Tendências em Matemática Aplicada e Computacional, **20**, N. 3 (2019), 417-427

www.scielo.br/tema
doi: 10.5540/tema.2019.020.03.0417

Theoretical and Numerical Aspects of a Third-order Three-point Nonhomogeneous Boundary Value Problem

A. L. M. MARTINEZ[1*], M. R. A. FERREIRA[2] and E. V. CASTELANI[3]

Received on December 03, 2018 / Accepted on February 21, 2019

ABSTRACT. In this paper we are considering a third-order three-point equation with nonhomogeneous conditions in the boundary. Using Krasnoselskii's Theorem and Leray-Schauder Alternative we provide existence results of positive solutions for this problem. Nontrivials examples are given and a numerical method is introduced.

Keywords: numerical solutions, third-order, boundary value problem, Krasnoselskii's Theorem.

1 INTRODUCTION

Multi-point boundary value problems there has been attention of several studies mainly focused on the existence of solutions with qualitative and quantitative aspects, we recommend [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15] and the references therein. It is well known that the Krasnoselskii's fixed point theorem, Avery-Peterson and Leggett-Williams theorems are massively used in this line.

In this paper, motived by [13], we discuss the existence of a positive solution for the third-order boundary value problem:

$$u''' + f(t,u,u') = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = u'(0) = 0, \quad u'(1) - \alpha u'(\eta) = \lambda, \tag{1.2}$$

where $\eta \in (0,1), \alpha \in [0, \frac{1}{\eta})$ are constants and $\lambda \in (0,\infty)$ is a parameter. Essentially, we combine Leray-Schauder Alternative and Krasnoselskii's theorem to show the existence of a positive solution for (1.1)-(1.2) without supposing superlinearity on f. Numerical solutions are poorly

*Corresponding author: André L. M. Martinez – E-mail: andrelmmartinez@yahoo.com.br – https://orcid.org/0000-0003-1888-648X

[1]Departamento Acadêmico de Matemática, Universidade Tecnológica Federal do Paraná, Cornélio Procópio, Paraná, Brazil – E-mail: andrelmmartinez@yahoo.com.br

[2]Colegiado de Matemática, Universidade Estadual do Norte do Paraná, Cornélio Procópio, Paraná, Brazil – E-mail: marceloraferreira@gmail.com

[3]Departamento de Matemática, Universidade Estadual de Maringá, Maringá, Paraná, Brazil – E-mail: evcastelani@uem.br

explored, thus we complement this work presenting a numerical study for (1.1)-(1.2) based on Banach's Contraction Principle.

2 BACKGROUND MATERIAL

We begin this section by stating the following results.

Theorem 1. *Let E be a Banach space, $C \subset E$ a closed and convex set, Ω an open set in C and $p \in \Omega$. Then each completely continuous mapping $T : \overline{\Omega} \to C$ has at least one of the following properties:*

$(A1)$ *T has a fixed point in $\overline{\Omega}$.*

$(A2)$ *There are $u \in \partial\Omega$ and $\lambda \in (0,1)$ such that $u = \lambda T(u) + (1-\lambda)p$.*

Theorem 2. *Let E be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are bounded open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that, either*

$(B1)$ $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_1$ *, and* $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_2$*, or*

$(B2)$ $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_1$ *, and* $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_1 \setminus \Omega_1)$.

The first theorem is a well-known Leray-Schauder alternative and the second theorem is due to Krasnoselskii, see [1].

Let us set an auxiliary problem that will be useful in our context.

$$u''' + f(t,x,x') = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u'(0) = 0, \quad u'(1) - \alpha u'(\eta) = \lambda. \tag{2.2}$$

Related to this problem we have an important lemma.

Lemma 3. *Let $x \in C^1[0,1] := \{x \in C^1[0,1], t \in [0,1]\}$, then we have a unique solution for (2.1)-(2.2). Moreover, this solution is expressed by*

$$u(t) = \int_0^1 G(t,s)f(s,x(s),x'(s))ds + \frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 G_1(\eta,s)f(s,x(s),x'(s))ds + \frac{\lambda t^2}{2(1-\alpha\eta)}, \tag{2.3}$$

where G is the Green's function:

$$G(t,s) = \frac{1}{2}\begin{cases} (2t - t^2 - s)s, & s \leq t \\ (1-s)t^2, & t \leq s \end{cases} \tag{2.4}$$

and

$$G_1(t,s) = \frac{\partial G(t,s)}{\partial t} = \begin{cases} (1-t)s, & s \le t \\ (1-s)t, & t \le s \end{cases}. \tag{2.5}$$

Proof. If $u(t)$ is solution of (2.1) , we can suppose that

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 f(s,x,x')ds + At^2 + Bt + C.$$

From condition (2.2), we have $B = C = 0$ and

$$A = \frac{1}{2(1-\alpha\eta)}\int_0^1 (1-s)f(s,x,x')ds - \frac{\alpha}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)f(s,x,x')ds + \frac{\lambda}{(1-\alpha\eta)}$$

Thus (2.1)-(2.2) has a unique solution. Furthermore $u(t) = -\frac{1}{2}\int_0^t (t-s)^2 f(s,x,x')ds + \frac{t^2}{2(1-\alpha\eta)}\int_0^1 (1-s)f(s,x,x')ds$

$$\begin{aligned}
&-\frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)f(s,x,x')ds + \frac{\lambda t^2}{2(1-\alpha\eta)} \\
&= -\frac{1}{2}\int_0^t (t-s)^2 f(s,x,x')ds + \frac{t^2}{2}\int_0^1 (1-s)f(s,x,x')ds \\
&\quad + \frac{\alpha\eta t^2}{2(1-\alpha\eta)}\int_0^1 (1-s)f(s,x,x')ds - \frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)f(s,x,x')ds + \frac{\lambda t^2}{2(1-\alpha\eta)} \\
&= \frac{1}{2}\int_0^t (-t^2+2st-s^2)f(s,x,x')ds + \frac{1}{2}\int_0^t (1-s)t^2 f(s,x,x')ds \\
&\quad + \frac{1}{2}\int_t^1 (1-s)t^2 f(s,x,x')ds + \frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^\eta (1-s)\eta f(s,x,x')ds \\
&\quad + \frac{\alpha t^2}{2(1-\alpha\eta)}\int_\eta^1 (1-s)\eta f(s,x,x')ds - \frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)f(s,x,x')ds + \frac{\lambda t^2}{2(1-\alpha\eta)} \\
&= \frac{1}{2}\int_0^t (2t-t^2-s)sf(s,x,x')ds + \frac{1}{2}\int_t^1 (1-s)t^2 f(s,x,x')ds \\
&\quad + \frac{\alpha t^2}{2(1-\alpha\eta)}\left(\int_0^\eta (1-\eta)sf(s,x,x')ds + \int_\eta^1 \eta(1-s)f(s,x,x')ds\right) + \frac{\lambda t^2}{2(1-\alpha\eta)} \\
&= \int_0^1 G(t,s)f(s,x,x')ds + \frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 G_1(\eta,s)f(s,x,x')ds + \frac{\lambda t^2}{2(1-\alpha\eta)}.
\end{aligned}$$

□

Defining $x(t) = u(t)$ in Lemma 3 is easy to see that the solution of (1.1)-(1.2) can be expressed as fixed point of the operator $T : C^1[0,1] \to C^1[0,1]$ defined by:

$$Tu(t) = \int_0^1 G(t,s)f(s,u,u')ds + \frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 G_1(\eta,s)f(s,u,u')ds + \frac{\lambda t^2}{2(1-\alpha\eta)}. \tag{2.6}$$

Remark 4. *Related to G and G_1 we have useful properties that will be used in the next section.*

- *For all* $(t,s) \in [0,1] \times [0,1]$ *:*

$$0 \leq G_1(t,s) \leq (1-s)s$$

- *For all* $(t,s) \in [0,1] \times [0,1]$*:*

$$G(t,s) \leq G_1(1,s) = \frac{1}{2}(1-s)s$$

3 POSITIVE SOLUTIONS

Let $E = \{u \in C^1[0,1] : u(0) = 0\}$, where $C^1[0,1]$ be the Banach space of continuously differentiable functions in $[0,1]$ equipped with

$$\|u\|_E = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$

Remark 1. *If $u \in E$ then Tu satisfies $Tu(0) = 0$. Besides $\|(Tu)'\|_\infty \geq \|Tu\|_E$.*

In order to prove the existence we need to consider some basic assumptions.

$(H1)$ There exist positive constants A, B and β such that

- $\max\limits_{(s,v_1,v_2)\in[0,1]\times[-\beta,\beta]\times[-\beta,\beta]} \{|f(s,v_1,v_2)|\} \leq \dfrac{\beta(1-\alpha\eta)6B}{1+\alpha(1-\eta)}$

- $\lambda \leq A\beta(1-\alpha\eta)$

- $A + B \leq 1.$

Lemma 2. *Suppose that $(H1)$ holds. Thus the problem (1.1)-(1.2) has a solution $u^* \in E$ with $\|u^*\|_E \leq \beta$.*

Proof. Let us consider the Theorem 1 with $p = 0$ and $\Omega = \{u \in E : \|u\|_E < \beta\}$.

We claim that T is continuous and completely continuous. In fact, the continuity follows immediately from the Lebesgue dominated convergence theorem and noting that

$$|T(u)(t) - T(u_n)(t)| \leq \int_0^1 G(t,s)\left|f(s,u(s),u'(s)) - f(s,u_n(s),u_n'(s))\right| ds +$$

$$+ \left|\frac{\alpha t^2}{2(1-\alpha\eta)}\right| \int_0^1 G_1(\eta,s)\left|f(s,u(s),u'(s)) - f(s,u_n(s),u_n'(s))\right| ds$$

$$\leq \int_0^1 G1(1,s)\left|f(s,u(s),u'(s)) - f(s,u_n(s),u_n'(s))\right| ds +$$

$$+\left|\frac{\alpha}{2(1-\alpha\eta)}\right|\int_0^1 G_1(\eta,s)\left|f(s,u(s),u'(s))-f(s,u_n(s),u_n'(s))\right|ds,$$

with $u_n, u \in E$. To show complete continuity we will use the Arzela-Ascoli's theorem. Let $\Omega \subseteq E$ be bounded, in other words, there exists $\Lambda_0 > 0$ with $\|u\| \leq \Lambda_0$ for each $u \in \Omega$. Now if $u \in \Omega$ we have

$$\begin{aligned}
|(Tu)'(t)| &= \left|\int_0^1 G_1(t,s)f(s,u,u')ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(\eta,s)f(s,u,u') + \frac{\lambda t}{1-\alpha\eta}\right| \\
&\leq \int_0^1 |G_1(t,s)f(s,u,u')ds| + \frac{\alpha t}{1-\alpha\eta}\int_0^1 |G_1(\eta,s)f(s,u,u')| + \left|\frac{\lambda t}{1-\alpha\eta}\right| \\
&\leq \max_{t\in[0,1]} \frac{1-\alpha\eta+\alpha t}{1-\alpha\eta}\int_0^1 |(1-s)sf(s,u,u')|ds + \left|\frac{\lambda t}{1-\alpha\eta}\right| \\
&\leq \frac{1+\alpha(-\eta+1)}{1-\alpha\eta}\int_0^1 |(1-s)s||f(s,u,u')|ds + \left|\frac{\lambda}{1-\alpha\eta}\right|.
\end{aligned}$$

Then $T\Omega$ is a bounded equicontinuous family on $[0,1]$. Consequently the Arzela-Ascoli theorem implies $T: E \to E$ is completely continuous.

In addition, suppose there are $u \in \partial\Omega$ and $\lambda \in (0,1)$ with $u(x) = \lambda Tu(x)$. According $(H1)$ we have:

$$\begin{aligned}
&\|Tu\|_E < \|(Tu)'\|_\infty = \max_{t\in[0,1]} |(Tu)'(t)|, \\
&\leq \max_{t\in[0,1]} \frac{1+\alpha(-\eta+1)}{1-\alpha\eta}\int_0^1 |(1-s)s||f(s,u,u')|ds + \left|\frac{\lambda}{1-\alpha\eta}\right| \\
&\leq \max_{(s,v_1,v_2)\in[0,1]\times[-\beta,\beta]\times[-\beta,\beta]} \frac{1+\alpha(-\eta+1)}{1-\alpha\eta}|f(s,v_1,v_2)|\int_0^1 (1-s)s\,ds + \left|\frac{\lambda}{1-\alpha\eta}\right| \\
&\leq \max_{(s,v_1,v_2)\in[0,1]\times[-\beta,\beta]\times[-\beta,\beta]} \frac{1+\alpha(-\eta+1)}{1-\alpha\eta}\frac{|f(s,v_1,v_2)|}{6} + \left|\frac{\lambda}{1-\alpha\eta}\right| \\
&\leq \frac{1}{1-\alpha\eta}\left[\frac{1+\alpha(1-\eta)}{6}\max|f(s,v_1,v_2)| + \lambda\right] \\
&\leq \frac{1}{1-\alpha\eta}\left[\frac{1+\alpha(1-\eta)}{6}\frac{\beta(1-\alpha\eta)6B}{1+\alpha(1-\eta)} + \lambda\right] \\
&\leq \frac{1}{1-\alpha\eta}\left[\beta(1-\alpha\eta)B + A\beta(1-\alpha\eta)\right] \\
&\leq \beta A + \beta B \leq \beta.
\end{aligned}$$

Therefore, $\|u\|_E < \beta$ and $(A2)$ in Theorem 1 cannot occur. Thus $(A1)$ holds and there is $u^* \in E$ such that $\|u^*\|_E \leq \beta$. ☐

Theorem 3. *Suppose that* $(H1)$ *holds and* $f(s,u,v) \geq 0$, $\forall (s,u,v) \in [0,1]\times[-\beta,\beta]\times[-\beta,\beta]$. *Then (1.1)-(1.2) has at least one positive solution* $u^* \in E$.

Proof. We start the proof defining the cone $K \subset E$ by

$$K = \{u \in E: \ u \geq 0, \ u(0) = 0, \ u'(0) = 0\}.$$

From $(H1)$ and the definition of G and G_1, we have that T applies K in K. As seen in the last result, T is completely continuous.

We shall apply Theorem 2. Thus, we will define $\Omega_1 = \{u \in E; \|u\|_E < \beta\}$, $\Omega_2 = \{u \in E; \|u\|_E < \alpha\}$ and we will show that the following conditions are true for all $u \in K$:

(a) if $\|u\|_E = \alpha$ then $\|Tu\|_E \leq \alpha$;

(b) if $\|u\|_E = \beta$ then $\|Tu\|_E \geq \beta$.

In fact, the demonstration of (a) is similar to the proof of the Lemma 2. To prove (b) is necessary to verify that there is $\overline{\gamma} > 0$ with

$$\|Tu\|_E \geq \|u\|_E, \ \ \forall u \in K \cap \partial\Omega_3,$$

where $\Omega_3 = \{u \in E; \|u\|_E < \overline{\gamma}\}$.

Let us assume that the inequality is false, that is, for every $\overline{\gamma}$ such that $\beta > \overline{\gamma} > 0$ there exists $u \in E$ with $\|u\|_E = \overline{\gamma}$ and $\|Tu\|_E < \overline{\gamma}$. Thus for all $n \in \{1, 2, \cdots\}$ with $\frac{1}{n} < \alpha$, we can find $u_n \in K$ such that

$$\|u_n\|_E = \frac{1}{n} \text{ and } \|Tu_n\|_E < \frac{1}{n}.$$

Then $\|u_n\| \longrightarrow 0$ and $\|Tu_n\| \longrightarrow 0$, when $n \longrightarrow \infty$. Being T continuous, we have $\|T0\|_E = 0$. On the other hand, using $(H1)$ and the definition of G and G_1 we have

$$\begin{aligned} \|T0\|_\infty &\geq \max_{t\in[0,1]} \left\{ \frac{\lambda t^2}{2(1-\alpha n)} \right\}, \\ &\geq \frac{\lambda}{2(1-\alpha n)} > 0 \end{aligned}$$

which is a contradiction. Therefore we have the result. □

Remark 4. *Note that the most important step in the proof of Theorem 3 is to impose conditions to conclude that* 0 *is not fixed point of* T.

Example 3.1. *Let us consider (1.1)-(1.2) with*

$$\begin{aligned} f(t,u,v) &= \frac{1}{4}t + u^2 + v^2 \\ \eta &= \frac{1}{10}, \ \alpha = \frac{1}{3}, \ \lambda = \frac{1}{4} \end{aligned}$$

Choosing the constants

$$\beta = 10, \ \ A = 0.54, \ \ B = 0.45,$$

we can easily verify that in these conditions the hypotheses $(H1)$ *are satisfied.*

Example 3.2. *Let us define*

$$\begin{aligned} f(t,u,v) &= \frac{1}{4}t + sin(u) + \frac{1}{4}cos(v) \\ \eta &= \frac{1}{9},\ \alpha = \frac{1}{6},\ \lambda\frac{14}{10} \end{aligned}$$

As before, choosing the constants

$$\beta = 2,\ \ A = 0.75,\ \ B = 0.2,$$

we can verify that $(H1)$ is satisfied.

4 NUMERICAL SOLUTIONS

In this section we show the existence and uniqueness for (1.1)-(1.2) using Banach Fixed Point Theorem. This approach is classical but very important to define numerical methods for our problem. Let us consider the iterative sequence

$$u^{k+1} = T(u^k)$$

and the basic assumptions
$(H2)$ $|f(s,u,u') - f(s,v,v')| \leq A\max\left\{|u(s)-v(s)|,|u'(s)-v'(s)|\right\}$;
$(H3)$ $\dfrac{-t^2+t}{2} + \dfrac{\alpha t\eta(-\eta+1)}{2(1-\alpha\eta)} \leq \dfrac{1}{A}$.

Theorem 1. *Suppose that $(H1)$, $(H2)$ and $(H3)$ are satisfied. Then (1.1)- (1.2) has a unique solution u with $\|u\|_E \leq \beta$. Moreover, $u^{k+1} = T(u^k) \to u$.*

Proof. Let us consider $u, v \in \Omega$ with $\|u\|_E \leq \beta$ and $\|v\|_E \leq \beta$. Then
$\|Tu - Tv\|_E = \|(Tu-Tv)'\|_\infty$

$$\begin{aligned} &= \left| \int_0^1 G_1(t,s)[f(s,u,u') - f(s,v,v')]ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(t,s)[f(s,u,u') - f(s,v,v')]ds \right| \\ &\leq A\max_s\left\{|u(s)-v(s)|,|u'(s)-v'(s)|\right\}\left(\int_0^1 G_1(t,s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G_1(\eta,s)ds\right) \\ &\leq A\max_s\left\{|u(s)-v(s)|,|u'(s)-v'(s)|\right\}\left(\frac{-t^2+t}{2} + \frac{\alpha t\eta(-\eta+1)}{2(1-\alpha\eta)}\right) \end{aligned}$$

Using $(H3)$ we obtain

$$\begin{aligned} &\leq A\max_s\left\{|u(s)-v(s)|,|u'(s)-v'(s)|\right\}\frac{1}{A} \\ &\leq \max_s\left\{|u(s)-v(s)|,|u'(s)-v'(s)|\right\} = \|u-v\|_E \end{aligned}$$ □

Motivated by the last result we can define Algorithm 1.

In sequence we are presenting some examples in order to establish the effectiveness of Algorithm 1. In tables, ε_u^k denotes $\|u^* - u^k\|_\infty$ where u^* is the exact solution, ε^k denotes $\|u^{k+1} - u^k\|_\infty$ and $\overline{\varepsilon}^k = \frac{\|u^{k+1}-u^k\|_\infty}{\|u^{k+1}\|_\infty}$. Still, "It" denotes "iteration".

Algorithm 1 Fixed-Point

1: Define an uniformly distributed mesh $\{x_j\}$ in $[0,1]$
2: Define an initial approximation $u_j^0 = u^0(x_j)$
3: **for** $k = 0,1,2,...,$ **do**
4: Compute $u_j'^k$ using finite diferences
5: Compute u_j^{k+1} using

$$u^{k+1} = T(u^k) \text{ and Trapezoidal Rule}$$

6: Test the convergence
7: **end for**

Example 4.1. *In this example, we consider*

$$\begin{aligned} f(x,u,u') &= -u' \\ \eta &= \frac{\pi}{4},\ \alpha = \frac{1}{10},\ \lambda = 0.770760306689242 \end{aligned}$$

The the analytical solution is $u^*(x) = 1 - \cos(x)$. The Table 1 contains results of application in Example 4.1.

We can make additional tests. From Theorem 3 we have a solution for Examples 3.1 and 3.2 but in both case, we do not know which they are. Let us apply Algorithm 1 in these problems. For this purpose, we can consider the condition

$$\frac{\|u^{k+1} - u^k\|}{\|u^{k+1}\|_\infty} < 10^{-4}$$

as stopping criterion for the algorithm. The results for these examples are presented in Table 2 and 3, respectively. The illustrations of these results are given in Figure 1 and 2.

Table 1: Algorithm 1 considering Example 4.1.

It	ε_u^k	ε^k	$\overline{\varepsilon}^k$
1	0.104585227251908	0.355112466879952	1.000000000000000
2	0.072538564385106	0.032046662866802	0.082773878760819
3	0.069264937033799	0.003273627351307	0.008384612437847
4	0.068925441261629	0.000339495772170	0.000868781674432
5	0.068890166416009	0.000035274845620	0.000090261428243

Table 2: Algorithm 1 considering Example 3.1.

It	ε_u^k	ε^k	$\overline{\varepsilon}^k$
1	-	0.168278823890335	1.000000000000000
2	-	0.007563461402919	0.043012756518181
3	-	0.000744660869474	0.004216964422657
4	-	0.000077049209989	0.000436134198292

Table 3: Algorithm 1 considering Example 3.2.

It	ε_u^k	ε^k	$\overline{\varepsilon}^k$
1	-	0.740971458506793	1.000000000000000
2	-	0.010141509530254	0.013876702158219
3	-	0.000276799190473	0.000378602975785
4	-	0.000007307338952	0.000009995000056

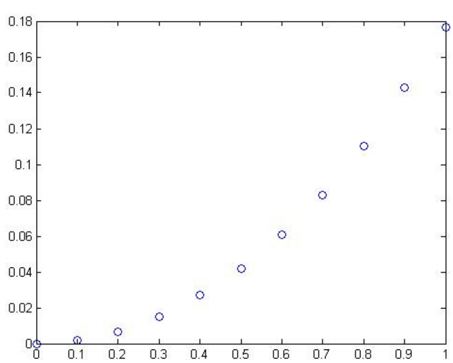

Figure 1: Numerical solution obtained from Example 1 using Algorithm 1.

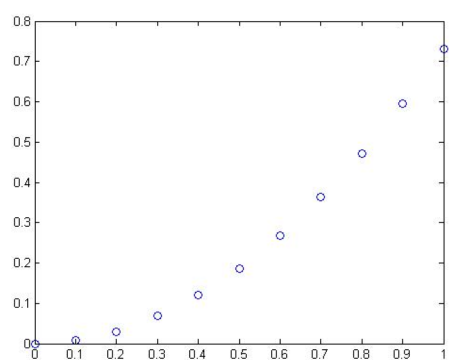

Figure 2: Numerical solution obtained from Example 2 using Algorithm 1.

RESUMO. Neste artigo, consideramos uma equação com três pontos de fronteira de terceira ordem com condições de contorno não homogêneas. Com uso do Teorema de Krasnoselskii e da Alternativa de Leray-Schauder, apresentamos resultados de existência para soluções positivas. Exemplos não triviais são fornecidos e um método numérico é introduzido.

Palavras-chave: soluções numéricas, terceira-ordem, problema de valor de contorno, Teorema de Krasnoselskii.

REFERENCES

[1] R.P. Agarwal, M. Meehan & D. O'Regan. "Fixed point theory and applications", volume 141. Cambridge university press (2001).

[2] D. Anderson. Multiple Positive Solutions for a Three-Point Boundary Value Problem. *Mathematical and Computer Modelling*, **27**(6) (1998), 49–57.

[3] D.R. Anderson. Green's function for a third-order generalized right focal problem. *Journal of Mathematical Analysis and Applications*, **288**(1) (2003), 1–14. doi:10.1016/S0022-247X(03)00132-X.

[4] D.R. Anderson & J.M. Davis. Multiple Solutions and Eigenvalues for Third-Order Right Focal Boundary Value Problems. *Journal of Mathematical Analysis and Applications*, **267**(1) (2002), 135–157. doi:10.1006/jmaa.2001.7756. URL http://linkinghub.elsevier.com/retrieve/pii/S0022247X0197756X.

[5] A. Boucherif & N. Al-Malki. Nonlinear three-point third-order boundary value problems. *Applied Mathematics and Computation*, **190**(2) (2007), 1168–1177. doi:10.1016/j.amc.2007.02.039. URL http://linkinghub.elsevier.com/retrieve/pii/S0096300307001476.

[6] H. Chen. Positive solutions for the nonhomogeneous three-point boundary value problem of second-order differential equations. *Math. Comput. Modelling*, **45**(7-8) (2007), 844–852. doi:10.1016/j.mcm.2006.08.004. URL http://dx.doi.org/10.1016/j.mcm.2006.08.004.

[7] Z.B. Fei & Xiangli. Existence of triple positive solutions for a third order generalized right focal problem. *Math. Inequal. Appl*, **9**(3) (2006), 2006.

[8] J. Graef & B. Yang. Multiple Positive Solutions to a Three Point Third Order Boundary Value Problem. *Discrete and Continuous Dynamical Systems*, **2005** (2005), 337–344.

[9] Y. Lin & M. Cui. A numerical solution to nonlinear multi-point boundary value problems in the reproducing kernel space. *Mathematical Methods in the Applied Sciences*, **34**(1) (2011), 44–47. doi:10.1002/mma.1327.

[10] Z. Liu & F. Li. On the existence of positive solutions of an elliptic boundary value problem. *Chinese Ann. Math. Ser. B*, **21**(4) (2000), 499–510. doi:10.1142/S0252959900000492. URL http://dx.doi.org/10.1142/S0252959900000492.

[11] R. Ma. Existence Theorems for a Second Order m-Point Boundary Value Problem. *Journal of Mathematical Analysis and Applications*, **211**(2) (1997), 545–555. doi:10.1006/jmaa.1997.5416. URL http://www.sciencedirect.com/science/article/pii/S0022247X97954160.

[12] Y.H. Ma & R.Y. Ma. Positive solutions of a singular nonlinear three-point boundary value problem. *Acta Math. Sci. Ser. A Chin. Ed.*, **23**(5) (2003), 583–588. doi:10.1016/j.jmaa.2004.10.029.

[13] Y. Sun. Positive solutions for third-order three-point nonhomogeneous boundary value problems. *Applied Mathematics Letters*, **22**(1) (2009), 45–51. doi:10.1016/j.aml.2008.02.002. URL http://linkinghub.elsevier.com/retrieve/pii/S0893965908000827.

[14] Q.L. Yao. The existence and multiplicity of positive solutions for a third-order three-point boundary value problem. *Acta Mathematicae Applicatae Sinica*, **19**(1) (2003), 117–122. doi:10.1007/s10255-003-0087-1.

[15] H. Yu, L. Haiyan & Y. Liu. Multiple positive solutions to third-order three-point singular semipositone boundary value problem. *Proceedings Mathematical Sciences*, **114**(4) (2004), 409–422.