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Notes

Critique on "Projectile Motion Analysis in the Presence of Dissipative Forces for the High Speed Regime" [Rev. Bras. Ens. Fis. 45, e20230060 (2023)]

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We have examined the recent publication titled "Projectile Motion Analysis in the Presence of Dissipative Forces for the High Speed Regime" [Rev. Bras. Ens. Fis. **45**, e20230060 (2023)] and identified an extremely unfortunate mistake in the equations of motion used for projectile motion. The correct formulation involves a set of coupled nonlinear differential equations that are well-established in the literature. Our findings indicate that incorporating linear drag in speed, alongside the independent horizontal and vertical motions governed by decoupled linear equations, offers a more pedagogical representation of the impact of drag forces on the trajectory and overall dynamics of moving objects, especially in undergraduate introductory courses.
Keywords: Projectile motion, Equations of motion, Drag force.

A fluid in which a moving body is immersed opposes its velocity due to the viscous friction between the fluid and the body's surface. This resistance creates a drag force that depends on the relative velocity between the body and the fluid. At very low speeds, this drag force is roughly proportional to the object's speed. However, at high speeds the drag force becomes more strongly influenced by the square of the object's speed (see, e.g. [1, 2]).

In a recent study published in Revista Brasileira de Ensino de Física, researchers investigated projectile motion by examining the effects of uniform gravity and drag force proportional to the square of the speed [3]:

$$-b|\vec{v}|^2\widehat{v}, \quad b>0. \tag{1}$$

They emphasized the importance of accounting for drag forces when studying the motion of objects and criticized the neglect of this factor in undergraduate introductory courses. However, their attempt to address this gap was flawed from the beginning as they employed incorrect basal equations. The authors used the following system of uncoupled differential equations for projectile motion in the XY-plane (Y-axis directed upwards):

$$\begin{aligned} -bv_x^2 &= m\frac{dv_x}{dt} \quad \text{(Eq. (4) in [3])} \\ -mg-bv_y^2 &= m\frac{dv_y}{dt} \quad \text{(Eq. (12) in [3])}, \end{aligned} \tag{2}$$

for the upward motion. How defensible is this system of equations for a drag force proportional to the square of the speed?

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It is conspicuous that the speed squared term in (1) encompasses both the horizontal and vertical components, illustrating the interdependence between horizontal and vertical motions. Indeed, by utilizing $\widehat{v}=\vec{v}/|\vec{v}|$ we can express the equation of motion as:

$$m\vec{g}-b|\vec{v}|\vec{v}=m\frac{d\vec{v}}{dt}, \tag{3}$$

which, in component form with the Y-axis directed upwards, can be accurately written as (see, e.g. [1, 4]):

$$\begin{aligned} -bv_x\sqrt{v_x^2+v_y^2} &= m\frac{dv_x}{dt} \\ -mg-bv_y\sqrt{v_x^2+v_y^2} &= m\frac{dv_y}{dt}, \end{aligned} \tag{4}$$

independently of the particular direction of the motion.

The clumsy system of coupled nonlinear differential equations expressed by (4) effectively captures the complex characteristics of projectile motion when a drag force proportional to the square of the speed is present. Analyzing these equations is generally challenging because such a system is not amenable to elementary treatments, and in most situations we should resort to approximation methods or numerical analysis in computer simulations (see, e.g. [4]). If $v_x(0)=0$, though, $v_x(t)=0$ for all t. This particular initial condition causes the equations to decouple. In this case, even if one of the equations is nonlinear, the system will still yield a closed-form solution for the vertical projectile motion (see, e.g. Problem 2-51 in [5], reference cited by the authors themselves as [12], taking note of the order of the authors).

Regarding projectile motion with linear drag force

$$-b|\vec{v}|\widehat{v} = -b\vec{v}, \quad b > 0, \tag{5}$$

it is characterized by a set of uncoupled linear differential equations (see, e.g. [6]):

$$\begin{aligned} -bv_x &= m\frac{dv_x}{dt} \\ -mg - bv_y &= m\frac{dv_y}{dt}. \end{aligned} \tag{6}$$

These uncoupled equations depict the separate horizontal and vertical movements of the projectile, resembling the behavior of projectile motion in the absence of drag force. However, it is particularly instructive to note a marked disaccord with the statements made in Sections 3.6 (p. 53) and 5.2 (p. 89) of Ref. [2].

Finally, we would like to highlight that even if one accommodates oneself to the incorrect equations of motion presented in [3], the methods employed by the authors demonstrate a level of proficiency that surpasses the typical expectations for a student enrolled in an undergraduate introductory course. However, it should be acknowledged that while the simplified scenario incorporating linear drag force may not accurately represent reality in high-speed situations, it still provides a valuable demonstration of how drag forces impact the trajectory and behavior of moving objects in a simpler analytical and exact manner. Furthermore, in addition to the reference [6], a relevant illustration of this can be found in [5], specifically Example 2.7. In conclusion, the linear drag force holds significant importance in certain undergraduate introductory courses.

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