A "SQUARE-ROOT RULE" FOR REINSURANCE

UMA "REGRA DE RAIZ QUADRADA" PARA RESSEGURO

MICHAEL R. POWERS
Professor and Director
of Advanta Center for Financial Services Studies
Temple University - USA
E-mail: Michael.powers@temple.edu

MARTIN SHUBIK
Seymour H. Knox Professor
of Mathematical Institutional Economics
Yale University - USA
E-mail: martin.shubik@yale.edu

ABSTRACT

In previous work, the authors derived a mathematical expression for the optimal (or "saturation") number of reinsurers for a given number of primary insurers (see Powers and Shubik, 2001). In the current article, we show analytically that, for large numbers of primary insurers, this mathematical expression provides a "square-root rule"; i.e., the optimal number of reinsurers in a market is given asymptotically by the square root of the total number of primary insurers. We note further that an analogous "fourth-root rule" applies to markets for retrocession (the reinsurance of reinsurance).

Keywords: *Primary insurance, reinsurance, retrocession, market size, square-root rule.*

RESUMO

Em um estudo anterior, os autores derivaram uma expressão matemática para a quantidade ótima (ou "de saturação") de resseguradoras para um dado número de seguradoras primárias (vide Powers e Shubik, 2001). Neste trabalho, mostramos analiticamente que, para grandes quantidades de seguradoras primárias, esta expressão matemática oferece uma "regra de raiz quadrada"; i.e., o número ótimo de resseguradoras em um mercado é dado assintoticamente pela raiz quadrada do número total de seguradoras primárias. Além disso, observamos que uma "regra de raiz quarta" análoga aplica-se aos mercados para retrocessão (o resseguro do resseguro).

Palavras-chave: *Seguro primário; resseguro; retrocessão; tamanho do mercado; regra de raiz quadrada.*

Received 26 August 2005 • Accepted 22 September 2006

1 INTRODUCTION

In the past two decades, the global reinsurance market has experienced a period of rapid change. Mergers and acquisitions have led to dramatic consolidation, reflected in a 61 percent decline in the number of domestic U.S. reinsurers (from 628 to 244) over the eleven years from 1990 through 2000 (see Venezian, Viswanathan, and Jucá, 2005). Since the early 1990s, experimentation with insurance-based securities, including various property-catastrophe indexes (see Powers and Powers, 1997) and catastrophe bonds, has provided novel alternatives to traditional reinsurance products.

These changes give rise to a number of fundamental questions:

- Is a reinsurance market necessary?
- If so, what is the optimal number of reinsurers?
- Is there a role for retrocession (i.e., the insuring of reinsurers)?
- If so, is there a theoretical or practical upper bound on the number of retrocession levels?

In previous work, the authors derived a mathematical expression for the optimal (or "saturation") number of reinsurers for a given number of primary insurers (see Powers and Shubik, 2001). In the current article, we show analytically that, for large numbers of primary insurers, this mathematical expression provides a "square-root rule"; i.e., the optimal number of reinsurers in a market is given asymptotically by the square root of the total number of primary insurers.

1.1 A Primary Insurance Market

We first review the formal model of a primary insurance market presented in Powers, Shubik, and Yao (1998) and Powers and Shubik (1998). This model employs a Cournot price-formation mechanism with arbitrary numbers of buyers and sellers, so that marginal changes in insurer solvency and competitive forces can be studied directly as the numbers of players change.

Consider a primary insurance market game with players consisting of m homogeneous customers, $i = 1,2, \ldots, m$, and n_0 homogeneous insurance firms, $i = 1,2, \ldots, n_0$. At time 0, let each customer (buyer) i have initial endowment $B_i(0) = V + A$ consisting of one unit of property with replacement value V and $A(\geq V)$ dollars in cash. Furthermore, let each insurer (seller) j have initial endowment $S_j(0) = R/n_0$ dollars of net worth, where R is the total amount of capital supplied by investors to the insurance market.

It is assumed that, during the policy period $[0, t]$, each customer's property is subject to a random loss with probability π, and that all losses are total. The random variable δ_i equals 1 if customer i suffers a property loss during $[0, t]$, and equals 0 otherwise, where the $\delta_i \sim$ i.i.d. Bernoulli(π).

To insure against a potential property loss in $[0, t]$, each customer i has the option of purchasing insurance from some insurer by making a strategic bid, $x_i \in [0, V]$, that represents the amount that he or she is willing to pay for insurance. Simultaneously, each insurer j has the option of offering to sell insurance by making a strategic offer, $y_j \in [0, cR/n_0]$, that represents the total dollar amount of risk that j is willing to assume, where $c > 1$ is a solvency constraint imposed by government regulators.

It is assumed that all bids and offers are submitted to a central clearinghouse that:

- calculates an average market price of insurance per exposure unit, $P_0(\mathbf{x}, \mathbf{y}) = \sum_{i'=1}^{m} x_{i'} / \sum_{j'=1}^{n} y_{j'}$;[1]
- collects all premium bids, x_i, and distributes them to the n_0 insurers in proportion to the insurers' respective coverage offers, y_j (i.e., insurer j receives the premium amount $y_j p_0(\mathbf{x}, \mathbf{y})$); and
- randomly assigns each customer i to an insurer $j(i)$ so that each insurer ends up with the same number of customers, μ_0 (i.e., it is assumed that n_0 divides m exactly and that $\mu_0 = m/n_0$).

Letting M_j denote the set of customers associated with insurer j, it is assumed that if customer $i \in M_j$ suffers a loss in $[0,t]$, then he or she will receive a loss payment in the amount $y_j \left(x_i / \sum_{i' \in M_j} x_{i'} \right)$ – i.e., an amount proportional not only to i's premium bid, x_i, but also to j's coverage offer, y_i. This loss payment is bounded above by V to reduce problems of moral hazard.

To recognize the possibility of insurer insolvency during $[0, t]$, let η_j be a Bernoulli random variable that equals 1 if insurer j becomes insolvent, and equals 0 otherwise. If there is an insolvency, it is assumed that government guaranty funds will pay a fixed proportion $g \in [0, 1]$ of all insurance claims made against the insolvent insurer.

1 To avoid potential problems of division by zero, it is assumed (as in Dubey and Shubik, 1978) that the clearinghouse furnishes at least one insurer, and one customer per insurer, that must make non-zero bids/offers.

1.2 A Market for Insurance and Reinsurance

In Powers and Shubik (2001), we extended the above model of a primary insurance market by introducing one or more levels of reinsurers. Although the second and higher levels of reinsurance are commonly referred to as "retrocession", the model denotes each level of reinsurance by its distance from the primary insurance market; thus, level "1" denotes the reinsurance of primary insurers, level "2" the reinsurance of level "1" reinsurers, etc.

In essence, the insurance/reinsurance model comprises an $(r + 1)$-stage strategic game in which there is first an interaction between the customers and the primary insurers, then an interaction between the primary insurers and the level 1 reinsurers, etc., through r levels of reinsurance. The solution to be considered is a perfect pure strategy non-cooperative equilibrium (PSNE) – "perfect" in the sense that the equilibrium in the overall game is also an equilibrium in every subgame.

For the insurance/reinsurance market game with one primary insurance market and $r \in \{1, 2, 3, \ldots\}$ levels of reinsurance, the following three assumptions provide the basic analytic framework.

Assumption 1: There are

(i) m homogeneous customers (buyers) in the primary market, each with utility function $u_B(w) = \dfrac{1 - e^{-\beta w}}{\beta}$,

(ii) n_0 homogeneous insurers (sellers) in the primary market, each with utility function $u_s(w) = \dfrac{1 - e^{-\sigma_0 w}}{\sigma_0}$, and

(iii) n_λ homogeneous reinsurers at level $\lambda \in [1, r]$, each with utility function $u_{s(\lambda)}(w) = \dfrac{1 - e^{-\sigma_\lambda w}}{\sigma_\lambda}$,

where $m > n_0 > n_1 > \ldots > n_r > 1$ and $\beta > \sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_r \geq 0$.

Assumption 2: The primary insurers make offers y_{j0} and bids $x_{j0}^{(1)}$, the reinsurers at level $\lambda \in [1, r-1]$ make offers $y_{j\lambda}^{(\lambda)}$ and bids $x_{j\lambda}^{(\lambda+1)}$, and the reinsurers at level r make offers $y_{jr}^{(r)}$, where: (1) all primary insurers and reinsurers make their offers independently of their bids, (2) price determinations, premium distributions, and random customer assignments are made at each level by a central clearinghouse, and (3) each reinsurer at level $\lambda \in [1, r]$ ends up with the same number of customers, μ_λ (i.e., it is assumed that n_λ divides $n_{\lambda-1}$ exactly and that $\mu_\lambda = n_{\lambda-1}/n_\lambda$).

Assumption 3: Letting $M_k^{(1)}$ denote the set of primary insurers associated with level 1 reinsurer k, it follows that if insurer $j \in M_k^{(1)}$ suffers losses associated with customers $h \in H_j \subseteq M_j$ in $[0, t]$, then it will receive a loss payment in the amount $y_k^{(1)} \dfrac{x_j^{(1)}}{\sum_{j' \in M_k^{(1)}} x_{j'}^{(1)}} \dfrac{\sum_{h \in H_j} x_h}{\sum_{i' \in M_j} x_{i'}}$ (i.e., an amount proportional not only to j's premium bid, $x_j^{(1)}$, but also to k's coverage offer, $y_k^{(1)}$), and that an analogous loss payment rule is applied at each higher level of reinsurance $\lambda \in [2, r]$.

2 MARKET EQUILIBRIUM

For the above insurance/reinsurance market game, we were able to show the existence of a unique type-symmetric pure strategy equilibrium in which price and quantity (for the primary market and r levels of reinsurance) are given implicitly by a system of nonlinear equations. (See Theorem 1 of Powers and Shubik, 2001.) From this result, it can be shown that both price and quantity decrease over the successive insurance/reinsurance levels.

2.1 Risk-Neutral Reinsurers

For the special case in which all reinsurers at levels $\lambda \in [L, r]$ are risk neutral (i.e., $\sigma_\lambda \to 0$ for $\lambda \in [L, r]$), we were able to: (1) show that equilibria do not exist for reinsurance levels $\lambda \in [L + 1, r]$, and (2) find explicit analytical forms for price and quantity in both the primary market and for all reinsurance levels $\lambda \in [1, L]$. Specifically, the equilibrium price at reinsurance level $\lambda \in [1, L]$ is given by

$$P_\lambda^* = \frac{\left(\dfrac{n_L}{n_L - 1}\right)\pi}{\displaystyle\prod_{\nu=\lambda+1}^{L} \left(\frac{\mu_\nu - 1}{\mu_\nu}\right)\left(\frac{n_{\nu-1} - 1}{n_{\nu-1}}\right)},$$

where $\prod_{\upsilon=L+1}^{L}\left(\frac{\mu_\upsilon - 1}{\mu_\upsilon}\right)\left(\frac{n_{\upsilon-1}-1}{n_{\upsilon-1}}\right) \equiv 1$, and

$$P_0^* = \frac{\left(\frac{n_L}{n_L - 1}\right)\pi}{\prod_{\nu=1}^{L}\left(\frac{\mu_\nu - 1}{\mu_\nu}\right)\left(\frac{n_{\nu-1}-1}{n_{\nu-1}}\right)}$$

denotes the equilibrium price in the primary insurance market. (See Corollary 1 of Powers and Shubik, 2001.)

2.2 Optimal Number of Reinsurers

For the case of $r = 1$, we used the above results to identify conditions under which the reinsurance market is "saturated" – i.e., under which it is no longer desirable, on the margin, to introduce an additional risk-neutral reinsurer rather than an additional risk-averse primary insurer. (See Section 5.2 of Powers and Shubik, 2001.) This was done by comparing the price of insurance in the primary insurance market under two alternatives.

The first alternative, denoted by A_1, is a primary insurance market with one level of reinsurance, where the primary market has n_0 insurers, and the reinsurance market has n_1 reinsurers. The second alternative, A_2, is the same primary insurance market, except that the number of primary insurers is increased by one (to $n_0 + 1$), while the number of reinsurers is decreased by one (to $n_1 - 1$). To identify the point at which the number of reinsurers has reached its optimal saturation level, we solved for the maximum value of n_1 such that $P_0^{(A_1)} < P_0^{(A_2)}$; i.e.,

$$n_1^* = Max\{n_1 : \pi \leq P_0^{(A1)} < P_0^{(A2)} \leq 1\}$$

$$= Max\left\{n_1 : \left(\frac{n_L}{n_1 - 1}\right)\left(\frac{n_0}{n_0 - 1}\right)\left(\frac{n_0}{n_0 - n_1}\right) < \left(\frac{n_1 - 1}{n_1 - 2}\right)\left(\frac{n_0 + 1}{n_0}\right)\left(\frac{n_0 + 1}{n_0 - n_1 + 2}\right)\right\}, \tag{1}$$

where the existence of a unique solution $n_1^* \in \{2, 3, ..., n_0 - 1\}$ is guaranteed by the fact that the inequality in (1) is satisfied for $n_1 = 2$, but not for $n_1 = n_0$.

Having computed n_1^* for values of n_0 in the interval [10,500], we presented a graph of these results, reproduced in Figure 1.

From a tabular display of the same results (see Table 1), it is easy to see that the solution to (1), n_1^*, is approximately equal to the square root of n_0 (although this observation was not made in Powers and Shubik, 2001).

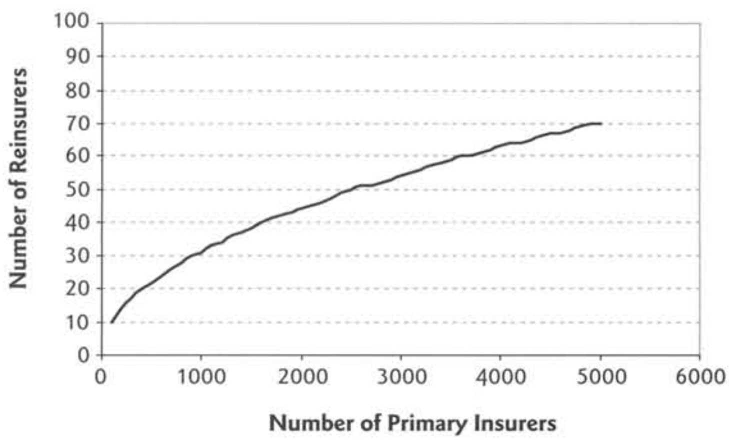

Figura 1
Insurance/Reinsurance Market

Tabela 1 Optimal Numbers of Reinsurers for Selected Numbers of Primary Insurers

n_0 (Number of Primary Insurers)	n_1* (Optimal Number of Reinsurers)
10	3
20	4
30	5
40	6
50	7
100	10
200	14
300	17
400	20
500	22
1,000	31
2,000	44
3,000	54
4,000	63
5,000	70
10,000	101

3 A SQUARE-ROOT RULE

We now show analytically that, for large numbers of primary insurers, the solution to problem (1) is indeed a "square-root rule"; i.e., the optimal number of reinsurers in a market is given asymptotically by the square root of the total number of primary insurers, as stated formally in the following result.

Theorem 1: For sufficiently large n_0, there exists a unique solution $n_1^* = n_1^*(n_0) \in \{2, 3, \ldots, n_0 - 1\}$ to problem (1), where

$$n_1^*(n_0) \sim \sqrt{n_0}$$

as $n_0 \to \infty$.

Proof: First, we extend problem (1) from the two-dimensional integer grid $\{(n_0, n_1) : 2 \leq n_1 \leq n_0\}$ to the corresponding two-dimensional real space $\{(a, x) : 2 \leq x \leq a\}$ by considering the inequality

$$\left(\frac{x}{x-1}\right)\left(\frac{a}{a-1}\right)\left(\frac{a}{a-x}\right) < \left(\frac{x-1}{x-2}\right)\left(\frac{a+1}{a}\right)\left(\frac{a+1}{a-x+2}\right). \tag{2}$$

Apart from its points of unboundedness, (2) is equivalent to the cubic polynomial inequality

$$f(x) = (a^2 - a - 1)x^3 + (a^3 - a^2 + 3a + 2)\,x^2 - (a^3 + a^2 + 3a + 1)x - (a^4 + a^3 - a^2 - a) < 0. \tag{3}$$

Thus, the unique solution specified by the theorem – if it exists – is given by $n_1^* = [x^*]$, where $x^* = x^*(a) \in (2, a)$ is a positive real root of $f(x)$ such that $f'(x) > 0$.

For large a, one can rewrite (2) as

$$\left(\frac{x}{x-1}\right)\left(\frac{x-2}{x-1}\right)\left(\frac{a-x+2}{a-x}\right)\left(\frac{a}{a+1}\right) < \left(\frac{a-1}{a}\right)\left(\frac{a+1}{a}\right) = 1 - O\left(a^{-2}\right),$$

from which it follows that we seek the roots $x(a)$ of

$$f(x) = [1-(x-1)^2\,O(a^{-2})]\,a^2 - [x^2-x-1-(x-1)^3\,O(a^{-2})]\,a - [x(x-1)^2 - x(x-1)^2\,O(a^{-2})] = 0 \tag{4}$$

Solving (4) for a as a function of x yields

$$a = \frac{[x^2-x-1-(x-1)^3\,O(a^{-2})]}{2[1-(x-1)^2\,O(a^{-2})} \pm \frac{\sqrt{[x^2-x-1-(x-1)^3\,O(a^{-2})]^2 + 4\,[1-(x-1)^2\,O(a^{-2})]\,[x(x-1)^2 - x(x-1)^2 O(a^{-2})]}}{2[1-(x-1)^2\,O(a^{-2})]} \tag{5}$$

Anticipating that there exists at least one positive root $x(a) = o(a^{2/3})$ to (4), we find exactly one solution to (5); namely,

$$a = \frac{\hat{x}^2 - O(\hat{x})}{2[1-o(\hat{x}^{-1})]} + \frac{\sqrt{\hat{x}^4 - O(\hat{x}^3)}}{2[1-o(\hat{x}^{-1})]} \sim \hat{x}^2$$

which implies

$$\hat{x}(a) \sim \sqrt{a}$$

For *sufficiently large* a, it is clear that $\hat{x}(a) \in (2, a)$.
To confirm that $\hat{x}(a)$ is the desired root of $f(x)$, we consider the local extrema of this polynomial, given by

$$f'(x) = 3(a^2-a-1)\,x^2 + 2(a^3-a^2+3a+2)x - (a^3+a^2+3a+1) = 0$$

or equivalently,

$$x = \frac{-(a^3-a^2+3a+2) \pm \sqrt{(a^3-a^2+3a+2)^2 + 3(a^2-a-1)(a^3+a^2+3a+1)}}{3(a^2-a-1)} \tag{6}$$

As $a \to \infty$, the larger solution to (6) is given by

$$x_U(a) = \frac{-(a^3-a^2+3a+2) + \sqrt{(a^3-a^2+3a+2)^2 + 3a^5 + O(a^3)}}{3a^2 - O(a)}$$

$$= \frac{\dfrac{[3a^5 + O(a^3)]}{2(a^3-a^2+3a+2)} - O(a)}{3a^2 - O(a)} = \frac{\dfrac{3}{2}a^2 - O(a)}{3a^2 - O(a)} \sim \frac{1}{2}$$

For *sufficiently large* a, it follows that $\hat{x}(a) > x_U(a)$, which implies $f'(\hat{x}) > 0$. Therefore, $x^*(a) = \hat{x}(a)$. □

4 DISCUSSION AND CONCLUSIONS

It is interesting to note that the asymptotic square-root rule for the optimal number of reinsurers – and indeed the exact solution $n_I{}^*$ given by (1) – depends only on the number of primary insurers. In a recent article, Venezian et al. (2005) tested the proposed square-root rule empirically, using the numbers of primary insurers and reinsurers from a group of eighteen to twenty different national insurance markets over a period of eleven years. Instructively, these authors found that their data are consistent with the square-root rule, and that year-to-year variations in their regression model are reasonably well explained by a measure of market risk aversion proposed by Madsen and Pedersen (2003). This suggests that, in addition to the square-root relationship, the number of reinsurers in a market during a specific time period may be associated with the risk aversion of investors during that period.

Beyond the world of ordinary reinsurance lie the misty realms of retrocession (second-order reinsurance), second-order retrocession (third-order reinsurance), and so on. Although our model in Powers and Shubik (2001) permitted an arbitrary number of reinsurance levels, we acknowledged that such clearly defined levels are not reflective of the real world. While a few specialized purveyors of retrocession do exist, higher-order reinsurance is typically provided by ordinary reinsurers through the packaging and repackaging of risk through various types of pooling arrangements.

It may be argued that the absence of distinct higher-order reinsurance markets is consistent with the analytical results of our game-theoretic model. Assuming that there exist at least two second-order reinsurers in the market, and again using the minimization of price in the primary insurance market as the optimality criterion, the expression for the optimal number of second-order reinsurers is given by

$$n_2^* = Max\left\{n_2 : \left(\frac{n_2}{n_2-1}\right)\left(\frac{n_1}{n_1-1}\right)\left(\frac{n_1}{n_1-n_2}\right)\left(\frac{n_0}{n_0-n_1}\right) < \left(\frac{n_2-1}{n_2-2}\right)\left(\frac{n_1+1}{n_1}\right)\left(\frac{n_1+1}{n_1-n_2+2}\right)\left(\frac{n_0}{n_0-n_1-1}\right)\right\}$$

By methods similar to those employed in the proof of Theorem 1, the following result may be obtained.

Theorem 2: For sufficiently large n_0, if $n_1 \sim \sqrt{n_0}$, then there exists a unique solution $n_2^* = n_2^*(n_0) \in \{2, 3, \ldots, n_1 - 1\}$ to problem (7), where

$$n_2^*(n_0) \sim \sqrt[4]{n_0}$$

as $n_0 \rightarrow \infty$.

In short, the number of retrocessionaires in a market should be approximately equal to the fourth-root of the number of primary insurers, which for most national insurance markets (other than that of the U.S.) is rather small. Thus, according to the model, one should not expect to see distinct significant retrocession markets, except perhaps in the U.S. This result agrees with empirical observation.

More generally, for all financial instruments involving risk and transaction costs, the principles dictating how many levels of paper are economically optimal need to be considered. We suspect that four or five is an extreme upper bound, and our result here conforms to this.

References

DUBEY, P.; SHUBIK, M.. A Theory of Money and Financial Institutions. 28. The Non-Cooperative Equilibria of a Closed Trading Economy with Market Supply and Bidding Strategies. *Journal of Economic Theory*, vol.17, n.1, p.1-20, February 1978.

MADSEN, C. K.; PEDERSEN, H. W.. An Examination of Insurance Pricing and Underwriting Cycles. In: 13TH INTERNATIONAL AFIR COLLOQUIUM, September 2003, Maastricht, The Netherlands.

POWERS, I. Y.; POWERS, M. R.. Seeking the Perfect Catastrophe Index, *Best's Review, Property/Casualty Edition*, p.101-103, December 1997.

__________, M. R.; SHUBIK, M.. On the Tradeoff between the Law of Large Numbers and Oligopoly in Insurance. *Insurance: Mathematics and Economics*, vol.23, n.2, p.141-156, November 1998.

__________. Toward a Theory of Reinsurance and Retrocession. *Insurance: Mathematics and Economics*, vol.29, n.2, p.271-290, October 2001.

POWERS, M. R.; SHUBIK, M.; YAO, S. T.. Insurance Market Games: Scale Effects and Public Policy. *Zeitschrift für Nationalökonomie (Journal of Economics)*, vol.67, n.2, p.109-134, June 1998.

VENEZIAN, E. C.; VISWANATHAN, K. S.; JUCÁ, I. B.. A 'Square-Root Rule' for Reinsurance? Evidence from Several National Markets. *Journal of Risk Finance*, vol.6, n.4, p.319-334, August 2005.

NOTE – address of the authors

Temple University
481 Ritter Annex (044-00)
Philadelphia – PA – USA
19122

Yale University
PO BOX 208200
New Haven – CT – USA
06520-8200