i + \Delta\Omega)]\,\hat{e}$, where ΔV and $\Delta\Omega$ are the increase in the speed and angular velocity of the Earth caused by the adhesion of the fuel. Momentum conservation of the spacecraft-fuel-Earth system expressed in the S frame requires

$$m_0(-V_i + R_e\Omega_i) - M_eV_i = mv_f + (m_0 - m) \times [-V_i - \Delta V + R_e(\Omega_i + \Delta\Omega)] - M_e(V_i + \Delta V) = 0. \quad (39)$$

The angular momentum of the system is also conserved. Here the spatial point that coincides with the Earth's center at the moment of launch (not the Earth's center itself) is taken as the reference point. Therefore,

$$R_em_0(-V_i + R_e\Omega_i) + kM_eR_e^2\Omega_i = R_emv_f + R_e(m_0 - m)[-(V_i + \Delta V) + R_e(\Omega_i + \Delta\Omega)] + kM_eR_e^2(\Omega_i + \Delta\Omega). \quad (40)$$

As in the preceding section, it is easy to verify that the increase in the Earth's translational kinetic energy is negligible in comparison with the increase in the spacecraft's kinetic energy.

From Eqs. (39) and (40), $\Delta\Omega$ can be found

$$\Delta\Omega = \frac{-m(V_i + v_f - R_e\Omega_i)}{R_e[kM_e + (k+1)(m_0 - m)]} = -\frac{m(v_f - R_e\Omega_i)}{kM_eR_e} < 0. \quad (41)$$

The increase in the rotational kinetic energy of the Earth is thus calculated to be

$$\Delta E_{kr} = \frac{1}{2}kM_eR_e^2\left[(\Omega_i + \Delta\Omega)^2 - \Omega_i^2\right] = -mR_e\Omega_i(v_f - R_e\Omega_i) < 0. \quad (42)$$

The spacecraft's accelerating eventually slows down the Earth's rotation when the spent fuel falls to the ground. In this sense, it is still reasonable to argue that the spacecraft gains its kinetic energy at the expense of the rotational kinetic energy of the Earth.

In a westward launch, the corresponding results are

$$\Delta\Omega = \frac{m(v_f + R_e\Omega_i) - mV_i}{R_e[kM_e + (k+1)(m_0 - m)]} = \frac{m(v_f + R_e\Omega_i)}{kM_eR_e} > 0, \quad (43)$$

and

$$\Delta E_{kr} = \frac{1}{2}kM_eR_e^2\left[(\Omega_i + \Delta\Omega)^2 - \Omega_i^2\right] = mR_e\Omega_i(v_f + R_e\Omega_i) > 0. \quad (44)$$

5. Conclusion

When one tries to solve an astronautics problem, such as finding the second and third cosmic velocities, in an inertial reference frame in which the spacecraft-Earth system is moving, the increase in the kinetic energy of the Earth should be considered, because the power of the gravitational pull from the spacecraft doing work on the Earth is not negligible in comparison with the power of the gravitational pull from the Earth doing work on the spacecraft.

The launching of an initially grounded spacecraft into space inevitably changes the Earth's rotation. Launching the spacecraft eastwards is more economical in terms of energy consumption because part of the rotational kinetic energy of the Earth is converted into kinetic energy of the spacecraft. For Verne's gun, the conversion is direct; for Tsiolkovsky's rocket, the conversion is indirect.

References

[1] T. Moore, *Six Ideas that Shaped Physics, Unit C: Conservation Laws Constrain Interaction* (WCB McGraw-Hill, New York, 2003), 2^{nd} ed, p. 60.

[2] B.K. Gao and T.Z. Xie, College Physics **11**, 46 (1991) (in Chinese).

[3] R.A. Serway and J.W. Jewett, *Principle of Physics* (Qinghua University Press, Beijing, 2004), 3^{rd} ed., p. 211.

[4] A.S. Qi and C.Y. Du, *Mechanics* (Higher Education Press, Beijing, 2005) 2^{nd} ed., p. 127 (in Chinese).

[5] Z.H. Hu, College Physics **7**, 26 (1984) (in Chinese).

[6] R.A. Serway and J.W. Jewett, *Principle of Physics* (Qinghua University Press, Beijing, 2004), 3^{rd} ed., p. 320.