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Characterizing Block Graphs in Terms of One-vertex Extensions

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ABSTRACT. Block graphs has been extensively studied for many decades. In this paper we present a new characterization of the class in terms of one-vertex extensions. To this purpose, a specific representation based on the concept of boundary cliques is presented, bringing about some properties of the graph.

Keywords: block graph, one-vertex-extension.

1 INTRODUCTION

Block graphs have been extensively studied for many decades with characterizations based on different approaches since the first one in 1963 until today. In this paper we present a new characterization of the class in terms of one-vertex extensions. As block graphs are a subclass of chordal graphs, properties of this class can be successfully particularized: a specific representation of block graphs based on the concept of boundary cliques is presented, bringing about some properties of the graph.

Harary [7] introduced the definition of a block graph based on structural properties and presented a classical characterization: the block graph $B(G)$ of a given graph G is that graph whose vertices are the blocks (maximal 2-connected components) $B_1,\dots,B_k$ of G and whose edges are determined by taking two vertices B_i and B_j as adjacent if and only if they contain a cut-vertex (its removal disconnects the graph) of G in common. A graph is called a *block graph* if it is the block graph of some graph.

Characterization 1. [7] *A graph is a block graph if and only if all its blocks are complete.*

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Howorka [8], Bandelt and Mulder [2] and Behtoei *et al.* [3] presented characterizations based on metric conditions.

Characterization 2. [8] *A graph is a block graph if and only if for every four vertices u,v,w,x, the larger two of the distance sums $d(u,v)+d(w,x)$, $d(u,w)+d(v,x)$, $d(u,x)+d(v,w)$ are equal.*

Characterization 3. [2] *A graph $G=(V,E)$ is a block graph if and only if for every vertices $u,v,w \in V$, G satisfies*

1. $I(u,v) \subseteq I(u,w) \cup I(v,w)$
2. $I(u,w) \subseteq I(u,v) \cup I(v,w)$
3. $I(v,w) \subseteq I(u,v) \cup I(u,w)$

where $I(x,y) = \{z \in V; d(x,y) = d(x,z) + d(z,y)\}$, $x,y \in V$.

Characterization 4. [3] *A graph G is a block graph if and only if it satisfies:*

1. *the shortest path between any two vertices of G is unique and*
2. *for each edge $e = uv \in E$, if $x \in N_e(u)$ and $y \in N_e(v)$, then, and only then, the shortest path between x and y contains the edge e, where $N_{e=uv}(v) = \{w \in V; d(v,w) < d(u,w)\}$.*

Bandelt and Mulder [1] presented a characterization based on forbidden subgraphs.

Characterization 5. [1] *A graph is a block graph if and only if it is $C_{n\geq 4}$-free and diamond-free.*

Mulder and Nebeský [11] characterized block graphs using an algebraic approach, a binary operation $+$ (*leap operation*) on a finite nonempty set V such that for $u,v,w \in V$,

1. $(u+v)+u = u$.
2. if $u \neq v$ then $(u+v)+v \neq u$.
3. if $u+v \neq v$ then $((u+v)+v)+u \neq u$.
4. if $u \neq v = u+v$, $u+w \neq v$ and $v+w \neq u$ then $u+w = v+w$.

The underlying graph $G_+ = (V, E_+)$ of $+$ is such that $uv \in E_+$ if and only if $u \neq v$, $u+v = v$ and $v+u = u$.

Intuitively, for any two vertices u and w in different blocks, the leap operation produces the cut-vertex z in the block of u on the way to w, i.e., $u+w = z$. If u and w are in the same block, then $u+w = w$.

Characterization 6. [11] *G is a block graph if and only if it is the underlying graph of a leap operation on V.*

Recently, the subject was resumed. Dress *et al.* [5] characterized block graphs in terms of their vertex-induced partitions: any partition of a given finite set V is a V-partition, and $P_V = (p_v)_{v\in V}$ is a V-indexed family of V-partitions. A family P_V is a compatible family of V-partitions if, for any two distinct elements $u, v \in V$, the union of the set in p_v that contains u and the set in p_u that contains v coincides with the set V, and $\{v\} \in p_v$ holds for all $v \in V$. Let $P(V)$ denote the set of all compatible families of V-partitions.

Characterization 7. [5] *There is a bijective function between the block graphs with vertex set V and $P(V)$.*

Mulder [10] presented a surprisingly simple characterization.

Characterization 8. [10] *The graph G is a block graph if and only if there exists a unique induced path between any two vertices in G.*

2 BACKGROUND

Basic concepts about chordal graphs are assumed to be known and can be found in Blair and Peyton [4] and Golumbic [6]. In this section, the most pertinent concepts are reviewed.

Let $G = (V(G), E(G))$, or simply $G = (V, E)$, be a connected graph, where $|E| = m$ and $|V| = n$. The *set of neighbors* of a vertex $v \in V$ is denoted by $N_G(v) = \{w \in V; vw \in E\}$ and its *closed neighborhood* by $N_G[v] = N_G(v) \cup \{v\}$. Two vertices u and v are *true twins* in G if $N_G[u] = N_G[v]$ and *false twins* in G if $N_G(u) = N_G(v)$. For any $S \subseteq V$, the subgraph of G induced by S is denoted $G[S]$. The set S is a *clique* if $G[S]$ is complete. A vertex $v \in V$ is said to be *simplicial* in G when $N_G(v)$ is a *clique* in G.

It is worth mentioning two kinds of cliques in a chordal graph G. A *simplicial clique* is a maximal clique containing at least one simplicial vertex. A simplicial clique Q is called a *boundary clique* if there exists a maximal clique Q', $Q \neq Q'$, such that $Q \setminus Q'$ is a set of simplicial vertices of G.

A *perfect elimination ordering* (*peo*) of a graph $G = (V, E)$ is a bijective function $\sigma : \{1, \dots n\} \to V$ such that $\sigma(i)$ is a simplicial vertex in the induced subgraph $G_i = G[\{\sigma(i), \dots, \sigma(n)\}]$, for $1 \leq i < n$. A *peo* is ultimately an arrangement of V in a sequence $\sigma(V) = [\sigma(1), \dots, \sigma(n)]$. It is well known that a graph G is chordal if and only if G admits a perfect elimination ordering.

3 BOUNDARY REPRESENTATION

In this section we present a representation of block graphs based on the concept of a perfect elimination ordering of the graph. As in a *peo*, where a vertex is eliminated when it is simplicial in the remaining graph, in this proposed representation, a maximal clique is eliminated when it is a boundary clique in the remaining graph. As all elements of the maximal clique are stored, the graph can be easily recovered. The representation is defined as follows; its structure is similar to the one presented in [9].

Let $G = (V,E)$ be a block graph with ℓ maximal cliques. A *boundary representation* of G is the sequence of pairs

$$BR(G) = [(P_1, s_1), \ldots, (P_\ell, s_\ell)]$$

such that

- $P_i \cup \{s_i\} = Q_i$, $i = 1, \ldots, \ell - 1$, is a boundary clique of graph $G[V \setminus (Q_1 \cup \ldots \cup Q_{i-1})]$ where
  - $P_i \subset V$ is the set of simplicial vertices of Q_i and
  - $s_i \in V$ is the cut-vertex of the clique Q_i;
- $P_\ell = Q_\ell$ is a maximal clique of G and $s_\ell = \oslash$ (the symbol $\oslash$ denotes the absence of the parameter).

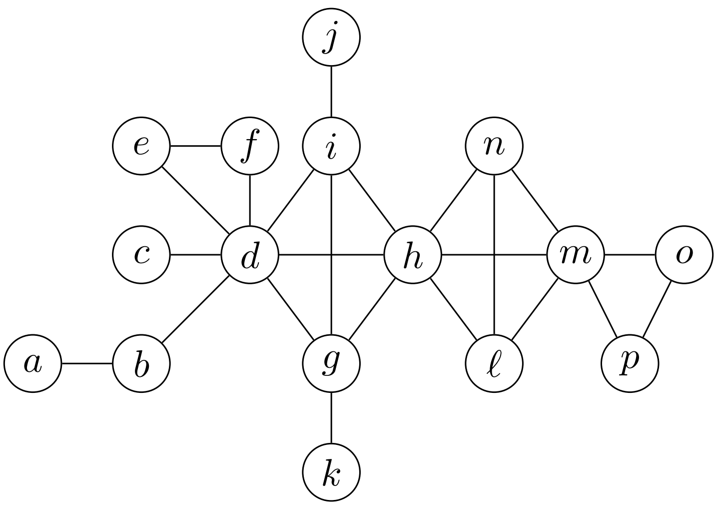

Figure 1: Block graph.

The algorithm to build the representation proceeds in stages. In each stage the boundary cliques of the current graph are determined. For each boundary clique, the simplicial vertices and its corresponding cut-vertex are recorded and all the simplicial vertices are removed from the graph. The process is repeated until only one clique remains. Note that it is possible to obtain a perfect elimination ordering of G in direct sense, unlike other well known algorithms (lexicographic breadth-first search [12], for instance).

The boundary representation makes possible to deduce some structural properties of the graph.

Property 1. $P_1 \cup \{s_1\}, P_2 \cup \{s_2\}, \ldots, P_\ell$ *are the maximal cliques of* G.

Property 2. *The set* $\{P_1, \ldots, P_\ell\}$ *is a partition of* V.

Property 3. *The sequence provided by all vertices of P_1, followed by all vertices of P_2, and so on, up to P_ℓ is a perfect elimination ordering of G. Observe that, since there is no order in the set P_i, $i = 1,\dots,\ell$, several sequences can be built.*

Employing the algorithm for the graph in Figure 1 we have:

$$BR(G) = [(\{a\},b),(\{c\},d),(\{e,f\},d),(\{k\},g),(\{j\},i),(\{o,p\},m),(\{b\},d),\\ (\{m,n,\ell\},h),(\{d,h,g,i\},\oslash)]$$

4 ONE-VERTEX EXTENSIONS

The concept of *one-vertex extension* was introduced by Bandelt and Mulder [1].

Let $G = (V,E)$ be a graph, $v \in V$ and $u \notin V$. An extension of G to a graph $G' = (V',E')$ is a *one-vertex extension* if it obeys one of the following three rules:

(α) $V' = V \cup \{u\}$ and $E' = E \cup \{vu\}$, i.e., $N_{G'}(u) = \{v\}$ (u is a pendant vertex).

(β) $V' = V \cup \{u\}$ and $E' = E \cup \{xu; x \in N_G[v]\}$, i.e., $N_{G'}[u] = N_{G'}[v]$ (u is a true twin of v).

(γ) $V' = V \cup \{u\}$ and $E' = E \cup \{xu; x \in N_G(v)\}$, i.e., $N_{G'}(u) = N_{G'}(v)$ (u is a false twin of v).

The special cases of (α), (β) and (γ) restricted to a simplicial vertex $v \in V$ are denoted by (α^*), (β^*) and (γ^*), respectively.

In order to generate a graph $G = (V,E)$, it is possible to establish a building sequence. A *one-extension sequence* (*oes*) of G is a sequence of triples

$$\Pi(G) = [\pi(1),\dots,\pi(n)]$$

being $\pi(i) = (e_i, v_i, u_i)$, $i = 2,\dots,n$, such that

1. $e_i \in \{(\alpha),(\beta),(\gamma),(\alpha^*),(\beta^*),(\gamma^*)\}$;
2. $v_i = u_j$, for some $j < i$;
3. $u_i \neq u_k$, $1 \le k \le i-1$;

and $\pi(1)$ is the special initial triple $(\oslash,\oslash,u_1)$.

Bandelt and Mulder [1] presented characterizations of distance hereditary graphs and ptolemaic graphs; the first one using the extensions (α), (β) and (γ), and the second one using (α), (β) and (γ^*). Theorem 4.1, presented below, shows a characterization of block graphs using one-vertex extensions.

Consider a graph G, $CV(G)$ the set of cut-vertices, $Simp(G)$ the set of simplicial vertices and $\mathbb{Q}(G)$ the set of maximal cliques of the graph.

Theorem 4.1. *A graph $G = (V,E)$ is a block graph if and only if there is a sequence $\Pi(G)$ of G composed by type (α) and type (β^*) extensions.*

Proof. Consider a block graph G with ℓ maximal cliques and its boundary representation $BR(G) = [(P_1,s_1),\ldots,(P_\ell,\oslash)]$. It is possible to construct a sequence $\Pi(G)$ by transversing the boundary representation in reverse order.

Let $(P_\ell,\oslash)$, $v \in P_\ell$ and $\pi(1) = (\oslash,\oslash,v)$. Consider $w \in P_\ell \setminus \{v\}$ and $\pi(2) = (\alpha,v,w)$. For $x \in P_\ell \setminus \{v,w\}$, let be the triple (β^*,w,x). So, there are the following elements of the sequence $\Pi(G)$: $\pi(i) = (\beta^*,w,x)$, $i = 3,\ldots,|P_\ell|$. Thus, a first maximal clique of G is obtained.

Let $(P_{\ell-j},s_{\ell-j})$, $j = 1,\ldots,\ell-1$. By the definition of boundary representation, $s_{\ell-j} \in Q_k$, $\ell-j+1 \le k \le \ell$. Consider $v \in P_{\ell-j}$. The graph obtained by the extension $\pi(1+\sum_{k=\ell-j+1}^{\ell}|P_k|) = (\alpha,s_{\ell-j},v)$ has v as a pendant vertex and $s_{\ell-j}$ as a cut-vertex. The vertices v and $s_{\ell-j}$ belong to a new maximal clique Q. For $x \in P_{\ell-j} \setminus \{v\}$, let be the triple (β^*,v,x). Thus, there are the following elements of the sequence $\Pi(G)$: $\pi(i) = (\beta^*,v,x)$, $i = 2+\sum_{k=\ell-j+1}^{\ell}|P_k|,\ldots,\sum_{k=\ell-j}^{\ell}|P_k|$. These extensions increase the clique Q to which vertex v belongs in G. Then, we obtain the one-extension sequence of G, $\Pi(G)$, composed by type (α) and type (β^*) extensions.

Conversely, consider $\Pi(G) = [(\oslash,\oslash,v)]$. The resulting graph G is a trivial graph $(\{v\},\emptyset)$ and it is a block graph.

Consider $H = (V(H),E(H))$ a block graph with $n-1$ vertices obtained by $\Pi(H) = [\pi(1),\ldots,\pi(n-1)]$ a sequence of (α) and (β^*) extensions. Let $v \in V(H)$, Q the maximal clique to which it belongs in H and $u \notin V(H)$.

Let $\Pi(G) = \Pi(H)\|\pi(n) = [\pi(1),\ldots,\pi(n-1),\pi(n)]$.

If $\pi(n) = (\alpha,v,u)$, two cases must be analyzed.

1. v is a simplicial vertex of H. Then, $CV(G) = CV(H) \cup \{v\}$ and $Simp(G) = (Simp(H) \setminus \{v\}) \cup \{u\}$.
2. v is a cut-vertex of H. Then, $CV(G) = CV(H)$ and $Simp(G) = Simp(H) \cup \{u\}$.

In both cases, the set of maximal cliques $\mathbb{Q}(G) = \mathbb{Q}(H) \cup \{vu\}$.

If $\pi(n) = (\beta^*,v,u)$, v must be a simplicial vertex in H. So, $CV(G) = CV(H)$, $Simp(G) = Simp(H) \cup \{u\}$ and $\mathbb{Q}(G) = (\mathbb{Q}(H) \setminus Q) \cup \{Q'\}$ where Q is a maximal clique such that $v \in Q$ and $Q' = Q \cup \{u\}$.

In any case, G is a block graph. □

The proof of Theorem 4.1 provides a possible one-extension sequence of a block graph. As an example, consider the block graph G in Figure 1 and the boundary representation of the same graph presented in Section 3. The one-extension sequence obtained from $BR(G)$ is

$$\begin{aligned}\Pi(G) = [& (\oslash,\oslash,d),(\alpha,d,h),(\beta^*,h,g),(\beta^*,h,i),(\alpha,h,m),(\beta^*,m,n),\\ & (\beta^*,m,\ell),(\alpha,d,b),(\alpha,m,o),(\beta^*,o,p),(\alpha,i,j),(\alpha,g,k),\\ & (\alpha,d,e),(\beta^*,e,f),(\alpha,d,c),(\alpha,b,a)].\end{aligned}$$

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RESUMO. Grafos bloco têm sido extensivamente estudados por muitas décadas. Neste artigo apresentamos uma nova caracterização da classe em termos de extensões em um vértice. Com esse objetivo, é definida uma representação especial, baseada no conceito de cliques limítrofes, ressaltando propriedades dos grafos bloco.

Palavras-chave: grafo bloco, extensão em um vértice.

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