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Articles

Equilibrium of a wood stack formed by close-packed cylindrical logs

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The static equilibrium of cylindrical wood pieces in a close-packed order is analyzed. By considering the coefficient of static friction μ_L between two adjacent logs and the coefficient of static friction μ_G between the logs and the ground, it is demonstrated that the more stringent condition for static equilibrium arises from μ_L. This holds true for both systems comprising three logs and those with six logs. The value of the lower threshold $\widetilde{\mu}_L$ for equilibrium is greater for a system with six logs compared to three logs. This problem could serve as a study project in Newtonian mechanics for first-year college students or for advanced high-school physics students.

Keywords: Newtonian mechanics, static equilibrium, Euclidean geometry.

1. Introduction

Real-world problems in physics often stem from everyday life experiences. However, there are instances where solving simple problems using classical physics may seem not very useful or instructive. For example, consider the scenario where we materially stack cylindrical wood pieces in a close-packed order as in Fig. 1. Eventually, the pile collapses. At this juncture, we either accept this fact as experimental evidence or delve into pencil-and-paper calculations to gain a deeper understanding of the system.

However, calculations can be time-consuming, and we may hesitate to invest significant effort in this endeavor. Moreover, we might question whether the outcome of lengthy calculations justifies the effort expended. I encountered this dilemma last October but then decided to dedicate an entire weekend in putting down some notes for the present work. At this stage, I hope that the solution of the problem at hand could be at least instructive.

In the present work we begin by examining a simplified version of the more complex system depicted in Fig. 1: a configuration comprising three identical logs of mass M and radius R arranged in a close-packing order as shown in Fig. 2. In this system, the coefficient of static friction between the bottom logs and the ground is denoted as μ_G, while the coefficient of static friction between two logs in contact is denoted as μ_L. Our objective is to find the intervals $\mu_G \geq \widetilde{\mu}_G$ and $\mu_L \geq \widetilde{\mu}_L$, where $\widetilde{\mu}_G$ and $\widetilde{\mu}_L$ represent threshold values of the coefficients of static friction, for which the system is seen in equilibrium.

In addition, a simple extension to six logs is also examined to provide insight into the potential outcomes

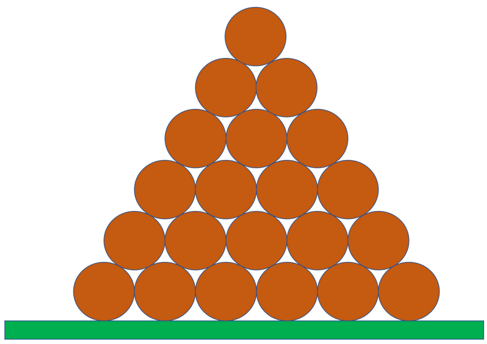

Figure 1: Identical cylindrical logs (the third dimension is not shown) piled in a close-packed order.

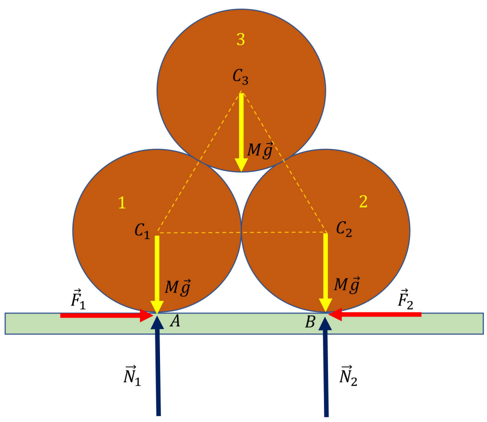

Figure 2: Three identical cylindrical logs of mass M and radius R are arranged in closed-packed order. All external forces are shown. The weights of the three logs are applied at the center of masses C_1, C_2, C_3. The friction forces $\vec{F}_1$ and $\vec{F}_2$ are applied in A and B, respectively. The normal forces $\vec{N}_1$ and $\vec{N}_2$ are also applied in A and B, respectively.

*Correspondence email address: rdeluca@unisa.it when the number of logs in the pile is increased. Similar problems have been investigated, particularly in the context of sand piles [1]. However, it is worth noting that in these cases, the elementary component of the system typically consists of a spherical body rather than a cylindrical one.

The work is organized as follows. In the following section the three-logs problem will be defined and solved by Newtonian mechanics [2, 3]. In the third section the generalization of the problem to the six-logs system will be sketched. Conclusions will be drawn in the last section.

2. Equilibrium Properties of the Three-Logs System

By considering the system in Fig. 2, we analyze the static properties of the three-logs system as follows. By taking $n = 3$ and by setting the sum of the horizontal and vertical components of the forces equal to zero, we have:

$$F_1 = F_2 = F, \tag{1a}$$

$$N_1 + N_2 - nMg = 0, \tag{1b}$$

where the friction forces $\vec{F}_1$ and $\vec{F}_2$ are applied in A and B, respectively, and the normal forces $\vec{N}_1$ and $\vec{N}_2$ are applied, in the order, in the same two points.

Calculating torques about point B, we find:

$$N_1 = \frac{n}{2}Mg. \tag{2}$$

Calculating torques about point A, we have:

$$N_2 = \frac{n}{2}Mg. \tag{3}$$

In this way, because of the solutions found in Eq. (2) and Eq. (3), by setting $N_1 = N_2 = N$, Eq. (1b) becomes an identity.

We now turn to analyze the internal forces on the top log, as shown in Fig. 3. By calculating torques about point C_3, we have:

$$F_{13} = F_{23} = F_3. \tag{4}$$

Considering now the sum of the horizontal components of all forces, because of Eq. (4), we find:

$$N_{13} = N_{23} = N_3. \tag{5}$$

Finally, by setting to zero the sum of the vertical components of all forces and by considering Eq. (4) and Eq. (5), we have:

$$F_3 + \sqrt{3}N_3 = Mg. \tag{6}$$

Notice that the angles between the internal forces in Fig. 3 and the horizontal or vertical axis are either 30° or 60°.

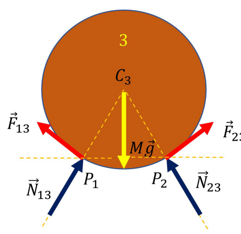

Figure 3: Force diagram for the top log. The only external force here is the weight, while the friction forces $\vec{F}_{13}$ and $\vec{F}_{23}$ and the normal reactions $\vec{N}_{13}$ and $\vec{N}_{23}$ are internal to the system.

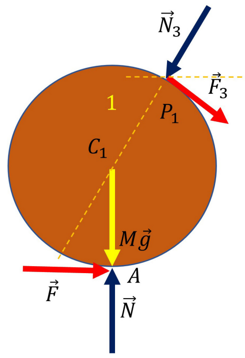

Figure 4: Force diagram for the left bottom log. The external forces are the following: the weight of the log $M\vec{g}$, the friction force $\vec{F}$, and the normal reaction $\vec{N}$. The friction force $\vec{F}_3$ and the normal reaction $\vec{N}_3$ are internal to the system.

Consider now the forces acting on the left bottom log, as shown in Fig. 4. A similar diagram can be drawn for the bottom right log. However, because of the symmetry of the problem, we may avoid considering both bottom logs.

In Fig. 4 we recognize the external forces $M\vec{g}$, $\vec{F}$, and $\vec{N}$, and the internal forces $\vec{F}_3$ and $\vec{N}_3$. We here specify that the latter force act along the dashed line passing through C_1. Notice also that the horizontal contact force between the two bottom logs has been neglected. In this way, by calculating torques about C_1, we have:

$$F_3 = F. \tag{7}$$

By now setting to zero the horizontal and vertical components of all forces acting on the bottom left log, we may write:

$$2F + \sqrt{3}F_3 - N_3 = 0, \tag{8a}$$

$$2N - \sqrt{3}N_3 - F_3 - 2Mg = 0. \tag{8b}$$

By combining Eq. (7), and (8a), we have:

$$N_3 = (2 + \sqrt{3})F_3. \tag{9}$$

On the other hand, by Eq. (2), or Eq. (3), and by Eq. (8b) we have:

$$F_3 = \frac{(n-2)}{2+\sqrt{3}}\frac{Mg}{2}. \tag{10}$$

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Therefore, by Eq. (9) we have:

$$N_3 = \frac{n-2}{2}Mg. \tag{11}$$

Assuming Newtonian friction between the logs, we may set

$$F_3 \leq \mu_L N_3, \tag{12}$$

so that, because of Eq. (9), the threshold value $\widetilde{\mu}_L$ of the coefficient of static friction μ_L is given by the following expression:

$$\widetilde{\mu}_L = \frac{1}{2+\sqrt{3}}. \tag{13}$$

By the same token, because of Eq. (2) or Eq. (3), and Eq. (7) and (10), the threshold value $\widetilde{\mu}_G$ of the coefficient of static friction μ_G is given by the following expression:

$$\widetilde{\mu}_G = \frac{1}{3(2+\sqrt{3})} = \frac{1}{3}\widetilde{\mu}_L. \tag{14}$$

Therefore, to have equilibrium of the wood stack, we need to satisfy the following two conditions:

$$\mu_L \geq \frac{1}{2+\sqrt{3}} \approx 0.268; \tag{15a}$$

$$\mu_G \geq \frac{1}{3\left(2+\sqrt{3}\right)} \approx 0.089. \tag{15b}$$

The condition on μ_L in Eq. (15a) is therefore more stringent than the condition on μ_G in Eq. (15b). Interestingly, the solution does not depend explicitly on n.

3. The Six-Logs System

By considering the six-logs system depicted in Fig. 5, we may try to generalize the results obtained in the previous section. First, the number $n = 6$ is now obtained by considering $k = 3$ logs in contact with the ground. By means of little Gauss's formula, we have $n = \frac{k(k+1)}{2}$. In Fig. 4 we recognize the external forces $M\vec{g}$, $\vec{F}_1$, $\vec{F}_2$, $\vec{N}_1$, $\vec{N}_2$, and $\vec{N}_3$.

By setting to zero the sum of the horizontal and vertical components of the external forces, we have:

$$F_1 = F_3 = F, \tag{16a}$$

$$N_1 + N_2 + N_3 - nMg = 0. \tag{16b}$$

By setting to zero the torques of all forces about point A, we write:

$$N_2 + 2N_3 - nMg = 0. \tag{17}$$

By now setting to zero the torques of all forces about point C, we obtain the same expression as Eq. (17) with N_3 substituted by N_1. In this way, we may write:

$$N_1 = N_3 = N. \tag{18}$$

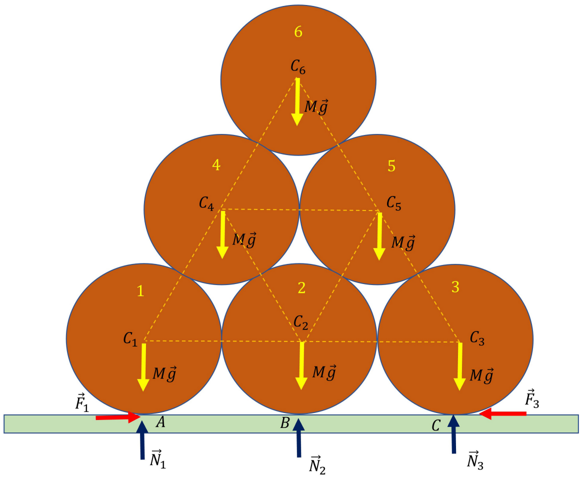

Figure 5: Six identical cylindrical logs of mass M and radius R are arranged in closed-packed order. The external forces are shown. The weights of the six logs are applied at the center of masses $C_1, C_2, \ldots C_6$. The friction forces $\vec{F}_1$ and $\vec{F}_3$ are applied in A and C, respectively, while the friction in B is neglected. The normal forces $\vec{N}_1$, $\vec{N}_2$, and $\vec{N}_3$ are applied in A, B, and C, respectively.

By the above, Eq. (16b) and Eq. (17) reduce both to the following expression:

$$N_2 + 2N = nMg. \tag{19}$$

In this case the system is statically indetermined since there are infinite solutions to Eq. (19). We thus can solve the system in terms of one variable as follows.

Let us start by considering the uppermost log. Here the force diagram is the same as in Fig. 3, except for the indices, which we show in Fig. 6. By calculating torques about point C_6, we have:

$$F_{46} = F_{56} = F_6. \tag{20}$$

Considering now the sum of the horizontal components of all forces, because of Eq. (20), we find:

$$N_{46} = N_{56} = N_6. \tag{21}$$

Finally, by setting to zero the vertical components of all forces and by considering Eq. (20) and Eq. (21), we

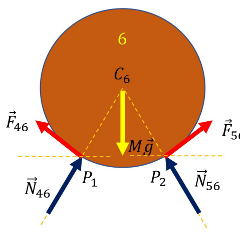

Figure 6: Force diagram for the upper top log. The only external force is the weight, while the friction forces $\vec{F}_{46}$ and $\vec{F}_{56}$ and the normal reactions $\vec{N}_{46}$ and $\vec{N}_{56}$ are internal to the system.

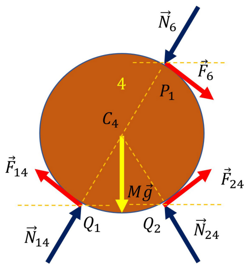

Figure 7: Force diagram for the log number 4. The only external force is the weight $M\vec{g}$. The friction forces $\vec{F}_6$, $\vec{F}_{14}$, and $\vec{F}_{24}$, and the normal reactions $\vec{N}_6$, $\vec{N}_{14}$, and $\vec{N}_{24}$ are internal to the system.

have:

$$F_6 + \sqrt{3}N_6 = Mg. \tag{22}$$

As expected, the above equation is seen to be similar to Eq. (6). We now turn our attention to log number 4 (log #4) and draw the force diagram in Fig. 7 to which we now refer. In what follows the horizontal contact forces between two adjacent logs will be neglected. By summing to zero the torques about C_4, we have:

$$F_6 + F_{14} - F_{24} = 0. \tag{23}$$

Setting to zero the sum of the horizontal and vertical components of all forces acting on log #4, we write, in the order:

$$\sqrt{3}(F_6 - F_{14} + F_{24}) - N_6 + N_{14} - N_{24} = 0, \tag{24a}$$
$$F_{14} + F_{24} - F_6 + \sqrt{3}(N_{14} + N_{24} - N_6) = 2\ Mg. \tag{24b}$$

We may use the above equations to reduce the number of variables to four, namely, F_6, F_{14}, N_6, and N_{14}, all in one equation:

$$3F_6 + F_{14} = \sqrt{3}(N_6 - N_{14}) + Mg. \tag{25}$$

The same can be done with log #5, for which we show the force diagram in Fig. 8. By following the same steps as before, we write the following equations for the forces in the diagram:

$$F_6 + F_{35} - F_{25} = 0. \tag{26a}$$
$$\sqrt{3}(F_{35} - F_6 - F_{25}) + N_6 + N_{25} - N_{35} = 0, \tag{26b}$$
$$F_{25} + F_{35} - F_6 + \sqrt{3}(N_{25} + N_{35} - N_6) = 2\ Mg. \tag{26c}$$

From Eq. (26a–c) we thus get:

$$4F_6 - F_{25} = \sqrt{3}N_{25} - Mg. \tag{27}$$

We now turn to the lowest level and show the force diagrams for logs #1, #2, and #3 in Fig. 9. Referring to the latter figure, we notice that, by taking torques about C_1, C_2, and C_3, we get, in the order:

$$F_{14} = F, \tag{28a}$$
$$F_{25} = F_{24}, \tag{28b}$$
$$F_{35} = F. \tag{28c}$$

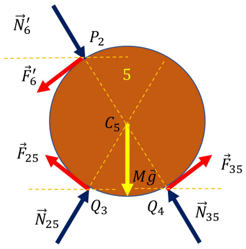

Figure 8: Force diagram for the log number 5. The only external force is the weight $M\vec{g}$. The friction forces $\vec{F}'_6$, $\vec{F}_{25}$, and $\vec{F}_{35}$, and the normal reactions $\vec{N}'_6$, $\vec{N}_{25}$, and $\vec{N}_{35}$ are internal to the system.

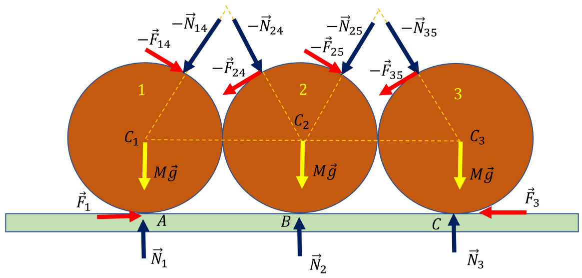

Figure 9: Force diagram for the logs number 1, 2, and 3. The external forces are the weight $M\vec{g}$ on each log, the friction forces $\vec{F}_1$ and $\vec{F}_3$, the normal forces $\vec{N}_1$, $\vec{N}_2$, and $\vec{N}_3$. The friction forces $-\vec{F}_{14}$, $-\vec{F}_{24}$, $-\vec{F}_{25}$, and $-\vec{F}_{35}$, and the normal reactions $-\vec{N}_{14}$, $-\vec{N}_{24}$, $-\vec{N}_{25}$, and $-\vec{N}_{35}$ are internal to the system.

By now considering log #1, setting to zero the sum of the x- and y-components of all forces acting on this log, we have, in the order:

$$\sqrt{3}F_{14} + 2F - N_{14} = 0, \tag{29a}$$
$$2N - 2Mg - \sqrt{3}N_{14} - F_{14} = 0. \tag{29b}$$

Considering now log #2 and setting to zero the sum of the x- and y-components of all forces acting on this log, we obtain:

$$(N_{24} - N_{25}) + \sqrt{3}(F_{25} - F_{24}) = 0, \tag{30a}$$
$$2N_2 - \sqrt{3}(N_{24} + N_{25}) - (F_{24} + F_{25}) - 2Mg = 0, \tag{30b}$$

In the above equations, because of Eq. (28b), the first reduces to state the symmetry requirement $N_{24} = N_{25}$, and $F_{24} = F_{25}$, while the second can be written as follows:

$$N_2 - \sqrt{3}N_{24} - F_{24} = Mg. \tag{31}$$

Finally, considering log #3 and setting to zero the sum of the x- and y-components of all forces acting on this log, we obtain the same equations as in (29a) and (29b), because of the symmetry requirement $N_{14} = N_{35}$.

In this way, we have written down the equations for static equilibrium. The solution in terms of Mg and N_6 is rather cumbersome and will be sketched in the

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Appendix. We here report the results:

$$F = \frac{2\sqrt{3}N_6 - Mg}{2+\sqrt{3}}, \tag{32a}$$

$$F_6 = Mg - \sqrt{3}N_6, \tag{32b}$$

$$F_{24} = \frac{(1+\sqrt{3})Mg - 3N_6}{2+\sqrt{3}}, \tag{32c}$$

$$N = 2\sqrt{3}N_6, \tag{32d}$$

$$N_{14} = 2\sqrt{3}N_6 - Mg, \tag{32e}$$

$$N_{24} = \frac{(4+3\sqrt{3})Mg - (8+3\sqrt{3})N_6}{2+\sqrt{3}}. \tag{32f}$$

The requirements to fulfill to have equilibrium are the following:

$$F \le \mu_G N, \tag{33a}$$

$$F_6 \le \mu_L N_6, \tag{33b}$$

$$F_{14} \le \mu_L N_{14} \tag{33c}$$

$$F_{24} \le \mu_L N_{24}. \tag{33d}$$

All the above conditions need to be satisfied to prevent the system to collapse. By making use of Eq. (32a–f), and of Eq (28a), the above relations can be written as follows:

$$[1-(2+\sqrt{3})\mu_G]N_6 \le \frac{Mg}{2\sqrt{3}}, \tag{34a}$$

$$\frac{Mg}{\mu_L+\sqrt{3}} \le N_6 \le \frac{(4+3\sqrt{3})\mu_L - (1+\sqrt{3})}{(8+3\sqrt{3})\mu_L - 3}Mg, \tag{34b}$$

$$\mu_L \ge \frac{1}{2+\sqrt{3}}, \tag{34c}$$

where (33b) and (33d) have been combined in (34b). We notice that Eq. (34c) is equal to Eq. (15a).

In the previous section we found that the most limiting condition on equilibrium is given by μ_L, we only analyze the problem to find the lowest possible value of μ_L, namely, $\widetilde{\mu}_L$. We shall only verify that, by considering $\mu_L = \widetilde{\mu}_L$, Eq. (34a) is verified.

According to Eq. (33c), the lowest possible value of μ_L might be $\frac{1}{2+\sqrt{3}}$. However, the latter value cannot satisfy Eq. (34b). In this way, we set:

$$\mu_L = \frac{a}{2+\sqrt{3}}, \tag{35}$$

with $a > 1$, and look for the lowest value of this quantity compatible with Eq. (34b). Therefore, we must have:

$$\frac{2+\sqrt{3}}{a+3+2\sqrt{3}}Mg \le N_6 \le \frac{(4+3\sqrt{3})a - (5+3\sqrt{3})}{(8+3\sqrt{3})a - (6+3\sqrt{3})}Mg. \tag{36}$$

To have a solution to the above equation, we must require:

$$\frac{2+\sqrt{3}}{a+3+2\sqrt{3}} \le \frac{(4+3\sqrt{3})a - (5+3\sqrt{3})}{(8+3\sqrt{3})a - (6+3\sqrt{3})}. \tag{37}$$

In this way, the possible values of the quantity a are given by the following second-degree inequality:

$$(4+3\sqrt{3})a^2 - (12+7\sqrt{3}) \ge 0. \tag{38}$$

Being $a > 0$, we finally have:

$$a \ge \sqrt{\frac{12+7\sqrt{3}}{4+3\sqrt{3}}} = \sqrt{\frac{15+8\sqrt{3}}{11}}. \tag{39}$$

By setting $a^* = \sqrt{\frac{15+8\sqrt{3}}{11}} \approx 1.62$, by Eq. (36), we have:

$$\widetilde{\mu}_L = \frac{a^*}{2+\sqrt{3}} \approx 0.434. \tag{40}$$

Comparing the above value with the threshold found in Eq. (15a), we notice that the coefficient of static friction between logs must increase, if we wish to maintain in equilibrium six logs, instead of three.

In this limiting condition, from Eq. (36) we have $N_6 = \frac{2+\sqrt{3}}{a^*+3+2\sqrt{3}}Mg$. Eq. (34a) thus reads:

$$[1-(2+\sqrt{3})\mu_G]\frac{2+\sqrt{3}}{a^*+3+2\sqrt{3}} \le \frac{1}{2\sqrt{3}}. \tag{41}$$

In this way, we may write:

$$\mu_G \ge \frac{3+2\sqrt{3}-a^*}{2(12+7\sqrt{3})} \approx 0.100. \tag{42}$$

Comparison of the value of the threshold given by the above equation with that in Eq. (40) confirms that the coefficient μ_L is more critical for equilibrium than μ_G, as also found in the three-log system. Moreover, comparing the results in Eq. (42) and in Eq. (15b), we may observe that the higher value of the threshold $\widetilde{\mu}_G$ means that a rougher ground is needed to provide equilibrium when the number of logs increase from three to six.

4. Conclusions

In this work, the equilibrium properties of a stack of three and six cylindrical wood pieces in a close-packed order have been studied. This investigation was inspired by a real-life experience of assembling wood pieces for a fireplace. Importantly, the analysis is applicable to similar arrangements of identical cylinders composed of other substances. As specified, our calculations are limited to stacks of three and six cylindrical logs for simplicity reasons. In fact, a more general analysis would imply solving $3n$ equations. This number can be determined as follows: two scalar equations related to second Newton's law and one to the torque equation for each cylinder. Moreover, when a more general analysis for $n > 6$ is made, one should consider that four contact points arise for some of the cylinders in the stack.

In the case of stacks consisting of three and six logs, the lower threshold values for the coefficients of static friction between two adjacent logs and between the logs and the ground, denoted as $\widetilde{\mu}_L$ and $\widetilde{\mu}_G$, respectively, have been calculated by Newtonian mechanics. For both stacks, the coefficients $\widetilde{\mu}_L$ and $\widetilde{\mu}_G$ are seen not to depend on the dimensions of the individual logs and satisfy the following inequality:

$$\widetilde{\mu}_L > \widetilde{\mu}_G. \tag{43}$$

The above relation indicates that the coefficient of static friction between logs is of greater significance for the equilibrium of the stack. Additionally, it has been found that by increasing the number of logs from three to six, the values of $\widetilde{\mu}_L$ and $\widetilde{\mu}_G$ increase. Consequently, in some cases, while it may be possible to have equilibrium for a stack of three logs, it might not be possible to maintain equilibrium for a stack of six logs made of the same type of wood.

The analysis presented in the present work can serve as a study project in Newtonian mechanics to first-year college students or to advanced high-school physics students.

Supplementary Material

The following online material is available for this article:
Appendix

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