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Non-harmonic response of relativistic particle to driving harmonic force

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We consider the properties of one-dimensional oscillations of a relativistic particle under a driving harmonic force. It is found that in the general case the speed oscillations are an infinite sum of odd sinusoidal and even cosinusoidal waves. In the limiting case of the ultra-relativistic particle, they represent a square wave. There is also the oscillation suppression for large values of the force amplitude and particular values of its initial phase. In this case, the particle performs only drift motion. The topic is addressed to undergraduates studying the applications of the theory of oscillations.
Keywords: Driven oscillator, relativistic particle, Fourier series, square wave.

1. Introduction

There are many paradoxes and extraordinary phenomena appearing in the theory of relativity [1]. Of particular interest are the effects of relativistic dynamics. For example, the magnitude and direction of the acceleration of a relativistic particle are determined not only by the force but also by its instantaneous velocity. Moreover, in the general case, the acceleration and force vectors do not coincide in direction [2]. It is also shown that a negative acceleration component can exist in the direction of the biggest force component and that acceleration does not decrease monotonically to zero [3].

In Ref. [4], the basic properties of projectile motion in special relativity were established. Shahin [5] has found that, unlike non-relativistic projectile motion, the launching angles that maximize both the horizontal range as well as the area under the trajectory are functions of the initial speed. The classical dynamics of the one-dimensional relativistic oscillator in the presence of both linear and Coulomb-like restoring force is explored in Ref. [6] and [7].

In this paper, we consider the properties of one-dimensional oscillations of a relativistic particle under a driving harmonic force. This type of oscillation can be realized, for example, by a charged particle located between the plates of a parallel-plate capacitor connected to an ac voltage. The motion of a charged particle in an oscillating electric field is also of interest in plasma physics, as many non-linear phenomena have a simple explanation in terms of the ponderomotive force (or Miller force) [8]. Finally, it has been suggested that the motion of charged particles in an oscillating electric field can be used in particle separation [9]. The issues covered in this paper will be useful to undergraduates studying the applications of the theory of oscillations.

2. Time Dependence of the Velocity

Let us consider a relativistic particle with the rest mass m that moves along the Ox-axis under the driving harmonic force

$$F_x = F_0 \cos\varphi, \tag{1}$$

where $\varphi = \omega t$; ω is the angular frequency and $F_0 > 0$ is the amplitude value of the oscillating force. We assume that at the initial time $t = 0$ $v_x = 0$, where $\vec{v}$ is the particle velocity. In this paper, we neglect the radiation damping force [10], which causes the weak damping of the oscillations over time. It allows us to apply the relativistic form of Newton's second law.

In special relativity, Newton's second law is expressed as

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = F_x \tag{2}$$

where

$$p_x = \frac{mv_x}{\sqrt{1 - v_x^2/c^2}} \tag{3}$$

is the relativistic momentum. Separating the variables in equation (2) and taking into account equations (1), (3) along with the initial condition $v_x(0) = 0$, we get:

$$\frac{v_x}{\sqrt{1 - v_x^2/c^2}} = \frac{F_0}{m\omega} \sin\varphi. \tag{4}$$

Solving equation (4) with respect to variable v_x, we derive:

$$u_x(\varphi) = \pm\frac{|\xi \sin\varphi|}{\sqrt{(\xi \sin\varphi)^2 + 1}} = \frac{\xi \sin\varphi}{\sqrt{(\xi \sin\varphi)^2 + 1}}, \tag{5}$$

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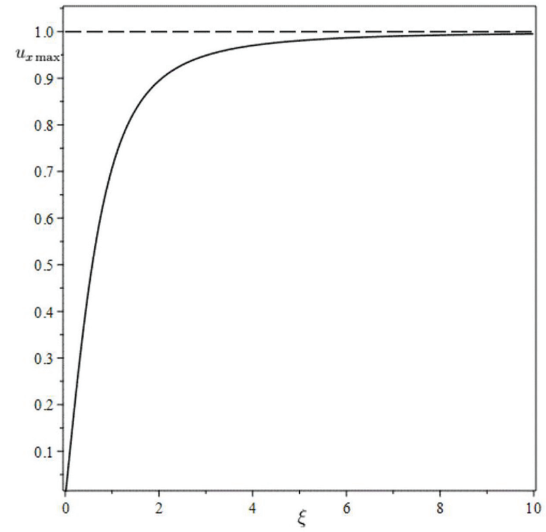

Figure 1: Dependence $u_{x\,\max}(\xi)$.

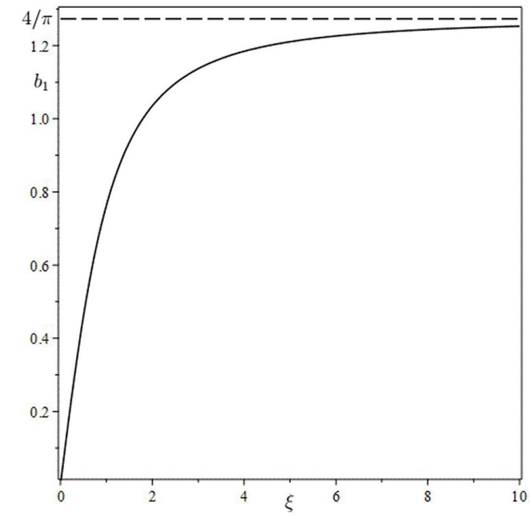

Figure 2: Dependence $b_1(\xi)$.

where $u_x = v_x/c$ is the dimensionless instant velocity; $\xi = F_0/(m\omega c) > 0$ is the dimensionless force amplitude. Therefore, the velocity of the particle oscillates in phase with the force, however, such an oscillation is anharmonic. According to equation (5) the oscillation amplitude is

$$u_{x\,\max} = \frac{\xi}{\sqrt{\xi^2+1}}. \tag{6}$$

The maximum value of this quantity is attained asymptotically at $\xi \to \infty$ (we want to remind the reader that dimensionless quantity ξ represents the amplitude of the force and not relative velocity amplitude so, the situation when $\xi \gg 1$ is quite realizable in practice) and equal to 1 (Figure 1).

In order to explore the features of oscillation (5), we expand the periodic function (5) into a Fourier series in the sine-cosine form:

$$u_x(\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\varphi) + \sum_{n=1}^{\infty} b_n \sin(n\varphi), \tag{7}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} u_x(\varphi)\,\mathrm{d}\varphi \tag{8}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u_x(\varphi)\cos(n\varphi)\,\mathrm{d}\varphi \tag{9}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u_x(\varphi)\sin(n\varphi)\,\mathrm{d}\varphi \tag{10}$$

The coefficient a_0 vanishes because the integrand in equation (8) is odd. For the same reason, all coefficients a_n are equal to zero. The integrand in equation (10) is even. It allows us to reduce the integration to the segment $[0, \pi]$. For even numbers n, the integrand $f(\varphi)$ in equation (10) also satisfies the equality:

$$f(\varphi + \pi) = -f(\varphi) \tag{11}$$

(in this case, function $f(\varphi)$ is said to be anti-periodic function with anti-period π [11]). Using this property and considering equation (10), we conclude that all coefficients b_n with even numbers n are equal to zero too. Thus,

$$u_x(\varphi) = \sum_{k=1}^{\infty} b_k \sin\{(2k-1)\varphi\}, \tag{12}$$

where

$$b_k = \frac{2}{\pi}\int_{0}^{\pi} u_x(\varphi)\sin\{(2k-1)\varphi\}\,\mathrm{d}\varphi. \tag{13}$$

For relatively small values of ξ $b_1 \approx \xi$ (Figure 2); all the other coefficients b_k $(k > 1)$ are negligibly small. Therefore, in this case, the particle performs harmonic oscillations, that is, it obeys the laws of classical Newtonian mechanics. As the force amplitude ξ increases, these coefficients increase as well. At any fixed value of ξ, each subsequent coefficient b_k is less than the previous one (Figure 3).

The most interesting situation occurs in the limiting case of large values of ξ. In this instance, all coefficients b_k asymptotically approach their maximum values

$$b_{k\,\max} = \lim_{\xi\to\infty} b_k(\xi) = \frac{4}{(2k-1)\pi}. \tag{14}$$

The resulting oscillation is a square wave, that is, a periodic wave that varies abruptly in amplitude between two opposite fixed values, spending equal times at each (Figure 4). It means that for this condition the speed of the oscillating particle remains almost constant and approximately equal to the speed of light during the whole half-period. In other words, the particle reaches

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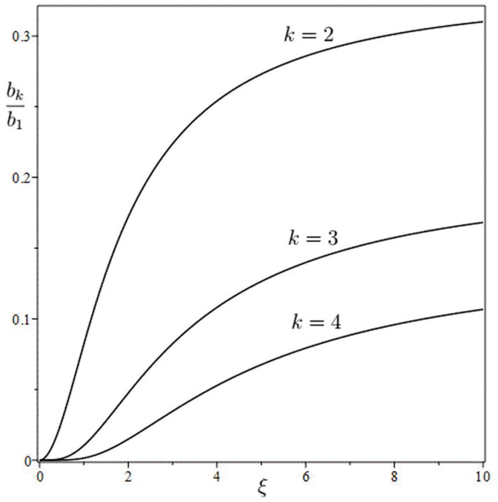

Figure 3: The ratio b_k/b_1 as a function of ξ at $k = 2, 3, 4$.

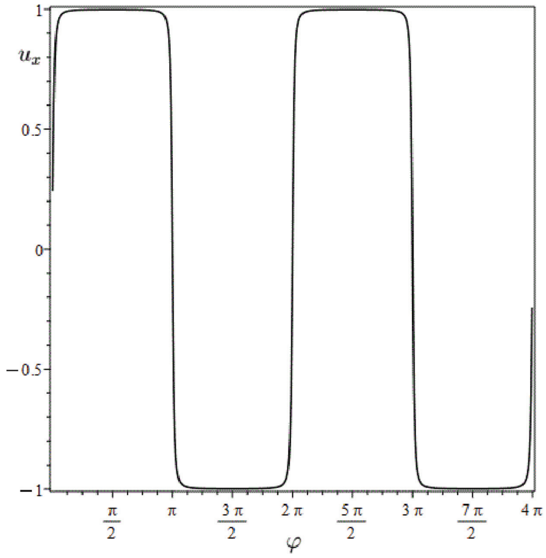

Figure 4: Dependence $u_x(\varphi)$ at $\xi = 20$.

relativistic velocities, having moved from the turning points ($u_x = 0$) even by a small (compared to the oscillation amplitude) distance to the equilibrium position ($F_x = 0$). Wherein, when the speed becomes close to the speed of light c, derivative $\mathrm{d}v/\mathrm{d}\mathcal{E}$ (where $\mathcal{E}$ is the sum of rest energy and kinetic energy) becomes small. Therefore, the total energy $\mathcal{E}$ can increase significantly with a small change in speed. It appears that this behavior is reminiscent a particle in a square well potential and it is a general property of various types of relativistic oscillators [6, 7].

3. Time Dependence of the Displacement

Using equation (5), relation $v_x = \mathrm{d}x/\mathrm{d}t$ and initial condition $x(0) = 0$, we get after integrating:

$$\rho_x(\varphi) = -\arctan\left\{\frac{\xi\cos\varphi}{\sqrt{(\xi\sin\varphi)^2+1}}\right\} + \arctan(\xi), \quad (15)$$

where $\rho_x = x\omega/c$ is the dimensionless coordinate. In the classical limit ($\xi \to 0$): $\rho_x \approx -\xi\cos\varphi$. In the ultra-relativistic limit ($\xi \to \infty$) the particle oscillation is a triangle wave (Figure 5).

According to equation (15), the oscillation amplitude is

$$\rho_{x\,\max} = \arctan\xi. \quad (16)$$

It is interesting that the maximum value of this quantity is attained asymptotically at $\xi \to \infty$ and equal to $\pi/2$ (Figure 6).

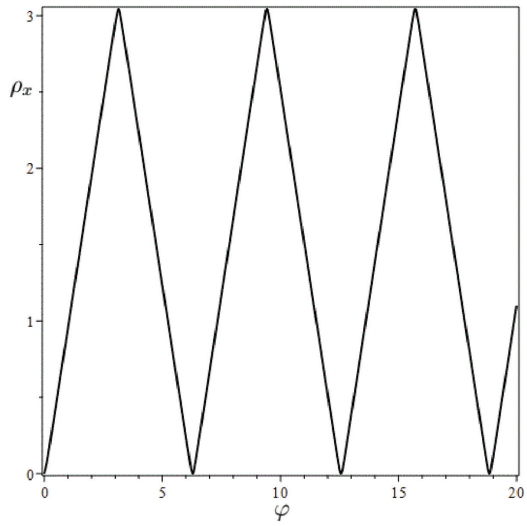

Figure 5: Dependence $\rho_x(\varphi)$ at $\xi = 20$.

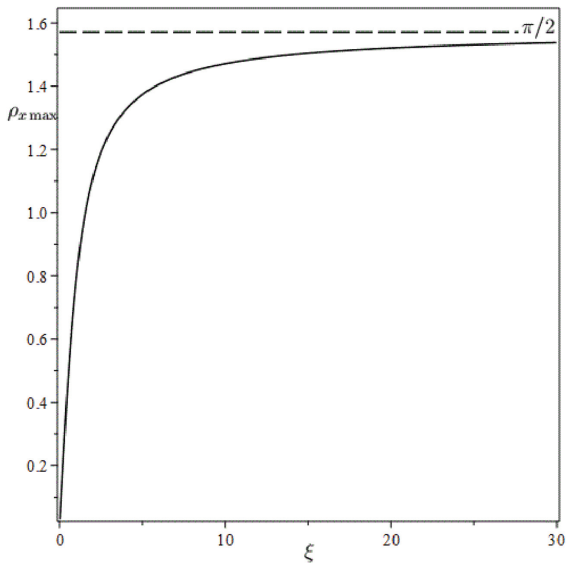

Figure 6: Dependence $\rho_{x\,\max}(\xi)$.

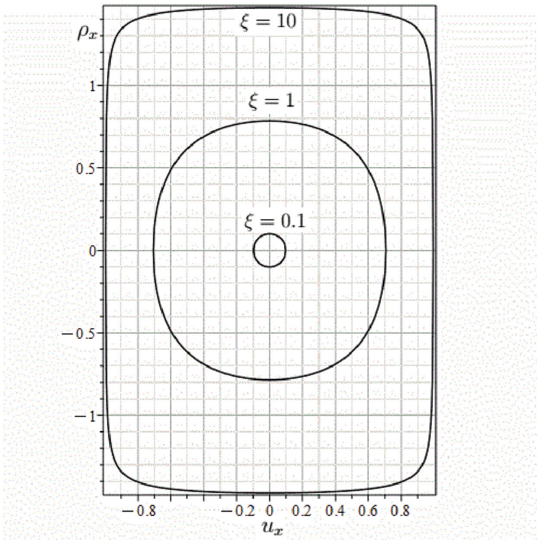

Figure 7: The phase trajectories at $\xi = 0.1, 1, 10$.

4. Phase Portrait of the Oscillations

Considering together equations (5) and (15), we find:

$$\rho_x = \pm \arctan \left\{ \sqrt{\xi^2 - (1+\xi^2)u_x^2} \right\}. \qquad (17)$$

In Figure 7 we present the set of phase portraits constructed using equation (17) for different values of the force amplitude ξ. At small values of this parameter, the phase trajectory is approximately a circle, whereas for $\xi \to \infty$ it degenerates into a rectangle with sides equal to 2 and π.

5. The General Case of Non-zero Initial Phase

The considered above analysis was based on the assumption that the initial phase φ_0 is equal to zero. Now, we consider the general case so that $\varphi = \omega t + \varphi_0$. In order to satisfy the initial condition $u_x(0) = 0$ we should replace in equation (4) $\xi \sin\varphi$ with $\xi(\sin\varphi - \sin\varphi_0)$. Then

$$u_x(\varphi) = \xi \frac{\sin\varphi - \sin\varphi_0}{\sqrt{\xi^2(\sin\varphi - \sin\varphi_0)^2 + 1}}. \qquad (18)$$

For relatively small values of ξ, only two non-zero coefficients are presented in series (7) of function (18). These are $a_0 \approx -2\xi\sin\varphi_0$ and $b_1 \approx \xi$. This classical case of particle motion is discussed in detail by Mohazzabi [12]. The main feature of such a motion is the existence of a steady drift, which is superimposed on the oscillation. The drift speed is evidently equal to $a_0/2$. Therefore, if the force is turned on randomly ($\varphi_0 \neq 0$), then the particle drifts away even if $u_x(0) = 0$. The semi-quantitative explanation of this phenomenon is also presented Ref. [12].

At relatively small values of ξ, the drift speed linearly depends on ξ (Figure 8). The function $a_0(\xi)$ has a horizontal asymptote at $\xi \to \infty$. It is interesting that there is an extremum of this function, which is achieved at some finite value of ξ. As φ_0 increases, this extremum rapidly shifts to the right (Figure 8).

In the general case, all coefficients a_n with odd numbers and coefficients b_n with even numbers are equal to zero. It means that

$$u_x(\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\{2k\varphi\} + b_k \sin\{(2k-1)\varphi\} \right]. \qquad (19)$$

Due to the presence of cosines with even numbers, the oscillations become asymmetric and lose their anti-periodicity (Figure 9). The behavior of the coefficients a_k, b_k is similar to that shown in Figure 3.

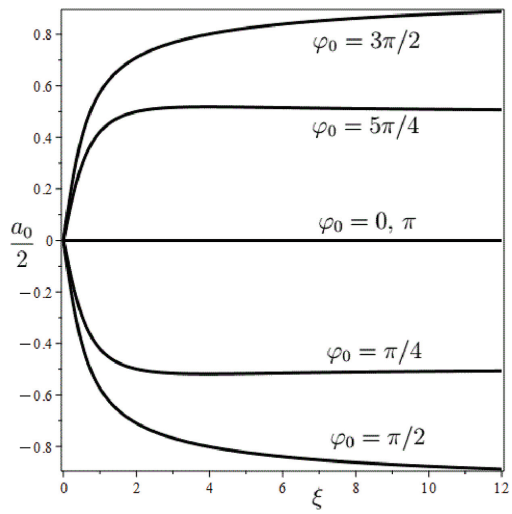

Figure 8: Dependence of the drift speed on ξ at different values of φ_0.

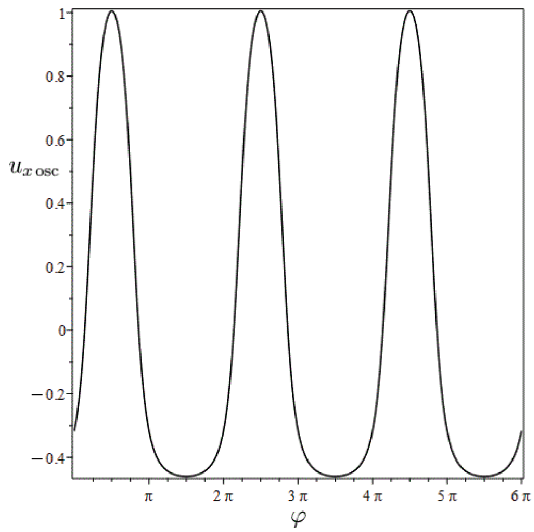

Figure 9: The oscillating part of u_x as a function of φ at $\xi = 2$ and $\varphi_0 = \pi/4$.

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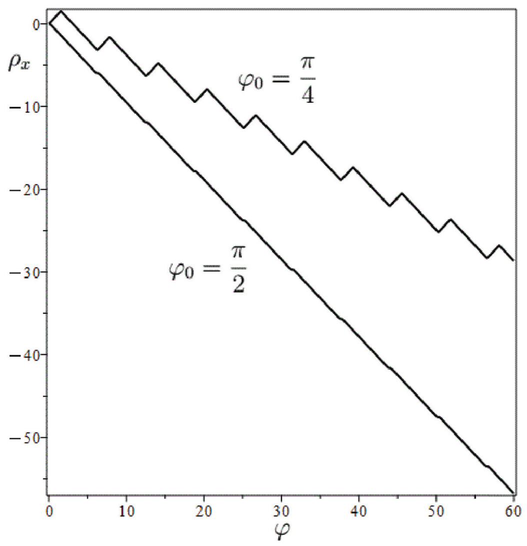

Figure 10: Dependence $\rho_x(\varphi)$ at $\xi = 50$.

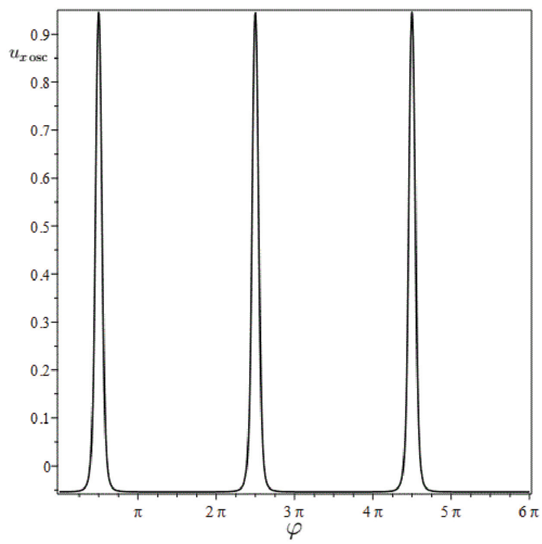

Figure 11: The oscillating part of u_x as a function of φ at $\xi = 50$ and $\varphi_0 = \pi/2$.

The most interesting peculiarity is that there is oscillation suppression for large values of ξ near $\varphi_0 = (2k-1)\pi/2$ (Figure 10). In this case, the particle moves all the time with an almost constant ultra-relativistic velocity equal to the drift velocity (Figure 11).

6. Conclusions

The driven relativistic oscillator is a topic that is possibly not explored deeply enough in undergraduate physics degree curricula. In this paper, we try to fill this gap. The topic may help students to better grasp such important mathematical concepts as the Fourier series, anti-periodic function, square and triangle waves, phase trajectory, etc. Finally, our consideration should help readers to probe the limits of the applicability of classical mechanics, and can be used in undergraduate courses or projects.

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