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Letter to the Editor

Solution for the simple harmonic oscillator problem via unilateral Fourier transform

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The solution of the simple harmonic oscillator problem is properly determined by means of the unilateral Fourier transform.

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The differential equation for the simple harmonic oscillator (SHO)

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = 0, \quad \omega > 0 \tag{1}$$

works as an excellent pedagogical tool for illustrating in a simple way several techniques for solving second-order differential equations such as power series expansion, and also Laplace transform (see, e.g. [1]) and Fourier series expansion [2] (see also [3]). Here, the differential equation for the SHO is approached by unilateral Fourier transform.

Let us begin with a brief description of the unilateral Fourier transform and a few of its properties. The direct Fourier sine and cosine transforms of $f(\xi)$ are denoted by $\mathcal{F}_s\{f(\xi)\} = F_s(k)$ and $\mathcal{F}_c\{f(\xi)\} = F_c(k)$, respectively, and are defined by the integrals (see, e.g. [1])

$$\begin{aligned} F_s(k) &= \mathcal{F}_s\{f(\xi)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty d\xi\, f(\xi)\sin k\xi, \\ F_c(k) &= \mathcal{F}_c f(\xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty d\xi\, f(\xi)\cos k\xi. \end{aligned} \tag{2}$$

The original function $f(\xi)$ can be recovered by the inverse unilateral Fourier transforms $\mathcal{F}_s^{-1}\{F_s(k)\}$ and $\mathcal{F}_c^{-1}\{F_c(k)\}$ expressed as

$$\begin{aligned} f(\xi) &= \mathcal{F}_s^{-1}\{F_s(k)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, F_s(k)\sin k\xi, \\ f(\xi) &= \mathcal{F}_c^{-1}\{F_c(k)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, F_c(k)\cos k\xi. \end{aligned} \tag{3}$$

We now observe that $f(\xi)$ retrieved by $F_s(k)$ must satisfy the homogeneous Dirichlet boundary condition at the origin, whereas $f(\xi)$ retrieved by $F_c(k)$ must satisfy the homogeneous Neumann boundary condition at the origin:

$$\begin{aligned} F_s(k) &\Rightarrow f(\xi)|_{\xi=0} = 0, \\ F_c(k) &\Rightarrow \left.\frac{df(\xi)}{d\xi}\right|_{\xi=0} = 0. \end{aligned} \tag{4}$$

Those boundary conditions are often overlooked in the literature [4–13] (see [14] for a merciless criticism). The unilateral Fourier transforms have the following derivative properties

$$\begin{aligned} \mathcal{F}_s\left\{\frac{d^2f(\xi)}{d\xi^2}\right\} &= -k^2F_s(k), \\ \mathcal{F}_c\left\{\frac{d^2f(\xi)}{d\xi^2}\right\} &= -k^2F_c(k), \end{aligned} \tag{5}$$

where the proper boundary conditions have already been used.

We are now ready to address the SHO delineated by the homogeneous Dirichlet boundary condition ($x(t)|_{t=0} = 0$) and the sine Fourier transform, or the homogeneous Neumann condition ($dx(t)/dt|_{t=0} = 0$) and the cosine Fourier transform. Using (2), with $X(k)$ denoting the unilateral Fourier transform of $x(t)$, one obtains

$$(k^2 - \omega^2)X(k) = 0. \tag{6}$$

The solution $X(k) = 0$ is certainly valid for $k \neq \pm\omega$. The complete solution for all k can be found by using the property $z\delta(z) = 0$ (see, e.g. [1]), where $\delta(z)$ is the Dirac delta symbol. Remembering that $0 \leq k < \infty$, one can write the solution as

$$X(k) = \sqrt{\frac{\pi}{2}}\delta(k-\omega) \times \begin{cases} N_s, & \text{for } \mathcal{F}_s\{x(t)\} \\ N_c, & \text{for } \mathcal{F}_c\{x(t)\}, \end{cases} \tag{7}$$

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where N_s and N_c are arbitrary constants. Using the property (see, e.g. [1])

$$\int_0^\infty dk\, F(k)\delta(k-k_0) = F(k_0), \quad k_0 > 0, \tag{8}$$

it is not difficult to see that each inverse Fourier transform yields one of the two linearly independent solutions of our problem:

$$x(t) = \begin{cases} N_s \sin \omega t, & \text{for } \mathcal{F}_s\{x(t)\} \\ N_c \cos \omega t, & \text{for } \mathcal{F}_c\{x(t)\}. \end{cases} \tag{9}$$

In conclusion, the complete solution of the SHO can be approached with simplicity via the unilateral Fourier transform method. To the best of author's knowledge, the SHO was never approached in this way.

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References

[1] E. Butkov, *Mathematical Physics* (Addison-Wesley, Reading, 1968).
[2] A.S. Castro, Rev. Bras. Ens. Fis. **36**, 2701 (2014).
[3] S.R. Oliveira, Rev. Bras. Ens. Fis. **39**, e3701 (2017).
[4] C. Rubio-Gonzalez and J.J. Mason, J. Appl. Mech. **66**, 485 (1999).
[5] C. Rubio-Gonzalez and J.J. Mason, Comput. Struct. **76**, 237 (2000).
[6] C. Rubio-Gonzalez and J.J. Mason, Int. J. Fracture **108**, 317 (2001).
[7] E. Lira-Vergara and C. Rubio-Gonzalez, Int. J. Fract. **135**, 285 (2005).
[8] C. Rubio-Gonzalez and E. Lira-Vergara, Int. J. Fract. **169**, 145 (2011).
[9] M. Nazar, M. Zulqarnain, M.S. Akram and M. Asif, Commun. Nonlinear Sci. Numer. Simulat. **17**, 3219 (2012).
[10] N. Shahid, M. Rana and I. Siddique, Bound. Value Probl. **48**, 1 (2012).
[11] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications* (CRC Press, New York, 2015), 3rd ed.
[12] J.C. Araújo and R.G. Márquez, Rev. Eletr. Paul. Mat. **11**, 136 (2017).
[13] J.C. Araújo and R.G. Márquez, TEMA (São Carlos) **20**, 95 (2019).
[14] A.S. Castro, Rev. Bras. Ens. Fis. **42**, e20200129 (2020).