

Obtaining a closed-form representation for the dual bosonic thermal Green function by using methods of integration on the complex plane

(Obtendo uma representação fechada para a função de Green térmica dual bosônica usando métodos de integração sobre o plano complexo)

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We derive an exact closed-form representation for the Euclidean thermal Green function of the two-dimensional (2D) free massless scalar field in coordinate space. This can be interpreted as the real part of a complex analytic function of a variable that conformally maps the infinite strip $-\infty < x < \infty$ ($0 < \tau < \beta$) of the $z = x + i\tau$ (τ : imaginary time) plane into the upper-half-plane. Use of the Cauchy-Riemann conditions, then allows us to identify the dual thermal Green function as the imaginary part of that function.

Keywords: thermal Green function, massless scalar field, residue theorem.

Nós deduzimos uma representação fechada exata para a função de Green térmica euclidiana do campo escalar livre sem massa bidimensional no espaço das coordenadas. Esta pode ser interpretada como a parte real de uma função complexa analítica de uma variável que realiza um mapeamento conforme da faixa infinita $-\infty < x < \infty$ ($0 < \tau < \beta$) do plano $z = x + i\tau$ (τ : tempo imaginário) no semiplano superior. O uso das condições de Cauchy-Riemann nos permite então identificar a função de Green térmica dual como a parte imaginária daquela função.

Palavras-chave: função de Green térmica, campo escalar sem massa, teorema dos resíduos.

1. Introduction

As remarked in Ref. [1], despite the fact that quantum field theories are usually formulated in coordinate space, calculations, in both $T = 0$ and $T \neq 0$ cases, are almost always performed in momentum space. However, when we are faced with the exact calculation of correlation functions we are naturally led to the prob-

lem of finding closed-form expressions for Green functions in coordinate space [2, 3].

A closed-form representation for the thermal Green function of the free massless scalar field² in 2D was firstly presented in Ref. [2], where attention has been focused on its relation with the bosonization of the massive Thirring model (MTM)³ using the imaginary-time formalism for finite temperature quantum field

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²A field $\phi(x)$ is called a *scalar* field, in contrast to a *vector* or *tensor* field, when under a Lorentz transformation $x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$, it transforms, trivially, according to $\phi(x) \rightarrow \phi'(x) = \phi(\Lambda^{-1}x)$. The quantization of such fields gives rise to spin-0 particles (*scalar bosons*), like the famous Higgs boson (a proposed elementary particle in the Standard Model of particle physics). However, it is a well-known fact that most particles in the universe have a nonzero intrinsic spin. These particles arise in field theory when we consider fields which transform non-trivially under Lorentz transformations. Indeed, fields with spin have more complicated transformation laws, since the various components of the fields rotate into one another under Lorentz transformations. A good example of such a field is the vector field $A^\mu(x)$ of electromagnetism, which transforms as $A^\mu(x) \rightarrow \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$. In field theory, spin-1 particles are described as the quanta of vector fields. Such *vector bosons* play a central role as the mediators of interactions (*force carriers*) in particle physics. Another good example comes from the Dirac equation, whose quantization gives rise to spin-1/2 particles (*fermions*). Last but not least, we must stress that, for each kind of field, we may conceive field theories describing massive and massless particles. For instance, in the case of vector (spin-1) fields, we may cite as important examples the gauge fields of the electromagnetic, the weak, and the strong interactions, whose corresponding vector bosons are, respectively, massless photons, massive W^\pm and Z^0 bosons, and massless gluons.

³The MTM is described by the Lagrangian density $\mathcal{L} = \bar{\psi}(i\gamma_\mu\partial^\mu - m)\psi - \frac{g}{2}(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi)$, where ψ is a two-component Dirac fermion field in (1+1)-dimensions, and γ^μ are Dirac gamma matrices. Notice that the interaction is the only local interaction possible since the model involves only four fermion variables. It is well known that it can be mapped, by a method called bosonization, into the sine-Gordon model of a scalar field, whose dynamics is determined by $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + 2\alpha\cos\eta\phi$, where the couplings in the two models are related as $g = \pi(4\pi/\eta^2 - 1)$; $m\bar{\psi}\psi = -2\alpha\cos\eta\phi$. Both models have been extensively studied.

theory [4]. Unfortunately, the authors omit valuable details of the calculations, which would be useful for graduate students and researchers working on this subject.

In the present work we present a simple and yet appealing step-by-step derivation of an exact closed-form representation for the thermal Green function of 2D free massless scalar theory in the coordinate space, at a level accessible to usual graduate students in physics. This has been obtained by using the imaginary-time formalism along with methods of integration on the complex plane and the software *Mathematica*. The peculiar form of this, allows us to easily recognize it as the real part of an analytic function, a fact that leads us to determine the corresponding dual thermal Green function as the imaginary part of that function, according to the Cauchy-Riemann conditions. This dual thermal Green function turns out to be a key ingredient for the obtainment of fermion correlators of the MTM at finite temperature, as shown in Ref. [5].

2. Two-dimensional thermal Green function

The Euclidean thermal Green function of the 2D free massless scalar theory can be written in the coordinate space ($\mathbf{r} \equiv (x, \tau)$) as [4]

$$G_T(\mathbf{r}) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{-i(kx + \omega_n \tau)}}{k^2 + \omega_n^2}, \quad (1)$$

where $\omega_n = 2\pi n/\beta$, $\beta = 1/k_B T$. At $T = 0$, $G_T(\mathbf{r})$ reduces to the 2D Coulomb potential.

The sum appearing in Eq. (1) may be evaluated by considering the following integral on the complex plane [6]

$$I_C = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz f(z) \delta_{BE}(\beta z), \quad (2)$$

where

$$f(z) = \frac{e^{z\tau}}{k^2 - z^2}, \quad \delta_{BE}(\beta z) = \frac{1}{e^{\beta z} - 1}, \quad (3)$$

and the integration contour \mathcal{C} is defined in Fig. 1-(a).

The function $f(z)$ has poles at $z = \pm k$, being therefore outside the contour \mathcal{C} . Those of $\delta_{BE}(\beta z)$, are situated at $z = i2\pi n/\beta = i\omega_n$, ($n = 0, \pm 1, \pm 2, \dots$) hence inside the contour \mathcal{C} .

It is easy to see, using the residue theorem [7] that the sum coincides with the integral I_C and, therefore, we have

$$\begin{aligned} I_C &= \sum_{n=-\infty}^{\infty} \text{Res} [f(i\omega_n) \delta_{BE}(\beta(i\omega_n))] \\ &= \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{e^{-i\omega_n \tau}}{k^2 + \omega_n^2}. \end{aligned} \quad (4)$$

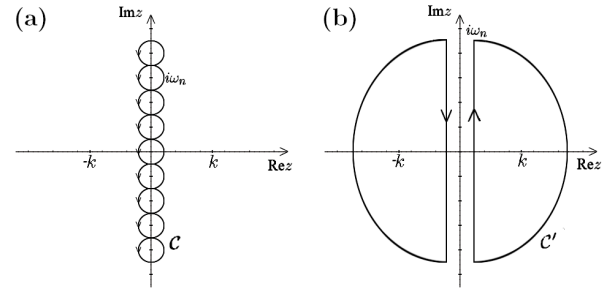


Figure 1 - Integration contour in the complex plane used in (a) Eq. (4) and (b) Eq. (5).

Deforming the contour \mathcal{C} into \mathcal{C}' shown in Fig. 1-(b) we have, using the residue theorem again:

$$\begin{aligned} I_{C'} &= -(\text{Res} [f(-k) \delta_{BE}(\beta(-k))] + \text{Res} [f(k) \delta_{BE}(\beta k)]) \\ &= \frac{\cosh \left[k \left(\tau - \frac{\beta}{2} \right) \right]}{2k \sinh \left[k \left(\frac{\beta}{2} \right) \right]}. \end{aligned} \quad (5)$$

Hence, since $I_C = I_{C'}$, we have

$$\frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{e^{-i\omega_n \tau}}{k^2 + \omega_n^2} = \frac{\cosh \left[k \left(\tau - \frac{\beta}{2} \right) \right]}{2k \sinh \left[k \left(\frac{\beta}{2} \right) \right]}, \quad (6)$$

which allows us to rewrite the expression (1) for $G_T(\mathbf{r})$ as

$$G_T(\mathbf{r}) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \frac{e^{-ikx} \cosh \left[k \left(\tau - \frac{\beta}{2} \right) \right]}{k \sinh \left[k \left(\frac{\beta}{2} \right) \right]}. \quad (7)$$

Observe that only the real part of the integral is non-vanishing. Making the change of variable $k \rightarrow (2\pi/\beta)y$, defining $a \equiv (2\pi/\beta)x$, $\theta \equiv [(2\pi/\beta)\tau - \pi]$, and introducing the regulator b we may rewrite Eq. (7) as

$$G_T(\mathbf{r}) = \lim_{b \rightarrow 0} \frac{1}{2\pi} \int_0^{\infty} dy \frac{y \cos ay \cosh \theta y}{(y^2 + b^2) \sinh \pi y}. \quad (8)$$

This integral, which can be found in Ref. [8], gives

$$G_T(\mathbf{r}) = \lim_{b \rightarrow 0} \left\{ \frac{e^{-|a|b} \cos b\theta}{4 \sin b\pi} + \sum_{k=1}^{\infty} (-1)^k \frac{k e^{-|a|k} \cos k\theta}{2\pi(k^2 - b^2)} \right\}. \quad (9)$$

The sum above may be evaluated in terms of hypergeometric functions ${}_2F_1$ with the help of *Mathematica* [9]. The result is

$$\begin{aligned}
 G^T(\mathbf{r}) = \lim_{b \rightarrow 0} & \left\{ \frac{e^{-|a|b} \cos b\theta}{4 \sin b\pi} + \frac{e^{-|a|-i\theta}}{8\pi(b^2-1)} \left[b {}_2F_1(1-b, 1, 2-b; -e^{-|a|-i\theta}) \right. \right. \\
 & + {}_2F_1(1-b, 1, 2-b; -e^{-|a|-i\theta}) + be^{2i\theta} {}_2F_1(1-b, 1, 2-b; -e^{-|a|+i\theta}) \\
 & + e^{2i\theta} {}_2F_1(1-b, 1, 2-b; -e^{-|a|+i\theta}) - b {}_2F_1(b+1, 1, b+2; -e^{-|a|-i\theta}) \\
 & + {}_2F_1(b+1, 1, b+2; -e^{-|a|-i\theta}) - be^{2i\theta} {}_2F_1(b+1, 1, b+2; -e^{-|a|+i\theta}) \\
 & \left. \left. + e^{2i\theta} {}_2F_1(b+1, 1, b+2; -e^{-|a|+i\theta}) \right] \right\}. \quad (10)
 \end{aligned}$$

Taking the $b \rightarrow 0$ limit in the expression above, we obtain (since ${}_2F_1(1, 1, 2; -z) = \frac{\ln(1+z)}{z}$)

$$\begin{aligned}
 G_T(\mathbf{r}; b) &= \frac{1}{4\pi b} - \frac{|a|}{4\pi} - \\
 & \frac{1}{4\pi} \ln \left[\left(1 + e^{-|a|-i\theta} \right) \left(1 + e^{-|a|+i\theta} \right) \right]. \quad (11)
 \end{aligned}$$

By inserting the expressions for a and θ and defining the regulator mass $\mu_0 \equiv (2\pi/\beta)e^{-1/2b}$, we may write the scalar thermal Green function, after some algebra, as

$$\begin{aligned}
 G_T(\mathbf{r}) = \lim_{\mu_0 \rightarrow 0} & -\frac{1}{4\pi} \ln \left\{ \frac{\mu_0^2 \beta^2}{\pi^2} \left[\cosh \left(\frac{2\pi}{\beta} x \right) - \right. \right. \\
 & \left. \left. \cos \left(\frac{2\pi}{\beta} \tau \right) \right] \right\}, \quad (12)
 \end{aligned}$$

which coincides with the result presented for the first time in Ref. [2].

Finally, notice that the Eq. (12) may be also written as

$$G_T(\mathbf{r}) = \lim_{\mu_0 \rightarrow 0} -\frac{1}{4\pi} \ln [\mu_0^2 \zeta(\mathbf{r}) \zeta^*(\mathbf{r})], \quad (13)$$

where ($z = x + i\tau$)

$$\zeta(\mathbf{r}) \equiv \zeta(z) = \frac{\beta}{\pi} \sinh \left(\frac{\pi}{\beta} z \right). \quad (14)$$

3. The dual thermal Green function

From Eq. (13) we can also see that the thermal Green function may be written as the real part of an analytic function of a complex variable ζ , namely

$$G_T(\mathbf{r}; \mu_0) = \text{Re} [\mathcal{F}(\zeta)] = \frac{1}{2} [\mathcal{F}(\zeta) + \mathcal{F}^*(\zeta)], \quad (15)$$

where $\mathcal{F}(\zeta) \equiv -(1/2\pi) \ln [\mu_0 \zeta(\mathbf{r})]$.

The imaginary part of $\mathcal{F}(\zeta)$ may be written as

$$\begin{aligned}
 \tilde{G}_T(\mathbf{r}) \equiv \text{Im} [\mathcal{F}(\zeta)] &= \frac{1}{2i} [\mathcal{F}(\zeta) - \mathcal{F}^*(\zeta)] \\
 &= -\frac{1}{4\pi i} \ln \left[\frac{\zeta(\mathbf{r})}{\zeta^*(\mathbf{r})} \right]. \quad (16)
 \end{aligned}$$

Now, from the analyticity of $\mathcal{F}(\zeta)$, then, it follows that its imaginary and real parts must satisfy the Cauchy-Riemann conditions, which are given by

$$\epsilon^{\mu\nu} \partial_\nu G_T = -\partial_\mu \tilde{G}_T, \quad \epsilon^{\mu\nu} \partial_\nu \tilde{G}_T = \partial_\mu G_T. \quad (17)$$

This property characterizes \tilde{G}_T as the dual thermal Green function.

4. Concluding remarks

We would like to make a few comments about Eq. (13). Firstly, we note that in the zero temperature limit ($T \rightarrow 0$, $\beta \rightarrow \infty$), we have $\zeta(z) \rightarrow z$ and $\zeta^*(z) \rightarrow z^*$ and, therefore, we recover the well-known Green function at zero temperature, namely

$$\lim_{\beta \rightarrow \infty} G_T(\mathbf{r}; \mu_0) = -\frac{1}{4\pi} \ln [\mu_0^2 z z^*] = -\frac{1}{4\pi} \ln [\mu_0^2 \|\mathbf{r}\|^2]. \quad (18)$$

Comparing Eq. (13) with Eq. (18) we can also see that the only effect of introducing a finite temperature is to exchange the complex variable z for $\zeta(z)$. Since $\zeta(z)$ is analytic, we conclude that the thermal Green function is obtained from the one at zero temperature by the following conformal mapping [7]: the infinite strip $0 < \tau < \beta$ and $-\infty < x < \infty$ is mapped into the region within the upper-half- ζ -plane. Notice that only the values $[0, \beta]$ of τ are relevant because this variable is periodic in β , as it should at finite T .

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