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Quantifying mean grades and uncertainties from the ratio of service variables: accumulation to thickness. Applied to a phosphate deposit in Southern Mato Grosso, Brazil

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Abstract

Estimating mineral resources of a stratiform mineral deposit is not always a simple activity. Some difficulties can arise in mineral deposits where the thickness is relatively much smaller than the dimensions in the horizontal plane, or because these deposits have suffered events after their genesis, such as thrusts and folds. Usually, in these cases, we use a new variable named accumulation as a product between the average grade and thickness. Then, from these service variables, accumulation and thickness are estimated in a regular grid. Dividing accumulation by thickness, we have the mean grade. However, the mean grade is just an approach because the uncertainties of variables are not considered in this division. This calls for applying the Taylor series to computing, not only the mean grade, but also the associated uncertainty. In this article, we present the satisfactory results of this method applied to quantify the mean grades and uncertainties of the blocks to a stratiform phosphate deposit located in southern Mato Grosso, Brazil.

Keywords: Taylor Series, geostatistic, uncertainty, service variable, mean grade, phosphaste.

1. Introduction

A variable named accumulation is commonly used for ore resource estimation in stratiform, stratabound, or tabular deposits. This variable is also known as the service variable (Rossi and Deutsch, 2014) and is defined as the product of the average grade in the borehole and the thickness of the mineralized section (Journel and Huijbregts, 1978; Chilès and Delfiner, 2012). Actually, this variable has been used in the valuation of gold mines and it represents direct measurements of gold per unit area (Krige, 1978). The mean grade of a block is computed by dividing the estimated accumulation by the estimated thickness (David, 1977; Dowd and Milton, 1987; Rossi and Deutsch, 2014). However, the resulting mean grade is just an approximation (Dowd and Milton, 1987, p. 42) because the quotient is a function defined by the ratio of accumulation to thickness. The uncertainty associated with the mean grade must take into consideration uncertainties regarding estimations of both accumulation and thickness. Actually, we want to know the probability distribution function of the mean grade resulting from the division of accumulation by thickness. This issue can be addressed by Monte Carlo simulation, but the main drawback of this method is that the results obtained are not readily transferable to a new situation due to the nature of this method; it does not provide a closed analytical form (Dettinger and Wilson, 1981). On the other hand, there is another approach that allows calculating the statistical moments of the mean grade distribution, which is known as the first and second moment method (Dettinger and Wilson, 1981).

This approach is based on second order Taylor Series expansion of the ratio function (accumulation by thickness). The mean grade and its variance can be calculated by mathematical expressions. Herein, the equations developed for the calculation of mean grade and variance by Yamamoto *et al*. (2018) are tested in a real database of research in a stratiform phosphate mineral deposit located in the southern state of Mato Grosso.

2. Review of previous studies

It is clear in previous studies that the average grade resulting from the division of estimated accumulation by estimated thickness is accepted as estimated grade (David, 1977; Krige, 1978; Journel and Hujbregts, 1978; Sinclair and Blackwell, 2002; Chilès and Delfiner, 2012; Rossi and Deutsch, 2014).

$$G^* = \frac{GT^*}{T^*} \quad (1)$$

For notational convenience we will consider hereafter the function grade as:

$$G = f(x,y) = \frac{X}{Y}$$

However, according to Dowd and Milton (1987), the average grade obtained by dividing estimated accumulation by estimated thickness is just an approximation and that it is valid only under restrictive conditions: accurate thickness estimates; use of paired data to estimate both accumulation and thickness. Herein, we provide a full equation to calculate the average grade.

For the average grade, we have to assess the uncertainty associated with the estimate. Previous studies have addressed this issue by assuming that true values are known at estimation locations (Journel and Huijbregts, 1978). Therefore, errors can be determined and the error of the average grade results from the propagation of errors of accumulation and thickness. The true global values G, X and Y are known and also the estimated values G*, X* and Y* (Journel and Huijbregts, 1978). Thus, we can derive error expressions:

$$\varepsilon_x = X^* - X \qquad \varepsilon_y = Y^* - Y \qquad \varepsilon_G = G^* - G$$

Since estimates are assumed to be unbiased, we have (Journel and Huijbregts, 1978):

$$E[\varepsilon_x] = 0 \qquad E[\varepsilon_y] = 0$$

According to these authors, estimation variances are also known:

$$\sigma_x^2 = E[\varepsilon_x^2] \qquad \sigma_y^2 = E[\varepsilon_y^2]$$

The relative estimation variance of the quotient $G=X/Y$ can be calculated as second order Taylor expansion (Journel and Huijbregts, 1978):

$$\frac{\sigma_G^2}{G^2} = \frac{\sigma_X^2}{X^2} + \frac{\sigma_Y^2}{Y^2} - 2\rho_{XY}\frac{\sigma_X}{X}\frac{\sigma_Y}{Y} \quad (2)$$

This equation has also been used by Dowd and Milton (1987) and by Sinclair and Blackwell (2002).

Expression (2) considers that all variables are roughly lognormally distributed and therefore it is considered a particular case of Tukey's formula (David, 1977):

$$Var[G] = \left(\frac{dG}{dX}\right)^2 Var[X] + \left(\frac{dG}{dY}\right)^2 Var[Y] + 2\left(\frac{dG}{dX}\right)\left(\frac{dG}{dY}\right) cov(X,Y) \quad (3)$$

Notice that Equation (3) is the formula for variance of function G that resulted from first order Taylor expansion.

The derivatives of the function $G=X/Y$ are:

$$\frac{dG}{dX} = \frac{1}{y}\frac{dG}{dY} = \frac{1}{y^2}$$

Replacing these derivatives in (3) and considering the variance around a point $\theta = (\mu_x, \mu_y)$, we have:

$$Var[G] = \frac{\sigma_x^2}{\mu_y^2} + \frac{\mu_x^2 \sigma_y^2}{\mu_y^4} - 2\frac{\mu_x Cov(X,Y)}{\mu_y^3} \quad (4)$$

Although equations (2) and (4) look different, we can show that Equation (2) was also derived from (3) as follows. Multiplying both sides of Equation (2) by $G^2=X^2/Y^2$ and knowing that $Cov(X,Y)=\rho_{xy}\sigma_x\sigma_y$, we have:

$$\sigma_G^2 = \frac{\sigma_x^2}{Y^2} + X^2\frac{\sigma_y^2}{Y^4} - 2\frac{XCov(X,Y)}{Y^3}$$

Since Equation (2) results from first order Taylor expansion and considering that the expansion is made about a point $\theta = (\mu_x, \mu_y)$, we can replace X and Y by μ_x and μ_y, respectively. Thus, we have the same equation as (4):

$$\sigma_G^2 = \frac{\sigma_x^2}{\mu_y^2} + \frac{\mu_x^2\sigma_y^2}{\mu_y^4} - 2\frac{\mu_x Cov(X,Y)}{\mu_y^3}$$

Actually, this is an advantage of the Taylor method because it allows computation of the variance of the function without knowing the shapes of input distributions (Maskey and Guinot, 2003).

2.1 Computing mean and variance from second order taylor expansion

Yamamoto *et al.* (2018) developed mathematical expressions for calculating mean and variance for arithmetically combined variables. For the ratio function the mean or the mathematical expectation around a point $\theta = (\mu_x, \mu_y)$ and the variance are calculated as:

$$E[f(x,y)] \approx \frac{\mu_x}{\mu_y} - \frac{\sigma_{xy}}{\mu_y^2} + \frac{\mu_x\sigma_y^2}{\mu_y^3} \tag{5}$$

$$\left.\begin{aligned} Var[f(x,y)] \approx\ & \frac{\sigma_x^2}{\mu_y^2} + \frac{\mu_x^2}{\mu_y^4}\sigma_y^2 - 2\frac{\mu_x}{\mu_y^3}\sigma_{xy} + \frac{1}{\mu_y^4}E\left[\left(\sigma_{xy} - (x-\mu_x)(y-\mu_y)\right)^2\right] + \frac{\mu_x^2}{\mu_y^6}E\left[\left((y-\mu_y)^2 - \sigma_y^2\right)^2\right] \\ & + \frac{2}{\mu_y^3}E\left[(x-\mu_x)[\sigma_{xy} - (x-\mu_x)(y-\mu_y)]\right] + 2\frac{\mu_x}{\mu_y^4}E\left[(x-\mu_x)\left[(y-\mu_y)^2 - \sigma_y^2\right]\right] \\ & - 2\frac{\mu_x}{\mu_y^4}E\left[(y-\mu_y)[\sigma_{xy} - (x-\mu_x)(y-\mu_y)]\right] - 2\frac{\mu_x^2}{\mu_y^5}E\left[(y-\mu_y)\left[(y-\mu_y)^2 - \sigma_y^2\right]\right] \\ & + 2\frac{\mu_x}{\mu_y^5}E\left[[\sigma_{xy} - (x-\mu_x)(y-\mu_y)]\left[(y-\mu_y)^2 - \sigma_y^2\right]\right] \end{aligned}\right\} \tag{6}$$

Where σ_{xy} is the covariance between x and y. Details of the mathematical development of Equations (5) and (6) can be found in Yamamoto *et al.* (2018).

Notice that Equation (5) is just an approach based on second order Taylor expansion. As shown in a previous item, the mean grade is usually calculated as the ratio of accumulation to thickness. Actually, keeping just the first term on the right side of Equation (5), means that we are calculating the mathematical expectation after the first order Taylor expansion. This is why Dowd and Milton (1987) drew attention to this approach. Evidently, Equation (5), which results from second order Taylor expansion is better than considering just the first order. Moreover, Equation (5) is still an approach because there is no exact formula for the mean of the ratio of x to y, since the function $f(x,y)=x/y$ is infinitely differentiable with respect to y. Yamamoto *et al.* (2018) also provided a formula for the mean after the third order Taylor expansion.

Equation (6) is composed of ten terms as a result from second order Taylor expansion for the variance of the function $f(x,y)=x/y$. The first three terms on the right side of Equation (6) are related to the first order expansion (equation 4). Yamamoto *et al.* (2018) showed that this equation is a much better approach than just keeping the first order terms.

Mean and variance after Equations (5) and (6), respectively, are valid as global statistics. Since we are interested in computing mineral resources in blocks of a 2D deposit, we must be able to compute local first and second order statistical moments (mean, variance and covariance) and high-order moments as required by Equation (6).

Because accumulation and thickness are additive variables, they can be estimated by kriging (Chilès and Delfiner, 2012). Theoretically, μ_x and μ_y are estimated separately from different sets of kriging weights. However, we cannot proceed with further calculations, especially second order moments, because of the risk of bias, as recognized by Chilès and Delfiner (2012). These authors suggest using the same basic variogram multiplied by a constant. Actually, accumulation and thickness are proportional variables and therefore their variograms are proportional to the same basic variogram function (Chilès and Delfiner, 2012). Indeed, these variables are intrinsically co-regionalized (Journel and Huijbregts, 1978). Considering that both variables are estimated using the same data configuration, the kriging weights will be the same (Chilès and Delfiner, 2012) μ_x and μ_y are calculated as follows:

$$\mu_x \Leftarrow GT^*(x_o) = \sum_{i=1}^{n}\lambda_i GT(x_i) \tag{7}$$

$$\mu_y \Leftarrow T^*(x_o) = \sum_{i=1}^{n}\lambda_i T(x_i) \tag{8}$$

The variances σ_x^2 and σ_y^2 are calculated by using the approach provided by the interpolation variance (Yamamoto, 2000).

$$\sigma_x^2 \Leftarrow S_{GT}^2(x_o) = \sum_{i=1}^{n} \lambda_i \left(GT(x_i) - GT^*(x_o)\right)^2 \quad (9)$$

$$\sigma_y^2 \Leftarrow S_T^2(x_o) = \sum_{i=1}^{n} \lambda_i \left(T(x_i) - T^*(x_o)\right)^2 \quad (10)$$

The covariance between accumulation and thickness can be computed as (Yamamoto *et al*. 2018):

$$\sigma_{xy} \Leftarrow Cov(GT,T) = \sum_{i=1}^{n} \lambda_i GT(x_i)T(x_i) - \left(\sum_{i=1}^{n} \lambda_i\, GT(x_i)\right)\left(\sum_{i=1}^{n} \lambda_i\, T(x_i)\right) \quad (11)$$

First and second order statistical moments are used to compute both mean and variances according to Equations (5) and (6), respectively. High-order moments are calculated as:

$$E[.] = \sum_{i=1}^{n} w_i(.)$$

3. Materials and methods

For this research, we used a geochemical database composed of 92 drillholes, which were sampled at one-meter intervals and generated 4,892 samples. The evaluation method proposed by this article required careful analysis of each drillhole for the selection of mineralized intervals in apatite phosphorus (considering the CaO / P_2O_5 ratio) and the insertion of some variables into the database: weighted mean of P_2O_5, thickness and accumulation.

Accumulation of P_2O_5 content by thickness resulted in 92 sample points, which the samples generated consider only x and y coordinates, not considering the z coordinate and placing all samples in a single horizontal plane.

Evaluation of mineral resources using a service variable makes use of some additional procedures. Thus, after adding these variables to the database, some tasks are required to obtain the mean grade and its variance of estimated blocks, such as statistical analysis of variable thickness and accumulation; calculation of experimental variograms of both variables and determination of directions of anisotropy; modeling an experimental variogram; ordinary kriging of both variables; calculation of the mean grade and uncertainty; displaying 2D maps of the mean grade and uncertainty.

It should be pointed out that ordinary kriging was applied because it is the best non-biased linear estimator, besides being able to calculate a local kriging variance and covariance through the equations by Yamamoto (2000), and of great importance for the calculation of uncertainty by the equations presented here deriving the Taylor Series.

4. Results and discussion

Analyzing frequency distributions for both variables, we conclude that they present positive asymmetry (Figure 1). Moreover, we can see that a quarter of the distribution has zeroes, because of negative drillholes (no phosphate apatite). Statistics for both variables are presented in Tables 1 and 2.

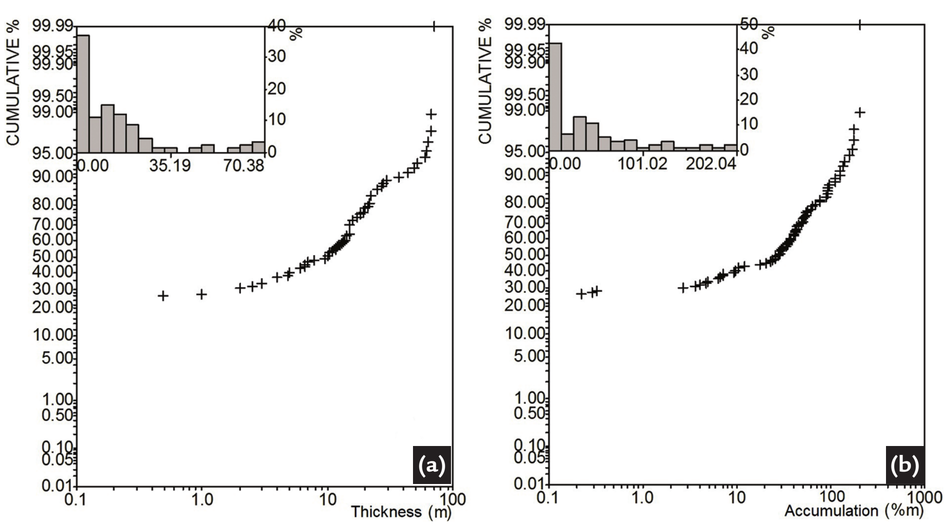

Figure 1
Accumulated frequency curve of the Thickness (a) and Accumulation (b) variables with histogram inserted.

Table 1
Statistics for thickness.

Thickness Variable (m)

# data	Mean (m)	Standard Deviation	Coefficient of Variation	Maximum	Minimum	Upper Quartile	Median	Lower Quartile
92	14.12	17.23	1.22	70.30	0.00	18.00	9.74	-99.00

Table 2
Statistics for accumulation.

Accumulation Variable (m%)

# data	Mean (%m)	Standard Deviation	Coefficient of Variation	Maximum	Minimum	Upper Quartile	Median	Lower Quartile
92	42.06	50.82	1.21	202.04	0.00	54.67	28.05	-99.00

In order to identify the directions of anisotropy, we computed the variogram map (Figure 2), which shows anisotropic directions between 70-80° and 160-170° in which the former is the direction of high continuity, showing lower variance.

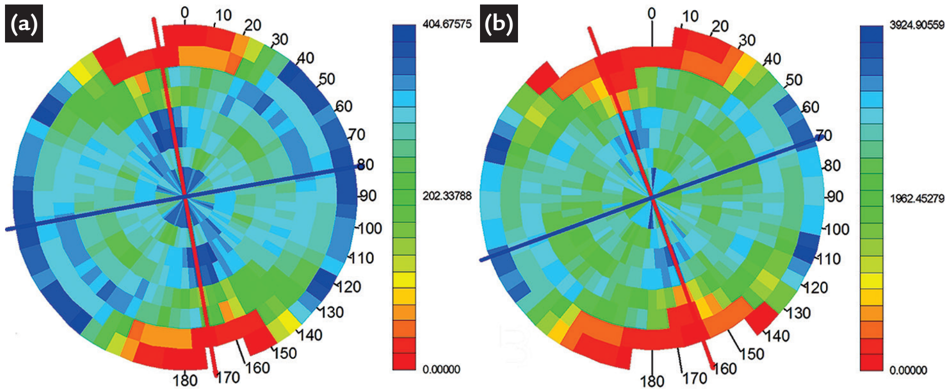

Figure 2
Variogram maps for Thickness (a) and Accumulation (b) variables.

Experimental variograms (Figure 3) were computed according to anisotropic directions as displayed in variogram maps. In addition, Figure 3 shows variogram models fitted to experimental variograms according to parameters presented in Tables 3 and 4.

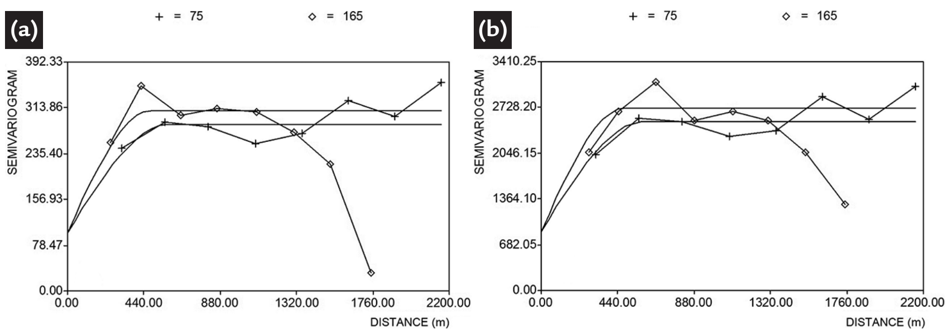

Figure 3
Experimental variograms for Thickness (a) and Accumulation (b) variables. Spherical theoretical model fit.

Table 3
Variogram parameters of Thickness variable (a).

Thickness Variable

Nugget Effect	Struct. 1	Range max.	Range min.	Sill	Struct. 2	Range max.	Range min.	Sill
100	(75°)	585	480	285	(165°)	$1*10^{30}$	480	309

Table 4
Variogram parameters of Accumulation variable (b).

Accumulation Variable

Nugget Effect	Struct. 1	Range max.	Range min.	Sill	Struct. 2	Range max.	Range min.	Sill
880	(75°)	585	480	2508	(165°)	$1*10^{30}$	480	2719.2

In Figure 3, it is possible to check the proportionality of the variograms by a constant k, according to Chilès and Delfiner (2012) who assert that proportional variables must have proportional variograms for the same basic variogram function.

Then, ordinary block kriging was carried out based on the variogram models presented. The block size for both directions is equal to 50 meters, resulting in 2,931 blocks. The blocks are defined in order to generate blocks with minerable dimensions, since the average distance of the samples is very large and would generate non-operational blocks.

Values of local variances and covariances were computed after expressions 9, 10 and 11, which were replaced in Equations 5 and 6 for calculation of the mean grade of P_2O_5 and associated uncertainty with the estimated blocks.

The results of this calculation can be displayed in 2D maps, as shown in Figures 4 and 5, for the mean grade of P_2O_5 and associated uncertainty, respectively. Remembering that dimension z is disregarded when the average hole content is accumulated by multiplying it to the thickness.

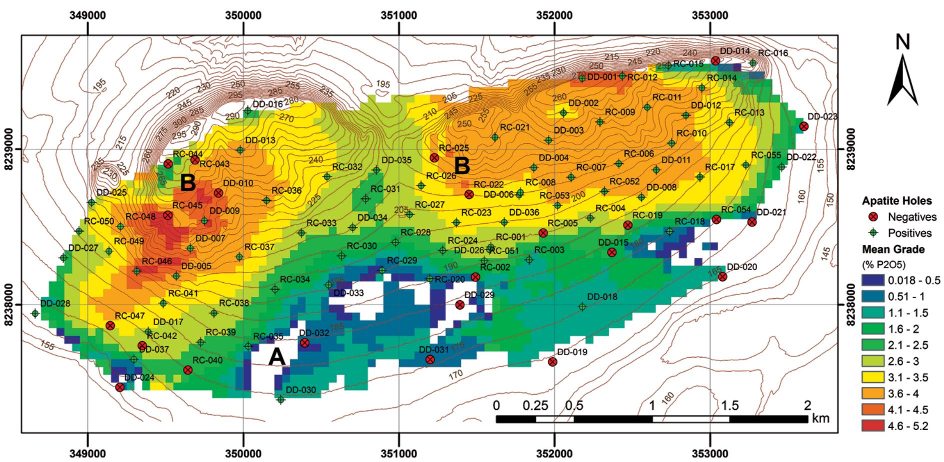

Figure 4
Map of mean grade of P_2O_5.

In Figure 4, there are empty blocks highlighted by the letter A. Actually, empty blocks resulted from negative mean grades because Equation 5 takes into account a subtraction of the second term that is greater than the first term. On the other hand, the letter B indicates regions of high grades but closer to negative drillholes. This happens because the thickness ($T^*= \mu_y$) is much less than the estimated accumulation ($GT^* = \mu_x$). Then in these cases care must be taken in interpreting the results, especially for high grades in regions of low probability.

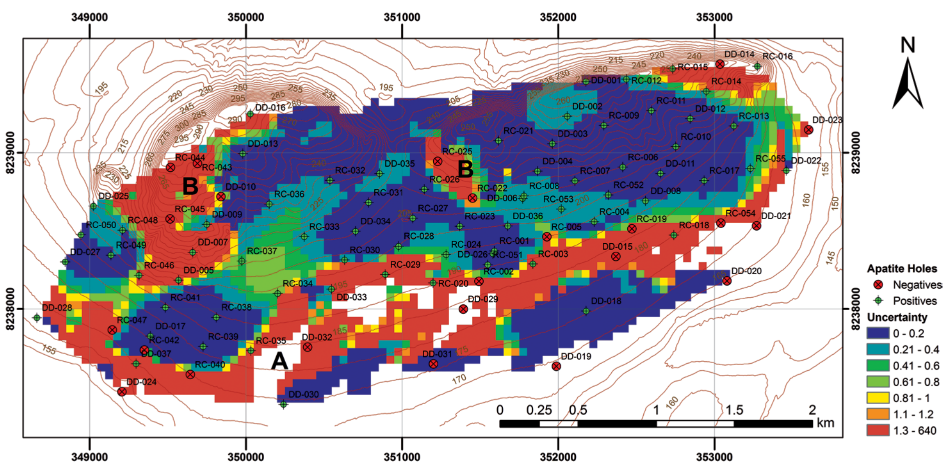

Figure 5
Map of uncertainties associated with the estimates.

A notable result can be seen in Figure 5, in which high uncertainties are associated with lithological boundaries. Moreover, the regions located by the letter B show high uncertainties associated with high mean grades. This means that these high mean grades are meaningless and should be disregarded when calculating the mineral resource.

5. Conclusion

This article provides an efficient method to compute not only the average grade from the ratio between accumulation and thickness, but also its uncertainty. Therefore, it represents a solution to the issue discussed in literature, which states that the ratio of accumulation to thickness is just an approach. It is important to note that the solution presented was possible because of the computation of local variances and co-variances. Finally, the application of the proposed method in a real database was able to show the efficiency of the service variable method and its use for ore resource estimation in stratiform deposits, obtaining fitting results with the drillholes and the mineralization, as well as solving the problem of the measurement of the local uncertainty associated with the estimates of grades.

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