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Comparison between simplified and FE models for short-term deflection in continuous RC beams

Comparação entre modelos simplificados e de elementos finitos para flecha imediata em vigas contínuas de concreto armado

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Abstract

In this work, a comparative study of different simplified methods and nonlinear finite element (FE) models used for calculating short-term deflections (vertical displacements of the longitudinal axis) in continuous reinforced concrete (RC) beams, under service loads, is performed. The simplified methods employed are the one proposed by Branson and the bilinear method recommended by the European Code CEB - Design Manual on Cracking and Deformations. Two finite element models are utilized: the first one with frame elements in which material nonlinearities are considered along the element and its cross section divided into layers, by using of constitutive relationships for steel and concrete, while the second one utilizes beam elements, with physical nonlinearity considered by means of moment-curvature diagrams, obtained from Branson equation. Several examples of continuous RC beams under service loads are analysed and the results obtained by the different models are compared taking as reference the nonlinear frame element model. A few conclusions and recommendations regarding the use of the different methods are drawn at the end of the work.

Keywords: deflections, beams, reinforced concrete, nonlinear analysis.

Resumo

Neste trabalho é realizado um estudo comparativo dos resultados de flecha (deslocamento vertical do eixo longitudinal) imediata de vigas contínuas de concreto armado sob cargas de serviço, obtidos por diferentes métodos simplificados e por modelos não lineares de elementos finitos (EF). Os métodos simplificados abordados são o proposto por Branson e o método Bilinear recomendado pelo CEB – Manual de projeto sobre Fissuração e Deformações. São utilizados dois modelos de EF: o primeiro com EF de pórtico plano, sendo a não linearidade dos materiais considerada ao longo do elemento e da seção discretizada em camadas, a partir de relações constitutivas do aço e do concreto, enquanto que o segundo utiliza EF de viga, sendo a não linearidade física considerada por meio de diagramas momento-curvatura, obtidos a partir da fórmula de Branson. Os resultados dos diferentes modelos são comparados entre si para diversas vigas de projeto, adotando como referência os resultados do modelo não linear de elementos de pórtico plano. A partir da análise dos resultados, extraem-se algumas conclusões e recomendações quanto ao uso dos métodos simplificados e modelos de EF estudados.

Palavras-chave: flecha, vigas, concreto armado, análise não linear.

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1.   Introduction

According to the Brazilian Code NBR 6118 [1], in order to verify the Serviceability Limit State (SLS) for allowable displacements, it becomes necessary in the structural design to estimate the maximum displacement that shall occur in the real structure (maximum vertical displacement or deflection of the bar axis). For this verification, simplified methods recommended by Design Codes or analytical models implemented into computer programs for structural analysis/design can be employed.

The calculation of the total deflection of a reinforced concrete (RC) beam can be divided into two parts, one of short-term and another one of long-term character, the latest one due mainly to shrinkage and creep effects. In this work, only short-term deflection of continuous beams is addressed, realizing that this is the first step in the total deflection verification.

When calculating the short-term deflection of RC beams in the SLS, the physical nonlinearity of the materials should be considered, especially the one caused by concrete cracking. In order to account for such nonlinearities, a nonlinear analysis of the RC beam needs to be performed, which has a greater degree of complexity as compared to linear analysis, since it requires more advanced programs and higher computational effort. There are several nonlinear methods that can be used to calculate deflections in RC beams, ranging from simplified models, as those recommended in Design Codes for RC structures, to very refined models, as the ones using finite elements and more elaborated constitutive models.

In nonlinear analysis, one of the most difficult parameter to determine is the contribution to the element stiffness of the intact concrete between cracks, an effect known as tension-stiffening. A nonlinear analysis is not required in simplified methods, since nonlinear effects, such as tension-stiffening, are considered in a simplified way; these methods can thus be rapidly utilized by engineers, either by means of hand-calculations or by using simple computational tools.

The availability of more refined models that allow calculating more accurately deflections of RC structural elements, representing more adequately their nonlinear behavior, combined with the recent development of advanced computational tools, arises the question about whether simplified methods should still be used to calculate structural deflections nowadays. However, if simplified methods provide satisfactory results as compared to more refined models, its use by structural engineers could then be justified in the sake of simplicity and ease of application.

Junges [2] performed a comprehensive study on the calculation of short-term deflections in simply-supported and continuous RC beams under service loads, by comparing several methods, simplified and refined ones. As the behavior of continuous beams differs from the one of simply-supported beams, due to the negative bending moment acting on the inner supports, which causes a greater extent of cracking in the beam spans, it is interesting to investigate separately the analytical results obtained by the different methods for continuous beams from the ones obtained by simply-supported beams. Results obtained for RC simply-supported beams have already been published earlier in Junges and La Rovere [3]. This work aims to present a summary of the part of the study performed by Junges [2] that regards RC continuous beams.

The simplified methods are described in the following section. The first method utilizes the equivalent moment of inertia equation proposed by Branson [4], applied in two different ways, one as recommended by the Brazilian Code NBR 6118 [1], and another one following the recommendations contained in the technical notes of NB-1 [5], published by Ibracon. The second method, indicated by CEB "Design Manual on Cracking and Deformations" [6], is known as Bilinear Method.

Two finite element (FE) models, described in Section 3, are included in this study: the first one uses plane-frame bar elements, and considers the material nonlinearities along the element axis and along its section, discretized into layers, by means of complete constitutive laws for concrete and reinforcing steel; while the second one uses beam elements, taking into account the physical nonlinearity by means of moment-curvature diagrams, obtained using Branson's equation.

The plane-frame finite element model, which is implemented into a computer program called ANEST/CA, is taken as the reference model in the comparative study of the several models/methods. This model was chosen as it has presented very good results in comparison with experimental results obtained from tests on RC structures - beams and plane-frames, especially for those structures with dominant flexural behavior, as showed by Stramandinoli [9]. For the case of continuous beams, however, few experimental results are available in the literature, from which three examples were selected to be presented here, in Section 4, in order to illustrate the choice of the reference model.

In view of the reduced number of experimental tests on continuous beams, theoretical examples of beams designed at service were analyzed in the comparative study, in such a way to perform a comprehensive study that allows analyzing the influence of different load types, reinforcement ratios, number of spans and beam geometry, in the results given by the different models and methods. A total of 60 continuous beams, of two and three spans, were designed and analyzed under service loads (called here *design beams*), by varying the beam geometry, load type and values, reinforcement ratio and concrete strength. Results of the different models/methods are compared, by taking as reference the nonlinear plane-frame FE model, as explained above.

This study intends to give some guidelines regarding the use of simplified methods and some FE models, by evaluating the variation in the results according to load type, reinforcement ratio and extension of cracked span. From these results, a few concluding remarks and recommendations on the use of the different methods/models studied are extracted at the end.

2.   Simplified methods to calculate short-term deflections in reinforced concrete (RC) beams

Two simplified methods to calculate short-term deflections in RC beams are utilized in this work: the Branson's equation, applied as indicated in the Brazilian Code NBR 6118 [1] and also following the technical notes given in NB-1 [5] published by Ibracon; and the Bilinear method, recommended by CEB [6]. These methods were implemented into a computer code named AVSer and are described as follows.

2.1 Branson's equation

The simplified method proposed by Branson adopts an equivalent moment of inertia (I_{eq}), where the moment of inertia is pondered between the uncracked (I) and the cracked (II) state, in such a way to consider that part of the beam is uncracked and also the tension-stiffening effect in the cracked part of the beam.

2.1.1 Branson' equation as indicated by NBR 6118

Since the version published in 2003, the NBR 6118 [1] recommends the use of Branson's equation to calculate short-term deflections in RC beams, by adopting a reference section to calculate the equivalent stiffness (EI_{eq}) of a beam's span:

$$EI_{eq} = E_{cs} \left\{ \left(\frac{M_{cr}}{M_a}\right)^m I_c + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^m \right] I_{II} \right\} = E_{cs} I_c \quad (1)$$

in which I_c is the moment of inertia of the gross concrete section; I_{II} is the moment of inertia of the cracked concrete section, in state II (neglecting the tensile stresses in concrete); E_{cs} is the secant modulus of elasticity of concrete, which can be obtained by Equation [2] from the initial tangent modulus E_{ci} (in MPa):

$$E_{cs} = 0{,}85 E_{ci} = 0{,}85 \cdot \alpha_E \cdot 5600 f_{ck}^{1/2} \quad (2)$$

where f_{ck} is the compressive strength of concrete in MPa, and α is a coefficient that varies according to the aggregate material adopted (from 0.7 to 1.2):
When one reference section is utilized for the entire span, one adopts $m=3$ to calculate EI_{eq}, in order to consider the tension-stiffening effect and also that part of the span remains in state I, whereas, when an individual section is taken, $m=4$ is adopted (Sabnis *et al*. [7]).
M_a is the bending moment at the critical section of the considered span, and M_{cr} is the cracking moment of the structural element:

$$M_{cr} = \frac{\alpha \cdot f_{ct} \cdot I_c}{y_t} \quad (3)$$

where y_t is the distance from the centroid to the most tensile fiber of the cross-section; α is a coefficient that relates in an approximate way the tensile strength in flexure (also known as modulus of rupture) with the tensile strength of concrete, being equal to 1.5 for rectangular sections; f_{ct} is the tensile strength of concrete, which, in the absence of experimental tests, can be estimated by the following equation, when checking maximum displacements in the SLS:

$$f_{ct} = f_{ct,m} = 0{,}3 \, f_{ck}^{2/3} \quad (4)$$

where $f_{ct,m}$ is the average tensile strength of concrete ($f_{ct,m}$ and f_{ck} in MPa).
When the acting bending moment, M_a, is less than the cracking

moment, M_{cr}, it means that the section is in state I, hence the stiffness of the gross section, $E_{cs} I_c$, must be utilized.

2.1.2 Branson´s equation according to the technical notes from NB-1

In the technical notes from NB-1 [5] published by Ibracon, it is suggested that, for the case of continuous beams where a higher accuracy is seek, the Branson's equation should be applied by adopting a pondered value for the equivalent stiffness, calculated according to Equation [5]:

$$(EI)_{eq} = \frac{1}{l} \left[(EI)_{eq,1} \cdot a_1 + (EI)_{eq,v} \cdot a_v + (EI)_{eq,2} \cdot a_2 \right] \quad (5)$$

in which $(EI)_{eq,1}$ is the equivalent flexural stiffness in stretch 1, $(EI)_{eq,v}$ is the equivalent flexural stiffness in the stretch corresponding to positive bending; and is the equivalent flexural stiffness in stretch 2 (see Figure [1]).
In each stretch of the beam, the equivalent flexural stiffness should be calculated in state II (EI_{II}) by considering the reinforcing bars at the most critical section, and by taking the acting moment M_a equal to M_1, M_v and M_2, respectively.

2.2 Bilinear method – CEB

The bilinear method is described in CEB "Design Manual on Cracking and Deformations" [6]. The short-term deflection is estimated by an intermediate value between the deflection value calculated using the stiffness in state I and the one calculated with the stiffness in state II, by means of an interpolation coefficient ζ, that accounts for the tensile contribution of concrete between cracks and also the portion of the span that remains uncracked, in state I:

$$W = (1 - \zeta) W_1 + \zeta W_2 \quad (6)$$

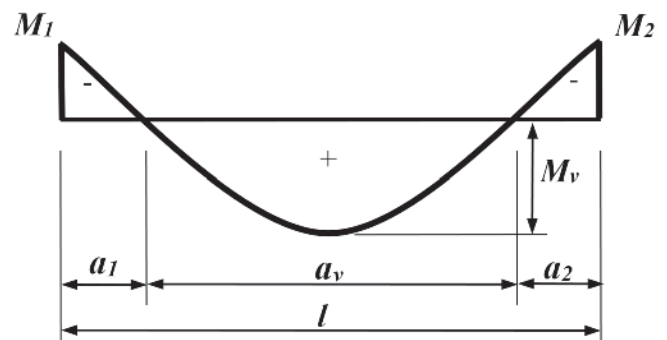where W_1 is the deflection calculated in state I; W_2 is the deflection

Figure 1
Parameters utilized to calculate the equivalent stiffness in a span of a continuous beam using Equation (5)

calculated in state II; and is a dimensionless coefficient, which is equal to zero for an uncracked section, when $M_a < M_{cr}$, and lies between 0 and 1 for cracked section, when $M_a > M_{cr}$, given by:

$$\zeta = 1 - \beta_1 \beta_2 \frac{M_{cr}}{M_a} \qquad (7)$$

in which b_1 is a coefficient that characterizes the quality of bonding of the rebars, and b_2 is a coefficient that represents the influence of duration of the applied loads (if applied in a sustained manner or for a large number of load cycles). According to Eurocode [8], for high-bond rebars and for non-cyclic loading, which is the case of the beams analyzed here, these coefficients can be simplified to:

$$\beta = \beta_1 \beta_2 = 1,0 \qquad (8)$$

M_{cr} is the cracking moment at the critical section, which according to CEB can be obtained by:

$$M_{cr} = \frac{y_t f_{ct}}{I_I} \qquad (9)$$

where f_{ct} is the tensile strength of concrete, defined by CEB as (f_{ct} and f_{ck} in MPa):

$$f_{ct} = f_{ctm} = 1,40 \left(\frac{f_{ck}}{10} \right)^{2/3} \qquad (10)$$

The coefficient can also be applied to interpolate directly the curvature instead of the deflection; this will be the procedure adopted in this work. It should be noted that Equation [4] given by NBR 6118 was extracted from Equation [10] given by CEB, hence the values found for the tensile strength given by these Codes practically coincide, but according to CEB the cracking moment is defined in Eq. [9] as a function of the tensile strength (given by Eq. [10]) whereas in NBR 6118 it is defined as a function of the tensile strength in flexure (Eq. [3]). However, in the comparative study of this work, these properties will be uniformed, thus the same equations, [3] and [4], will also be used for the Bilinear method.

3. Finite element models for nonlinear analysis of reinforced concrete beams

Two finite element (FE) models are utilized; a summary of the first one is described in subsection 3.1, and the second one is presented in subsection 3.2.

3.1 Finite element model for nonlinear analysis of reinforced concrete plane frames

This model was developed by Stramandinoli [9] for physical and geometrical nonlinear analysis of reinforced concrete (RC) planes

and beams, and it was implemented into a computer program called ANEST/CA. The model utilizes a bar element with three nodes - two external nodes with three degree of freedom (DOF) each (two in-plane translations and one rotation), and one internal node with only one DOF (axial displacement), which is included to properly capture the variation of the neutral axis along the element due to material nonlinearities. The Euler-Bernoulli beam assumptions are adopted, and the physical nonlinearities are considered along the element in the Gauss-integration points (three points are adopted) and along the element section by means of the Fiber Model, with the section discretized into layers. Each layer is assumed to be under a uniaxial stress-state, neglecting shear stresses. The constitutive model given by the CEB-FIP MC-90 [10] is adopted for concrete under compression. Under tension, the tension-stiffening model proposed by Stramandinoli and La Rovere [11] is utilized, which considers the interaction between concrete and reinforcement after cracking, by changing the constitutive law of concrete in the post-cracking range through an exponential decay curve, defined by a α parameter, which is a function of the reinforcement ratio and of the steel-to-concrete modular ratio. After yielding of reinforcement takes place the tensile stress in concrete drops abruptly to zero. The reinforcing steel is assumed to behave as an idealized elastic-plastic material, taking strain-hardening into account. The iterative Newton-Raphson method is utilized to solve the nonlinear equilibrium equations of the structure, with loads being applied incrementally. Further details on the model formulation can be found in Stramandinoli [9] and in Junges [2], and also in Stramandinoli and La Rovere [12].

This FE model was selected among the studied models as the reference one in the comparative study. This choice can be justified as this model is the most refined one, and also because it has been validated by comparison with several experimental results obtained from tests on RC beams (simply-supported and continuous) and plane frames by Stramandinoli [9] and Junges [2]. Although for continuous beams the comparison with experimental results was limited, due to the lack of tests available in the literature, the FE model displayed satisfactory results in the examples analyzed by these authors. Among these examples, a few were selected to be presented here in Section 4.

3.2 Beam-element model utilizing constitutive models from bending moment – curvature relations

This model consists on discretizing the beams in several beam elements of small length, using the matrix formulation of structural analysis to obtain the nodal displacements and member-end forces, disregarding shear deformation. For each element the secant stiffness (EI_{sec}) is calculated from the bending moment – curvature graph (taking the average of moments acting on the element ends in absolute value), by using the Branson's equation with the exponent m equal to 4. The iterative secant method is utilized to achieve convergence when solving the nonlinear equilibrium equations. This model, called here BE-Branson model, was implemented into a computer program named AVSer, developed by the fist author, and is limited to analysis of beams up to yielding of the reinforcing steel.
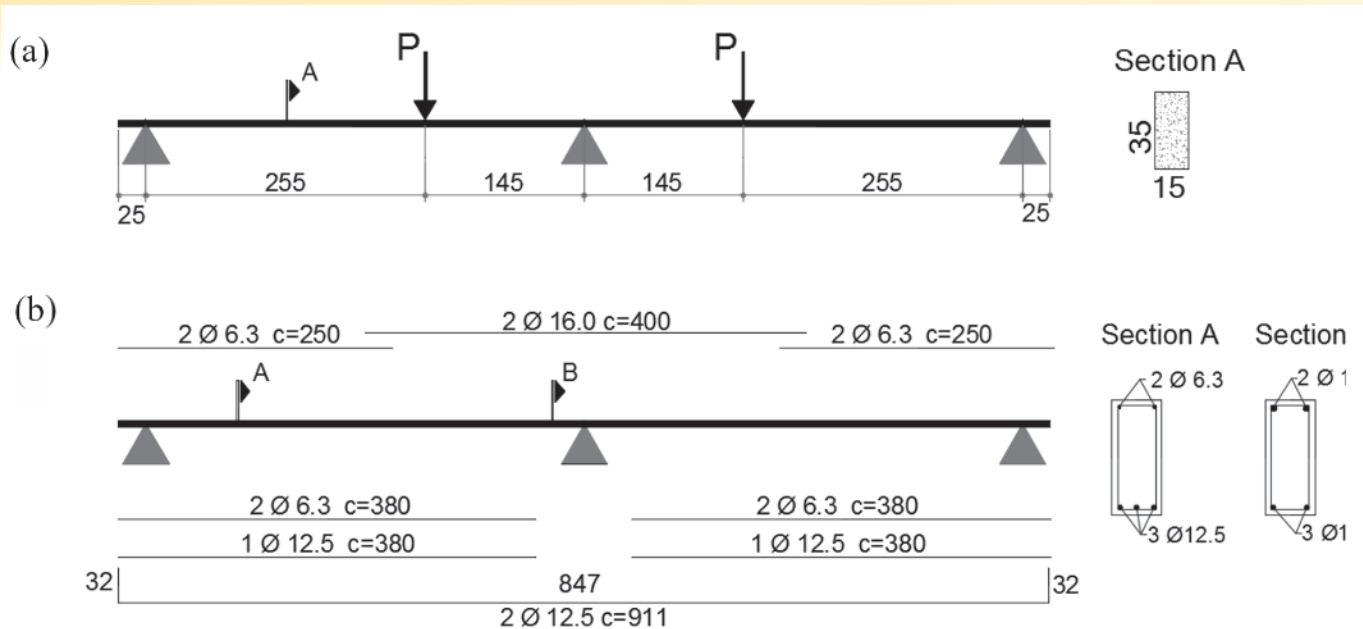
Figure 2
Beam tested by Cruz (14): a) geometry and loading pattern; b) longitudinal reinforcement detailing (dimensions in cm)

4. Comparison between experimental and numerical results obtained from the nonlinear FE model implemented into ANEST/CA program

The FE model for nonlinear analysis of RC structures implemented into ANEST/CA program has already been verified by Stramandinoli [9] in comparison with several experimental testing on beams and plane frames. The numerical model has shown a very good approximation of the structural behavior observed in the experiments, especially for those structures with dominant flexural behavior. In her comparative study, Junges [2] has also shown the good performance of the nonlinear FE model of ANEST/CA with respect to experimental tests; among the investigated models, it was the one that better approached the structural behavior of the tested beams at service, particularly for simply-supported beams. For continuous beams, there are only a handful of experimental results available in the literature, and the ANEST/CA model still showed a good approximation of the behavior at service, but not always it could capture the behavior at ultimate obtained from the

experimental tests, probably due to the opening of inclined cracks at the most critical sections (in the inner supports), which are subjected to combined bending and shear forces. As this comparative study is limited to deflections in beams under service loads, the choice of the ANEST/CA model (AM), using Bernoulli-bar elements as the reference model, can then be justified for this work. It should be pointed out that for those cases of continuous beams in which shear becomes important, plane finite elements or bar elements based on Timoshenko beam theory, as, for instance, the one developed by Stramandinoli and La Rovere [13], should be utilized. In order to illustrate the performance of the ANEST/CA model in the analysis of RC continuous beams, three examples of tests on continuous beams, of two spans, were selected in this work.
The first beam was tested by Cruz [14], who aimed to study the capacity of stress redistribution in RC continuous beams. The beam, detailed in Figure [2], is composed of two spans of 4.0 m, with a steel ratio of 0.82% for the longitudinal reinforcement at the section of maximum bending moment, and a reinforcing steel ratio of 1.46% at the central support section. In the analysis using the model from ANEST/CA program, the beam was discretized into 17 bar elements and the cross section into 20 layers along its height.

Table 1
Material properties for beams tested by Cruz (14) and Silva (15)

| Author/ beam	Concrete				rebar ϕ (mm)	f_y (MPa)	E_s (MPa)	ε_u	s.h.
	f_{cm} (MPa)	$f_{ct,m}$ (MPa)	E_{cs} (MPa)	ε_0					
Cruz (14)	40.80	2.92	27836.84	0.0021	6.3	647.0	196000	0.025	0.0219
Cruz (14)	40.80	2.92	27836.84	0.0021	12.5	580.0	193000	0.024	0.0197
Cruz (14)	40.80	2.92	27836.84	0.0021	16.0	547.0	202000	0.057	0.0266
Silva (15): V1 and V3	30,20	2,40	23124,00	0,0021	6.3	540.3	200000	0.02	0.02
Silva (15): V1 and V3	30,20	2,40	23124,00	0,0021	10.0	505.7	200000	0.02	0.02
Silva (15): V1 and V3	30,20	2,40	23124,00	0,0021	12.5	562.6	200000	0.02	0.02

The material properties are listed in Table [1], in which f_{cm} is the average compressive strength of concrete and e_o is the strain corresponding to f_{cm}; f_y is the yield stress and E_s is the elastic modulus of steel, e_u is the ultimate deformation and *s.h.* is the strain-hardening parameter of steel after yielding (equal to , where is the elastic-plastic modulus of steel).

Silva [15], in his study on the use of plastic analysis in statically-undetermined RC beams, tested several RC beams, from which two continuous beams of two spans: V1 and V3. Geometry, loading and details of longitudinal reinforcement of both beams are displayed in Figure [3]. The longitudinal reinforcing steel ratio of the section at midspan is 1.26% for beam V1 and 0.80% for V3, and for the section at the central support is 1.64% and 0.59% for beams V1 and V3, respectively. Two equal specimens (A and B) have been
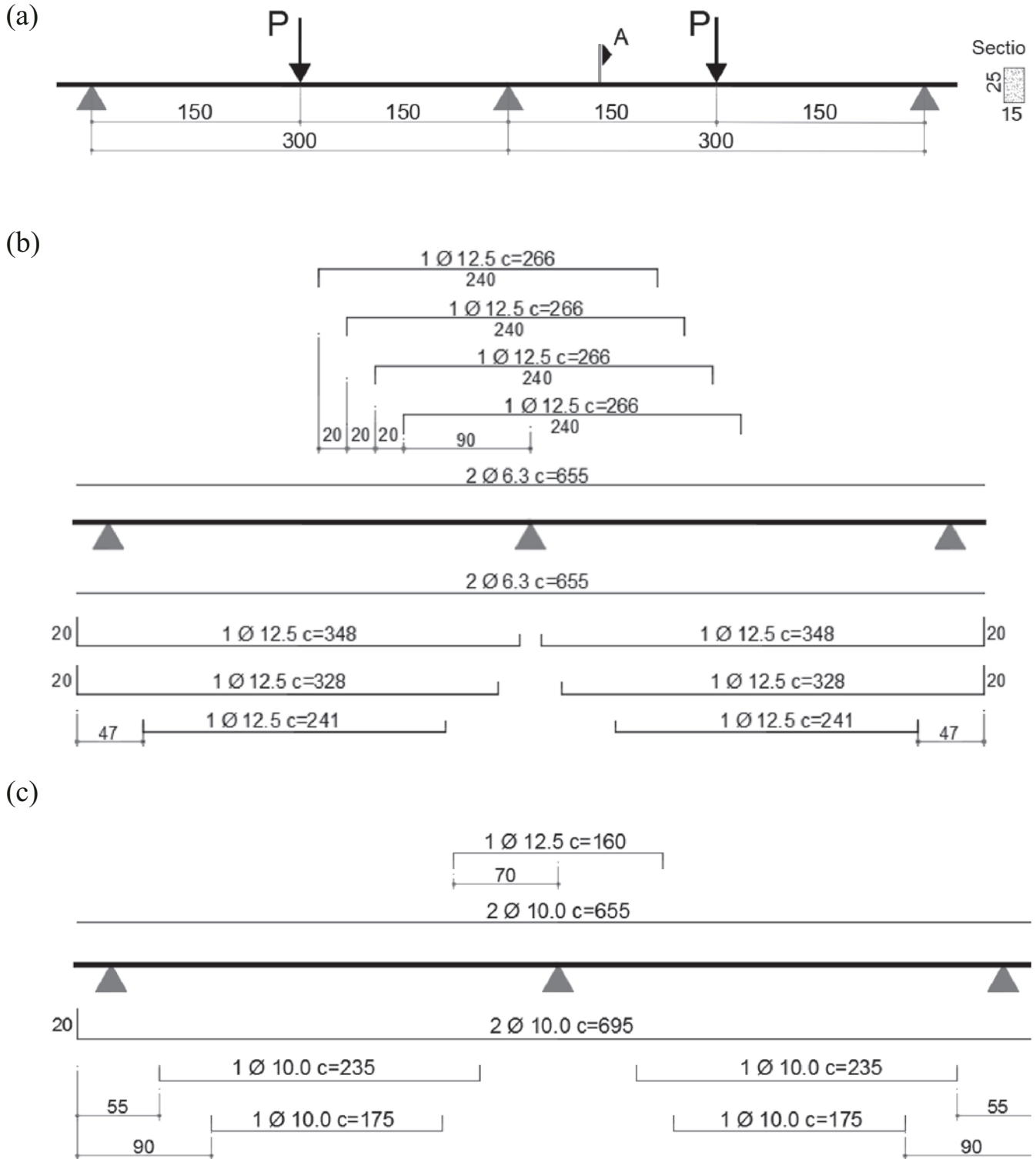
Figure 3
Beams tested by Silva (15): a) geometry and loading pattern; b) longitudinal reinforcement detailing for beam V1; c) longitudinal reinforcement detailing for beam V3 (dimensions in cm)
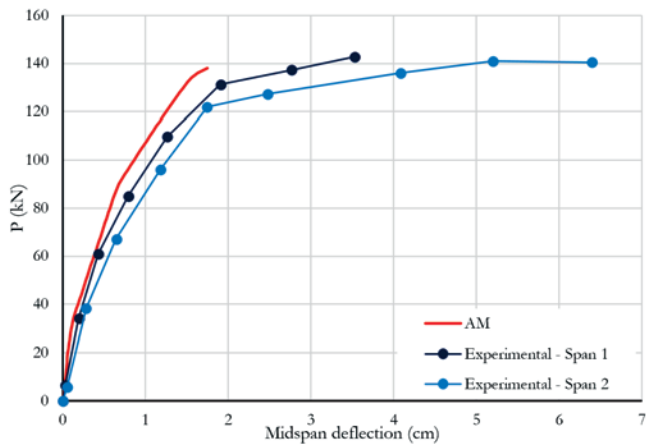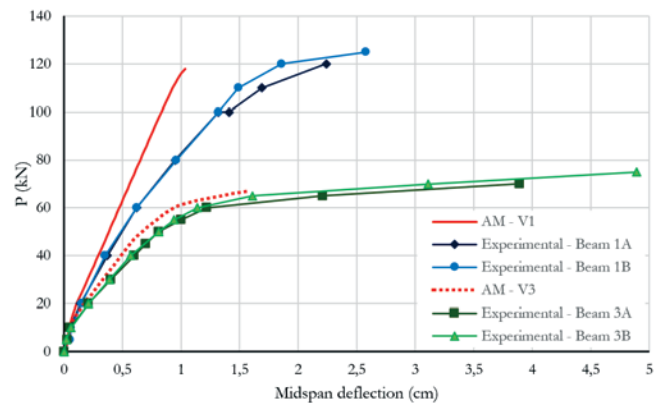
Figure 4
Applied load *versus* midspan deflection for the beam tested by Cruz (14)

Figure 5
Applied load *versus* deflection for beams V1 and V3 tested by Silva (15)

tested for each beam. In the analysis using program ANEST/CA, the beams V1 and V3 were discretized into 14 and 17 elements, respectively, along the beam axis, and the cross section of both beams were discretized into 20 layers along their height. Table [1] contains further information related to material properties. Regarding the finite element meshes utilized in the numerical analysis of the tested beams, they were chosen based on parametric studies performed earlier by Stramandinoli [9] and Junges [2]. Stramandinoli [9] analyzed one simply-supported beam under 3-point bending and another one under 4-point bending; convergence of the numerical solution upon mesh refinement was achieved in the first beam for meshes of at least 10 elements, and, in the second beam, for meshes with a minimum of 4 elements. Junges [2] analyzed a continuous beam of two spans under uniformly distributed loading, obtaining convergence of the solution upon mesh refinement for meshes with at least 10 elements in each span. The authors also follow the recommendation given by Bazant *et al.* [16] of not utilizing elements with length smaller than the height of their cross section. Stramandinoli [9] also investigated the effect of number of layers on the section discretization and concluded that beyond 10 layers there is no change in the numerical solution. It should still be noted that mesh refinement influences mainly the post-yielding behavior of the beam, in the final range prior to failure, while in this work emphasis is given to the behavior at service.

Figure [4] shows the graph of load *versus* midspan deflection for the beam tested by Cruz [14], with the experimental results measured at each span and the numerical result obtained by the ANEST/CA model (AM). By analyzing the curves in the range prior to yielding of the steel (up to approximately 120 kN), a good correlation between numerical and experimental results is verified.

Table 2
Steel properties used for the design beams

| f_y (MPa)	E_s (MPa)	ε_u	s.h.
500	210000	0.02	0.001

Results obtained for the beams V1 and V3 tested by Silva [15] are shown in Figure [5]. A good correlation between results from the numerical model and those obtained experimentally is also noted for beam V3, which is less reinforced longitudinally, but the same does not occur for beam V1, which is more reinforced longitudinally, where a greater difference is displayed between the curves obtained numerically and experimentally. This greater difference is due to shear effects, which becomes more important in this beam with a higher reinforcement ratio. These effects are not taken into account in the Bernoulli beam model used in ANEST/CA, as demonstrated by Stramandinoli [9], who analyzed this same beam V1 using the Timoshenko beam model (Stramandinoli and La Rovere, [13]), getting a much better approximation of the structural behavior obtained in the experimental tests. It should also be pointed out that it is difficult to reproduce exactly in the numerical model the test conditions, since the load and supports are not concentrated at a single point, and moreover there are differences between measurements of displacements obtained at each face of the beam during the tests. Hence, small differences between the load-displacement curves obtained numerically and experimentally can be considered acceptable.

5. Comparison between finite element models and simplified methods for deflection calculation in RC continuous beams

A total of 60 RC continuous beams designed at service, called here design beams, from which 44 of two spans and 16 of three spans, were analyzed. In the analyses the continuous beams were separated into groups, each one with beams of same geometry and concrete properties. Some groups were further divided into subgroups according to the loading type (uniformly distributed or concentrated load at midspan). Each subgroup is composed of four beams, where loading values and reinforcement ratio are varied. In this way, the variation in the results obtained by the different models/methods, according to reinforcement ratio, extension of cracked span, load type, beam geometry and material
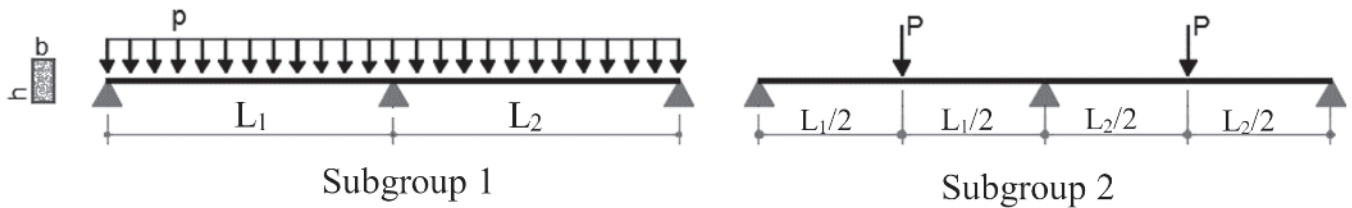
Figure 6
Loading patterns and variables used to define the geometry of two-span beams

properties, could be evaluated. For all *design beams* the longitudinal reinforcing steel utilized was CA-50, of which properties are shown in Table [2]. The material properties of concrete are calculated from the characteristic compressive concrete strength, f_{ck}, defined for a particular beam group, according to the equations given in NBR 6118 [1], and these calculated values are adopted for all models/methods. In the calculus of E_{cs} the coefficient α_E is taken equal to 1.

In order to calculate deflections in the *design beams* using the FE model implemented into ANEST/CA program, (which is the reference model, named here AM), the beam element sections are discretized into 20 layers and the tension-stiffening effect is considered only in the 5 layers under tension, starting from the extreme layer (the most tensile one), until yielding of reinforcement takes place (following the recommendation from CEB for sections under bending, see details in Stramandinoli [9]). In the analyzes using the different models, the same number of elements were used in the beam discretization, following the same recommendations regarding the meshes and number of layers, as seen before in the comparison with beams tested experimentally. In addition, in the parametric study performed by Junges [2], a mesh study was also undertaken for the model called BE-Branson, in which convergence of the numerical solution was achieved for meshes with at least 10 elements in each span, as found before for the reference model, AM.

For the analyses using the BE-Branson model, the coefficient *m* was taken equal to 4, whereas, for the simplified methods Branson-NBR 6118 and Branson-Ibracon, the usual value *m*=3 was adopted. For the methods Branson-NBR 6118 and Bilinear-CEB, for each span an equivalent stiffness, EI_{eq}, is calculated, taking as basis the most critical section in the span (with greater bending moment, in absolute value). Deflections in the *design beams* are calculated for quasi-permanent load combination (QPC), following the recommendations given in the NBR 6118 Code. For those beams subjected to concentrated load (P_{QPC}), a uniformly distributed load due to self-weight is also applied.

5.1 Continuous beams of two spans

Results from continuous beams of two spans are initially presented. Geometry and loading type of the analyzed beams are illustrated in Figure [6], and concrete properties and beam geometry used for a given group of beams are detailed in Table [3]. Only in the groups VCA-G1 and VCA-G2 the beam geometry is nonsymmetric.

Table [4] gives data relative to service loading on each beam (p_{QPC} and/or P_{QPC}), the longitudinal reinforcement areas: A_s ($M_{máx}$), related to the section of maximum positive bending moment in the span, and A_s (M_{min}), related to the section of minimum bending moment (inner support), with the respective reinforcement ratios. The last column of this table shows the extent of cracking in the beam span, in percentage, which is submitted to a moment M_a greater than the cracking moment M_{cr}.

Table [5] contains data relative to the beams of groups VCA-G1 and VCA-G2. As these beams are nonsymmetric, so is the

Table 3
Concrete properties, cracking moment and geometry for the groups of two-span beams

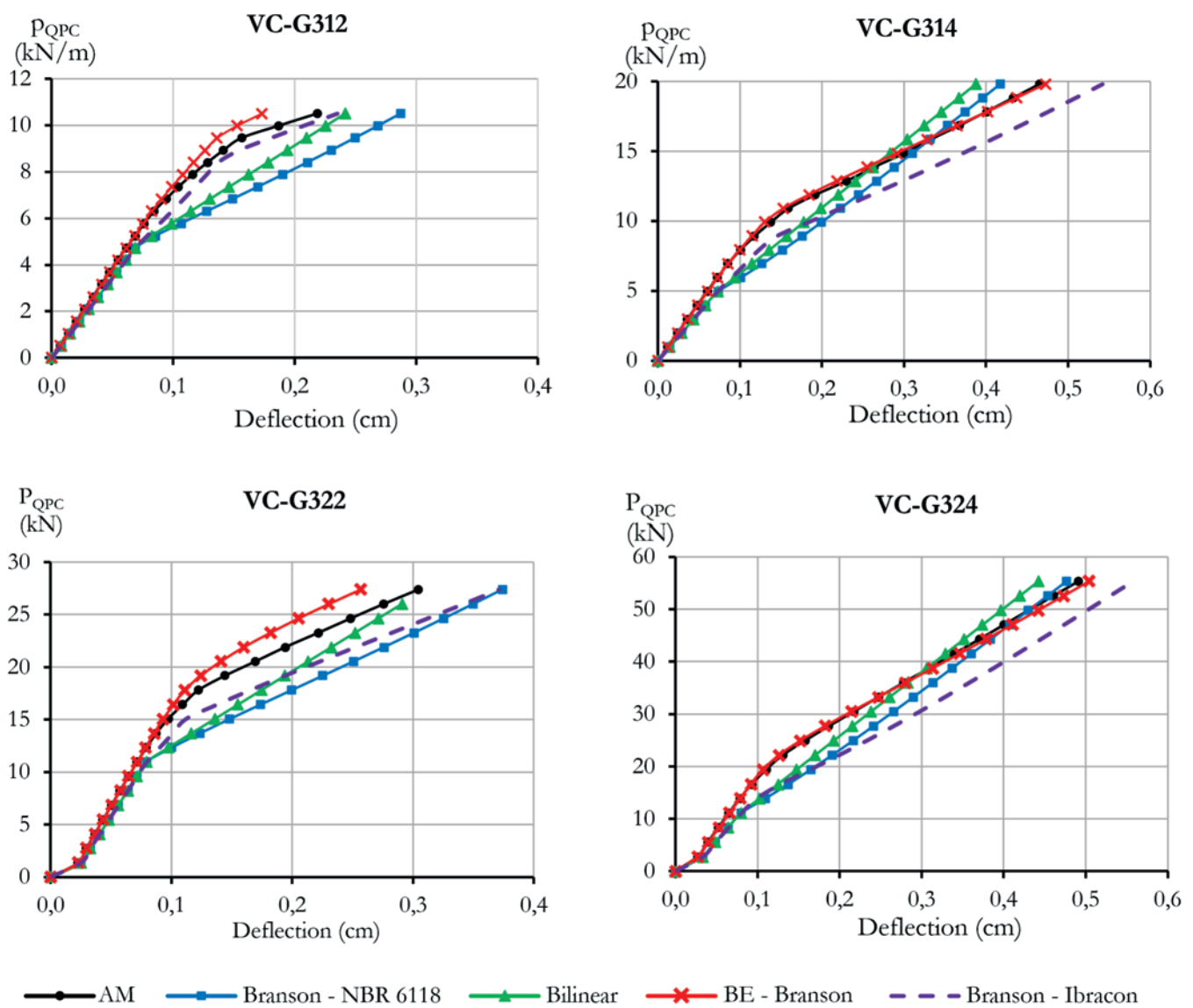| Group	f_{ck} (MPa)	f_{cm} (MPa)	$f_{ct,m}$ (MPa)	E_{cs} (MPa)	ε_0	M_{cr} (kN.m)	L_1 (m)	L_2 (m)	b (m)	h (m)
VC-G1	20	26.60	2.21	21287.4	0.002	5.97	3.00	3.00	0.12	0.30
VC-G2	25	31.60	2.56	23800.0	0.002	11.78	4.00	4.00	0.15	0.35
VC-G3	25	31.60	2.56	23800.0	0.002	12.31	4.50	4.50	0.12	0.40
VC-G4	25	31.60	2.56	23800.0	0.002	20.52	5.00	5.00	0.20	0.40
VC-G5	25	31.60	2.56	23800.0	0.002	32.06	6.00	6.00	0.20	0.50
VC-G6	30	36.60	2.90	26071.6	0.002	52.14	7.00	7.00	0.20	0.60
VCA-G1	25	31.60	2.56	23800.0	0.002	6.93	4.00	3.00	0.12	0.30
VCA-G2	25	31.60	2.56	23800.0	0.002	14.36	5.00	4.00	0.14	0.40

Table 4
Details of load and longitudinal reinforcement for the symmetrical two-span beams

| Beam	p_{QPC} (kN/m)	P_{QPC} (kN)	A_s (M_{max})		A_s (M_{min})		$M_{QPC} > M_{cr}$ (% span)
			rebars	ρ (%)	rebars	ρ (%)	
VC-G111	4.70	–	2 φ 6.3	0.175	2 φ 8.0	0.28	0.00
VC-G112	8.50	–	2 φ 8.0	0.278	2 φ 12.5	0.68	8.00
VC-G113	12.30	–	2 φ 10	0.436	4 φ 10	0.87	49.00
VC-G114	16.10	–	2 φ 12.5	0.683	3 φ 12.5	1.03	63.67
VC-G121	0.90	6.60	2 φ 6.3	0.175	2 φ 8.0	0.28	0.00
VC-G122	0.90	14.85	2 φ 10.0	0.436	2 φ 10	0.44	25.67
VC-G123	0.90	23.10	3 φ 10.0	0.656	4 φ 10.0	0.87	84.67
VC-G124	0.90	31.35	3 φ 12.5	1.025	3 φ 12.5	1.03	63.67
VC-G211	5.11	–	2 φ 8.0	0.190	2 φ 10	0.30	0.00
VC-G212	8.91	–	2 φ 10.0	0.299	4 φ 10.0	0.60	7.25
VC-G213	12.71	–	3 φ 10.0	0.450	2 φ 16.0	0.77	43.00
VC-G214	16.51	–	4 φ 10.0	0.598	3 φ 16.0	1.15	59.75
VC-G311	5.85	–	2 φ 8.0	0.210	2 φ 12.5	0.51	3.56
VC-G312	10.50	–	2 φ 12.5	0.513	3 φ 12.5	0.77	43.11
VC-G313	15.15	–	4 φ 10.0	0.654	3 φ 16.0	1.26	64.44
VC-G314	19.80	–	2 φ 16.0	0.838	4 φ 16.0	1.68	74.00
VC-G321	1.20	11.65	2 φ 10.0	0.327	2 φ 10	0.33	1.11
VC-G322	1.20	27.38	4 φ 10.0	0.654	3 φ 12.5	0.77	46.22
VC-G323	1.20	42.52	4 φ 12.5	1.025	3 φ 16.0	1.26	66.44
VC-G324	1.20	55.34	3 φ 16.0	1.256	4 φ 16.0	1.68	72.44
VC-G411	9.48	–	3 φ 10.0	0.295	4 φ 10.0	0.39	6.60
VC-G412	13.48	–	4 φ 10.0	0.393	4 φ 12.5	0.62	38.40
VC-G413	17.48	–	3 φ 12.5	0.461	4 φ 16.0	1.01	57.20
VC-G414	21.48	–	4 φ 12.5	0.615	5 φ 16.0	1.26	66.60
VC-G511	10.93	–	3 φ 10.0	0.235	4 φ 12.5	0.49	7.50
VC-G512	16.55	–	3 φ 12.5	0.369	4 φ 16.0	0.80	48.83
VC-G513	22.17	–	3 φ 16.0	0.603	5 φ 16.0	1.01	64.50
VC-G514	27.79	–	4 φ 16.0	0.804	7 φ 16.0	1.41	72.50
VC-G521	2.50	18.17	3 φ 10.0	0.246	4 φ 10.0	0.31	0.00
VC-G522	2.50	45.43	4 φ 12.5	0.492	5 φ 12.5	0.62	40.17
VC-G523	2.50	72.68	4 φ 16.0	0.804	5 φ 16.0	1.01	60.67
VC-G524	2.50	109.02	6 φ 16.0	1.207	7 φ 16.0	1.41	72.83
VC-G611	12.00	–	4 φ 10.0	0.262	3 φ 16.0	0.50	6.14
VC-G612	21.00	–	3 φ 16.0	0.503	6 φ 16.0	1.01	53.00
VC-G613	27.00	–	4 φ 16.0	0.670	8 φ 16.0	1.34	65.29
VC-G614	36.00	–	6 φ 16.0	1.006	7 φ 20.0	1.83	75.00

Table 5
Details of load and longitudinal reinforcement for the beams from groups VCA-G1 and VCA-G2

Beam	p_{QPC} (kN/m)	A_s (M_{max})		A_s (M_{min})		A_s ($M_{max'2}$)		$M_{QPC} > M_{cr}$ (% span)
		rebars	ρ (%)	rebars	ρ (%)	rebars	ρ (%)	
VCA-G11	4.28	2 φ 8.0	0.281	2 φ 10	0.44	2 φ 6.3	0.172	0.00
VCA-G12	7.65	2 φ 10	0.436	3 φ 10.0	0.66	2 φ 6.3	0.172	50.75
VCA-G13	12.15	4 φ 10.0	0.872	3 φ 12.5	1.03	2 φ 8.0	0.281	71.25
VCA-G14	16.65	4 φ 12.5	1.367	3 φ 16.0	1.68	2 φ 10	0.436	79.25
VCA-G21	7.25	2 φ 10	0.280	3 φ 10.0	0.42	2 φ 8.0	0.180	4.40
VCA-G22	11.40	3 φ 10.0	0.420	3 φ 12.5	0.66	2 φ 10	0.280	56.60
VCA-G23	15.55	3 φ 12.5	0.659	3 φ 16.0	1.08	2 φ 10	0.280	69.80
VCA-G24	19.70	4 φ 12.5	0.879	4 φ 16.0	1.44	2 φ 12.5	0.439	76.60

Figure 7
Applied load *versus* maximum deflection for beams VC-G312, VC-G314, VC-G322 and VC-G324 from the VC-G3 group
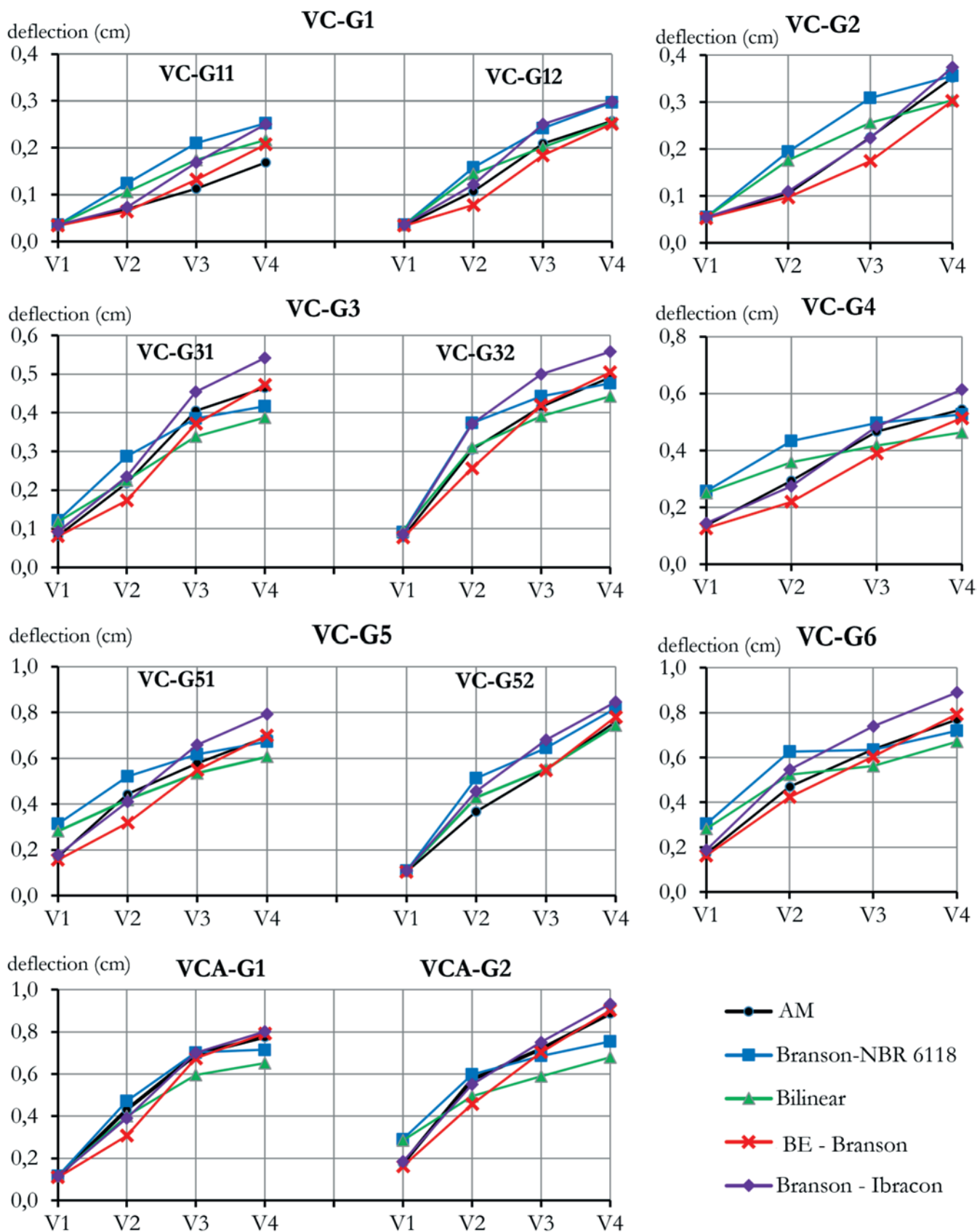
Figure 8
Maximum deflection values obtained by the different models/methods for the two-span symmetrical beams

Table 6
Concrete properties, cracking moment and geometry for the groups of three-span beams

Group	f_{ck} (MPa)	f_{cm} (MPa)	$f_{ct,m}$ (MPa)	E_{cs} (MPa)	ε_0	M_{cr} (kN.m)	L_e (m)	L_c (m)	b (m)	h (m)
VC3V-G1	25	31.60	2.56	23800.00	0.002	15.39	4.50	4.50	0.15	0.40
VC3V-G2	25	31.60	2.56	23800.00	0.002	32.06	5.50	5.50	0.20	0.50
VC3V-G3	25	31.60	2.56	23800.00	0.002	27.16	4.50	6.00	0.14	0.55
VC3V-G4	25	31.60	2.56	23800.00	0.002	15.58	3.00	5.00	0.12	0.45

longitudinal reinforcement, hence the reinforcement areas shown are at the section of maximum bending moment in the first span, A_s ($M_{máx\ l1}$), at the section of the inner support A_s (M_{min}), and at the section of maximum bending moment in the second span, A_s ($M_{máx\ l2}$). Results obtained by each method/model are analyzed by means of "applied load *versus* deflection" graphs, at the section of maximum deflection. Some of these graphs are shown in the following.

Figure [7] show the graphs for four beams from the group VC-G3, where it can be observed the differences in results obtained by the different methods/models under study. Among the methods/models investigated, the BE-Branson model was the one that displayed the "load- deflection" curve closest to the one obtained by the reference model (AM), showing very similar stiffness in the post-cracking range of the curve (state II), and also a close value of the load corresponding to the change of state, from I to II, at the analyzed section, resulting thus in similar values of final deflection, corresponding to the final applied load. In the "load-deflection" curve, the simplified methods Bilinear, Branson-NBR and Branson-Ibracon displayed a lower value of the load corresponding to the change of state, from I to II, at the analyzed section, as compared to the other models, resulting in this way in higher values of final deflection. It can also be observed that, in general, the beam stiffness in the post-cracking range of the curves (state II) given by the methods Branson-Ibracon and Branson-NBR approaches the one given by the reference model, AM; only the Bilinear-CEB method has shown a stiffer behavior of the model after cracking of concrete, resulting in lower values of final deflections.

Aiming to evaluate the deflection values obtained for all beams, the maximum deflection values obtained in each beam were plotted in graphs separated by groups and subgroups, in such a way to visualize the variation of maximum deflection from one beam to another, and also from one method to another. Figure [8] shows such graphs for the groups of beams of two spans and symmetric geometry. It can be noted that, in general, the Branson-Ibracon method displayed the highest values of deflections for beams V3 and V4 inside the groups, i.e., the beams with higher reinforcement ratios and extent of cracking in the span, whereas the Bi-linear method presented the lowest values for these beams. As for the beams with lower reinforcement ratios, the Branson-NBR method presented the highest values of deflections. Generally the BE-Branson model followed by the Branson-Ibracon method have shown final deflections closest to the ones obtained by the nonlinear finite element model of ANEST/Ca program, which was taken in this work as reference.

5.2 Continuous beams of three spans

Continuous beams of three spans were also analyzed, since they are very often utilized in reinforced concrete buildings, and no studies on short-term deflections for this type of beams have been found in the literature. Another reason for studying three-span beams is to better evaluate the Branson-Ibracon simplified method, in which three values of stiffness will be utilized for the central span of the beam, due to the shape of the bending-moment diagram. Sixteen beams of three spans divided into four groups, all subjected to uniformly distributed loading, were investigated.
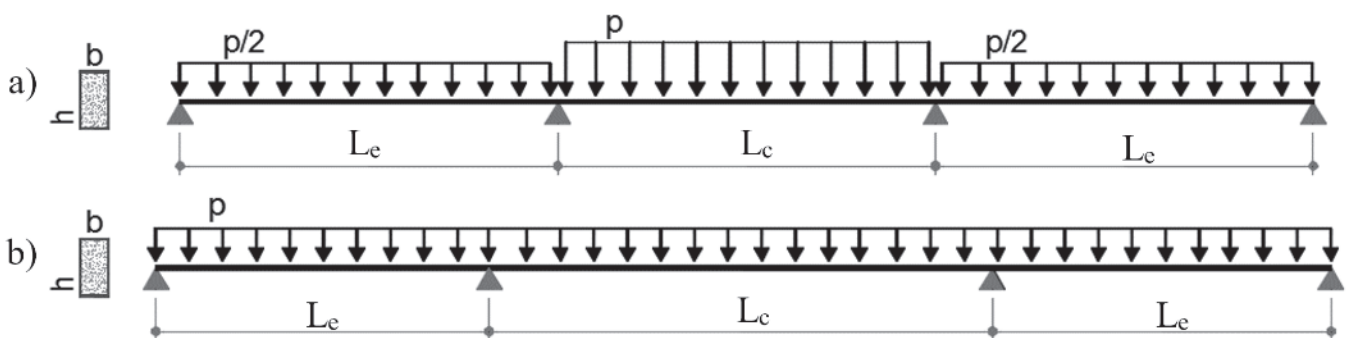
Figure 9
Loading patterns and variables used to define the geometry of three-span beams: a) groups VC3V-G1 and VC3V-G2; b) groups VC3V-G3 and VC3V-G4

Geometry of the beams and the two loading pattern utilized for uniformly distributed loading are illustrated in Figure [9].

Table [7] shows the data of each beam – the uniformly distributed load applied (p_{QPC}), the longitudinal reinforcement area at the section of maximum positive moment in the extreme spans, A_s ($M_{máx,le}$), and at the center of the central span, A_s ($M_{máx,lc}$), and the reinforcement area at the section of the inner supports, A_s (M_{min}).
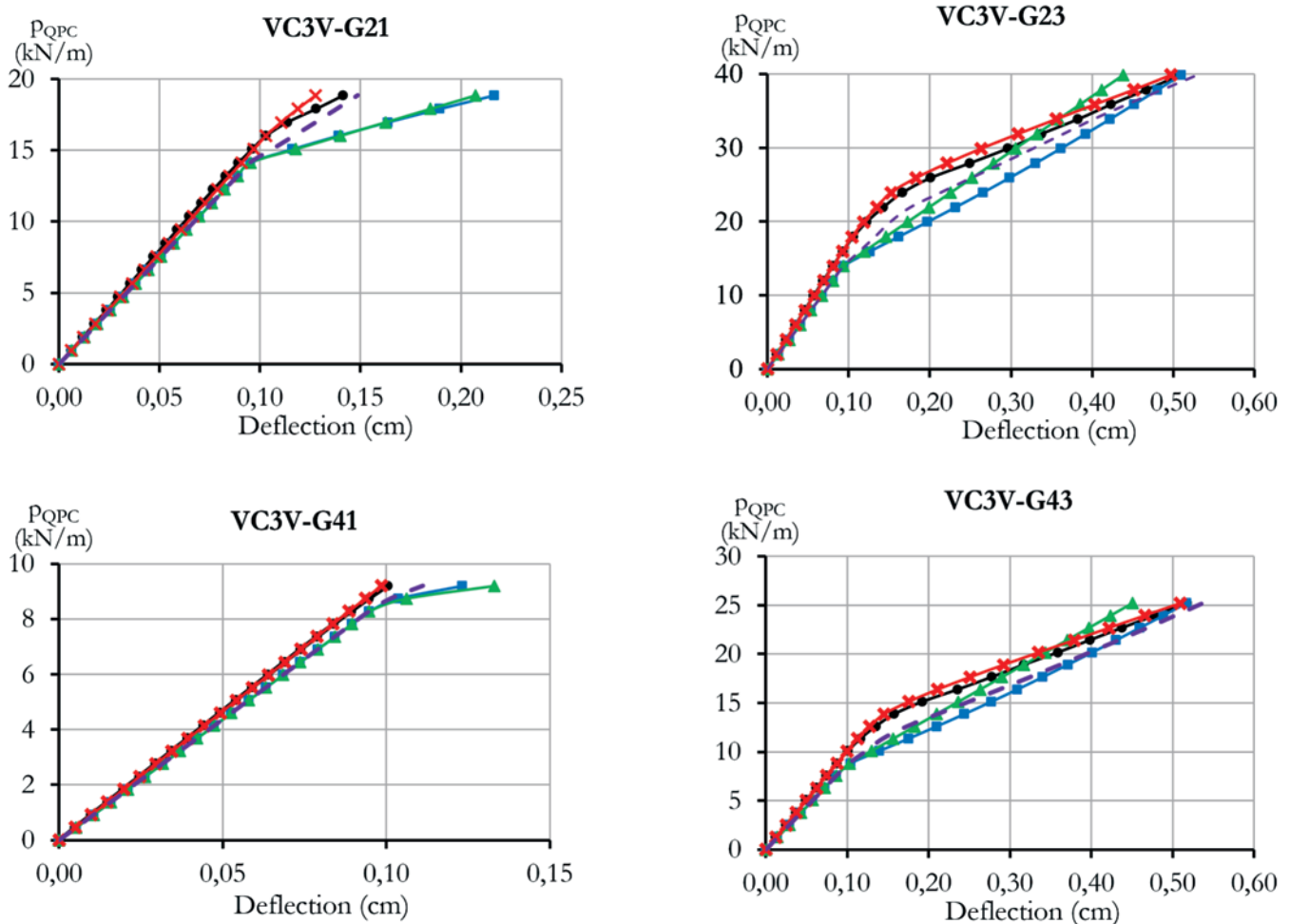
Some graphs of "applied load *versus* maximum deflection" are shown in Figure [10]. For these beams, a similar behavior to the continuous beams of two spans was observed, though a smaller difference between the load value corresponding to the change of state from I to II at the analyzed sections, obtained by the simplified methods and the FE models (BE-Branson and AM), was noted for the three-span beams as compared to the two-span ones.

Figure [11] contains the graphs with the maximum deflection value corresponding to the total applied load, calculated for each beam by each model/method. The same observations made before for two-span beams can be reproduced for the case of three-span beams.

5.3 Discussion of results

In order to analyze the results, the differences (in percentage) between the maximum short-term deflection found by the different methods and models investigated and the one obtained using the reference model from ANEST/CA program (AM) were calculated by considering the deflection values corresponding to the total service load (called final deflections). The average values of percentage differences, separated by method/model, for all 60 continuous beams analyzed are presented in Table [8]. The table contains the average of beams with negative percentage difference (Average (-)), which means beams that displayed lower deflections as compared to those obtained using AM; the average of beams with positive percentage difference (Average (+)); the total average (Average), and, in the last column, the standard deviation found for the percentage differences. The average values obtained by each method can be visualized in Figure [12].

From the average calculated for all beams, the BE-Branson model was the model that showed closest results to the ones obtained using AM (-5.49%), and also the least dispersion in the results,

Figure 10
Applied load *versus* maximum deflection for beams VC3V-G21, VC3V-G23, VC3V-G41 and VC3V-G43

Table 7
Details of load and longitudinal reinforcement for the three-span beams

Beam	p_{QPC} (kN/m)	A_s ($M_{max}I_e$)		A_s (M_{min})		A_s ($M_{max}I_c$)		$M_{QPC} > M_{cr}$ (% span)
		rebars	ρ (%)	rebars	ρ (%)	rebars	ρ (%)	
VC3V-G11	9.00	2 φ 8.0	0.168	2 φ 10.0	0.262	2 φ 8.0	0.168	0.00
VC3V-G12	19.00	2 φ 10	0.262	3 φ 12.5	0.615	3 φ 10.0	0.393	43.56
VC3V-G13	29.00	3 φ 10.0	0.393	3 φ 16.0	1.005	3 φ 12.5	0.615	65.33
VC3V-G14	39.00	4 φ 10.0	0.523	5 φ 16.0	1.675	3 φ 16.0	1.005	74.67
VC3V-G21	18.85	2 φ 10	0.157	3 φ 12.5	0.369	2 φ 12.5	0.246	8.00
VC3V-G22	34.60	4 φ 10.0	0.314	4 φ 16.0	0.804	4 φ 12.5	0.492	59.27
VC3V-G23	39.85	3 φ 12.5	0.369	5 φ 16.0	1.006	3 φ 16.0	0.603	64.73
VC3V-G24	50.35	4 φ 12.5	0.492	7 φ 16.0	1.408	4 φ 16.0	0.804	72.73
VC3V-G31	11.90	2 φ 10	0.204	4 φ 10.0	0.408	3 φ 10.0	0.306	6.67
VC3V-G32	18.55	3 φ 10.0	0.306	3 φ 16.0	0.783	4 φ 10.0	0.408	36.67
VC3V-G33	25.20	3 φ 12.5	0.479	4 φ 16.0	1.044	4 φ 12.5	0.639	58.00
VC3V-G34	35.18	4 φ 12.5	0.639	6 φ 16.0	1.568	3 φ 16.0	0.783	71.00
VC3V-G41	9.20	2 φ 8.0	0.187	2 φ 10.0	0.291	2 φ 10.0	0.291	2.00
VC3V-G42	17.20	2 φ 8.0	0.187	4 φ 10.0	0.581	3 φ 10.0	0.437	48.00
VC3V-G43	25.20	2 φ 8.0	0.187	3 φ 16.0	1.117	3 φ 12.5	0.683	70.40
VC3V-G44	33.20	2 φ 10	0.291	4 φ 16.0	1.489	3 φ 16.0	1.117	72.40

with most cases resulting in smaller deflections as compared to the ones given by AM. In the load-deflection curve, this model presented not only the point of change from state I to II coinciding in general with the one given by the reference model, AM, but also a very similar stiffness in the post-cracking range, which explains the close values obtained for the final deflection values. It can be noted from Table [8] that, for this particular model, the total number of beams analyzed is 58 instead of 60; this was due to the fact that for the beams VC3VG42 and VCG522 there were convergence problems in the iterative procedure used to solve the nonlinear equilibrium equations, which will not be discussed here (see details in Junges [2]).

From Table [8] it can be realized that all simplified methods have presented in average deflections values larger than the ones presented in average deflections values larger than the ones ob-

tained using the reference model AM. Among the simplified methods, the Bilinear was the one that displayed the best results, with the second lower average (7.98%) relative to the reference AM, though showing a high dispersion in the results (27.03 of standard deviation). The Branson-Ibracon method, despite obtaining an average difference a bit higher than the Bilinear method (9.68%), the corresponding standard deviation was considerably lower, hence it is a method with lower dispersion in results. Regarding the Branson-NBR method, it was the one showing the higher average difference, with most deflection values larger than the ones calculated using AM (48 beams). As the Branson-Ibracon method considers up to three sections for determining the equivalent stiffness of a span, it showed a better approximation to the AM and BE-Branson model as compared to the Branson-NBR method,

Table 8
Average of percentage differences (in %) between the final deflections found by the different methods and the reference model, AM

Method	Average (-)	nb. of beams	Average (+)	nb. of beams	Average	Standard deviation
Branson-NBR	-5.05	12	27.43	48	20.93	27.95
Branson-Ibracon	-6.16	8	12.11	52	9.68	11.26
Bilinear-CEB	-11.71	30	27.63	30	7.96	27.03
BE-Branson	-9.44	43	5.82	15	-5.49	10.86
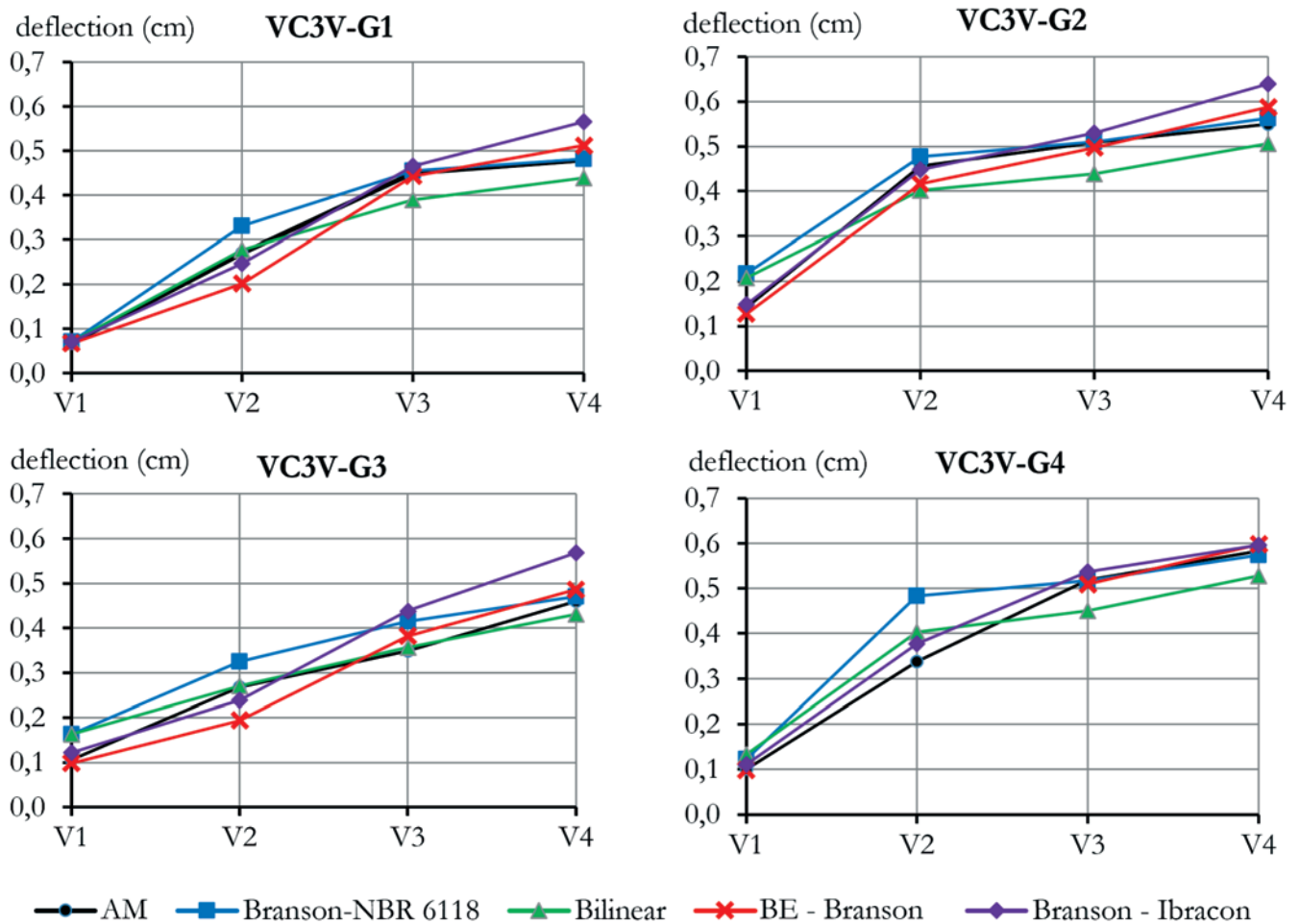
Figure 11
Maximum deflection values obtained by the different models/methods for the three-span beams

which adopts only one section to calculate the stiffness.

Hence it can be stated that the BE-Branson model was the one that displayed the closest results to the ones given by the ANEST/CA model. This approximation is better in comparison with the simplified methods that also utilize the Branson's equation (Branson-NBR and Branson-Ibracon), since the BE-Branson model takes into account the stiffness variation along the span, which occurs due to the variation of cracking level according to the acting moment and also the reinforcement ratio. It should also be noted that

for the BE-Branson model the exponent m in Branson's equation was taken equal to 4 instead of equal to 3, in such a way to disregard that part of the beam lies in state I, since the stiffness is calculated for an element of small length, which either is fully cracked or it lies in state I.

Table [9] presents the average and standard deviation for the percentage differences of deflection values separated into three groups: beams of two spans under uniformly distributed loading, beams of two spans under concentrated loads, and beams of three

Table 9
Average of percentage differences (in %) between the final deflections found by the different methods and the reference model, AM, for the two-span beams under distributed load and concentrated load, and for the three-span beams

Method	Two-span beams under distributed load		Two-span beams under concentrated load		Three-span beams	
	Average	Standard deviation	Average	Standard deviation	Average	Standard deviation
Branson-NBR	25.42	34.39	16.16	14.94	15.55	19.29
Branson-Ibracon	8.67	12.54	15.10	7.11	7.62	10.33
Bilinear-CEB	10.57	32.95	5.08	12.97	4.90	21.82
BE-Branson	-6.62	11.48	-4.77	9.35	-3.61	10.89
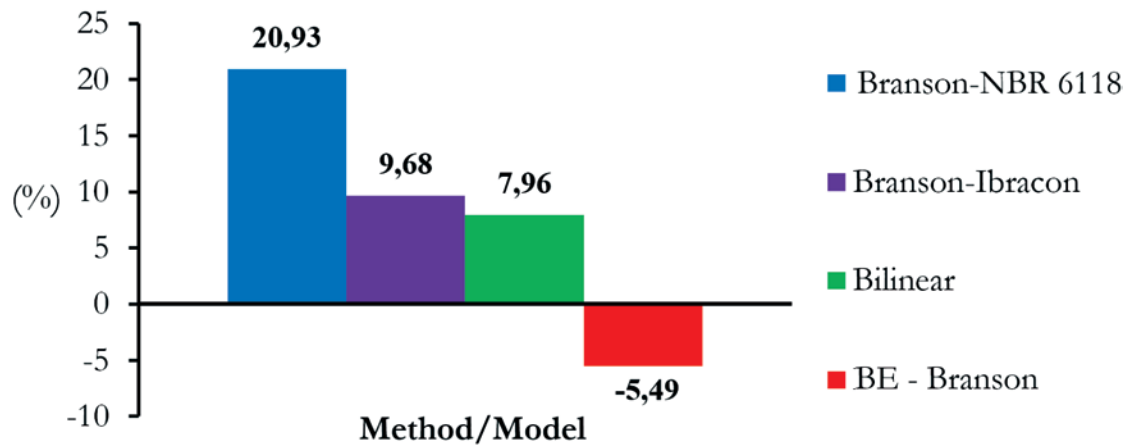
Figure 12
Graph of average percentage differences (%) between the final deflections found by the different methods and by the reference model, AM

spans under uniformly distributed loading. The average differences contained in Table [9] are illustrated in Figure [13].

By analyzing the average differences separated by cases, the BE-Branson model keeps being the one that approaches better the reference model AM, but some changes in the results given by the simplified methods with respect to the global analysis can be observed. Among the simplified methods, the Bilinear method still displays the lower difference relative to the reference model AM for the beams of two spans under concentrated loads and for the beams of three spans, keeping a high dispersion in the results. Nevertheless, for the case of beams of two spans under uniformly distributed loading, the Branson-Ibracon was the simplified method that approached better the reference model AM.

Results of deflections were also analyzed with respect to reinforcement ratio and extent of cracking in the span under total service load. It was observed that for beams with low reinforcement ratios, which are also the ones that displayed a lower extent of cracking in the span under total service load, the results of the methods and model under study showed a high dispersion relative to the ones

given by the AM, especially for the simplified methods, which in general display larger values of deflections relative to the refined FE models. As for beams with higher reinforcement ratios, above 1% in average, the variability of percentage differences of final deflections (corresponding to the total service load) relative to the ones obtained by the reference model AM decreases. This is due to the fact that, as these beams are quite cracked, the contribution of intact concrete between cracks for the beam stiffness is small, an effect (tension-stiffening) that is more rigorously modelled by the AM, hence the differences of the methods/models with respect to AM decrease. More details on this study can be found in Junges [2].

6.   Conclusions

This work aimed to perform a comprehensive study on different simplified methods and refined models used to calculate short-term deflections applied to continuous reinforced concrete beams at service, by comparison of results obtained from the different methods/models, taking as reference the nonlinear FE model implemented
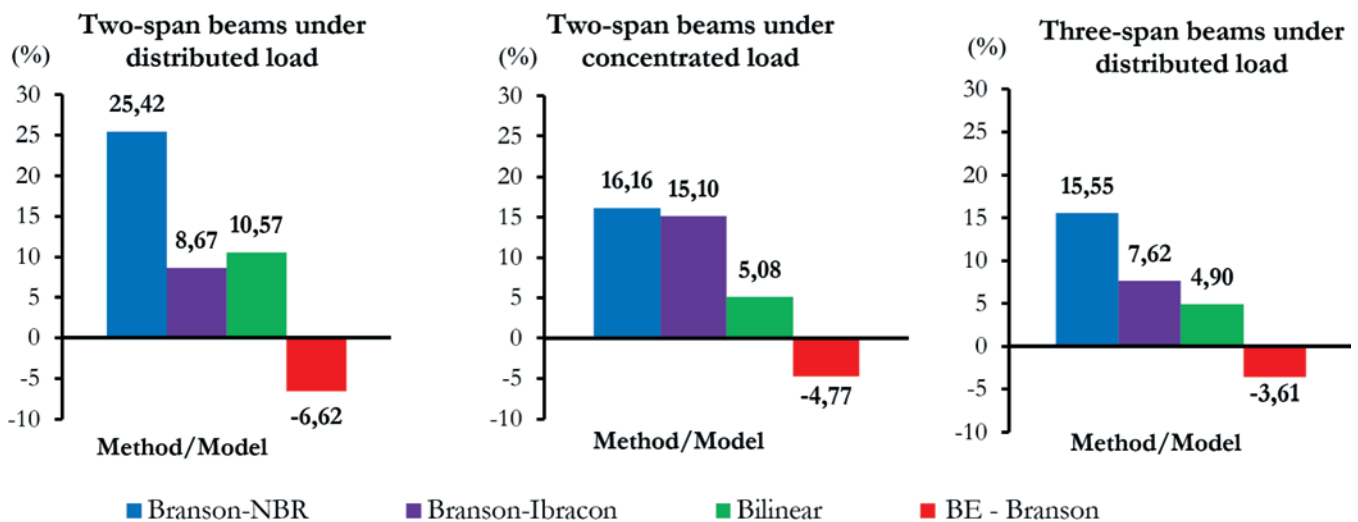
Figure 13
Graph of average percentage differences (%) between the final deflections found by the different methods and by the reference model, AM, separated by cases (beam geometry/loading)

into ANEST/CA program (named AM). A few conclusions, separated by method/model, were extracted, as listed below:

a) BE-Branson model: obtained in average the closest results to the ones given by AM, with less dispersion in the results, and also "load-deflection" curves more similar to the ones given by AM.

b) Branson-NBR method: final deflection values quite dispersive and generally larger than the ones obtained using AM, especially for continuous beams of two spans under uniformly distributed loading, with the load corresponding to the change of state from I to II considerably lower than the one obtained by AM.

c) Branson-Ibracon method: results of final deflections fairly close to the ones given by AM, with average difference of 9.68%, showing less difference for the case of beams under uniformly distributed loading.

d) Bilinear method: results of final deflections close to the ones obtained by AM (average difference of 7.96%), though with highly dispersive results; it has also shown a load corresponding to the change of state from I to II considerably lower as compared to the other models, similarly to the Branson-NBR method, but with a higher stiffness in the post-cracking range.

Concluding, regarding the refined models investigated, the ANEST/CA model (AM) is the most complete one and it has shown a good approximation to experimental results, as seen in Section 4, as also demonstrated by Stramandinoli [9] and Junges [2] for simply-supported beams. It is thus recommended for a more accurate calculation of short-term deflections. Optionally, the BE-Branson model (using $m=4$ in Branson's equation) can also be utilized, since it has shown satisfactory results and it is of ease implementation in computer programs, being the model that has shown the closest results to the ones given by AM for RC continuous beams, as seen in this work, as well as for simply-supported beams, as shown by Junges and La Rovere [3]. Among the simplified methods, the Branson-NBR method, in which the Branson's equation is applied according to NBR 6118 and by adopting only one reference section when calculating the equivalent stiffness in a span, has brought to the highest dispersive results and the largest average difference relatively to the reference model. Regarding the Branson-Ibracon method, in which the Branson's equation is applied as recommended by NBR 6118, but by calculating the equivalent stiffness in each span as indicated in the technical notes from NB-1 [5], published by Ibracon, it has displayed the least dispersive results, showing a close approximation to the reference model. This simplified method is thus recommended for estimations of short-term deflection in continuous beams. Despite displaying in average the closest results to the AM ones in comparison with the other simplified methods, the Bilinear method, recommended by CEB, showed the highest dispersion in results, being recommended only for the cases of beams under concentrated loads, for which it presented the least dispersion in results and the smaller average difference relative to the reference model, AM.

7. Acknowledgements

8. References

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