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Stability Boundary Characterization of Nonlinear Autonomous Dynamical Systems in the Presence of Saddle-Node Equilibrium Points[1]

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Abstract. A dynamical characterization of the stability boundary for a fairly large class of nonlinear autonomous dynamical systems is developed in this paper. This characterization generalizes the existing results by allowing the existence of saddle-node equilibrium points on the stability boundary. The stability boundary of an asymptotically stable equilibrium point is shown to consist of the stable manifolds of the hyperbolic equilibrium points on the stability boundary and the stable, stable center and center manifolds of the saddle-node equilibrium points on the stability boundary.

Keywords. Stability Region, Stability Boundary, Saddle-Node Equilibrium Point.

1. Introduction

Asymptotically stable equilibrium points of many practical nonlinear dynamical systems are not globally stable. As a consequence, the determination of stability regions (region of attraction or basin of attraction) of asymptotically stable equilibrium points is a fundamental problem in nonlinear system theory [10] with great importance in several applications [19, 17, 4]. The exact stability region is of difficult determination and, over the last thirty years, a great number of methods were proposed for estimating the stability region of attractors of nonlinear dynamical systems [16].

Some recent methods, such as those developed in [8] and [4], explore a topological characterization of the stability boundary (the boundary of the stability region) to obtain good estimates of the stability region. Therefore, developing characterizations of stability boundaries of nonlinear dynamical systems is of fundamental importance for developing efficient tools for stability region estimation.

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Under some reasonable assumptions, the stability boundary of an asymptotically stable equilibrium point was characterized in terms of the stable manifolds of a set of unstable equilibria (and/or closed orbits) lying on this boundary [6]. These existing characterizations of stability boundaries were proved under the key assumption that all the equilibrium points on the stability boundary are hyperbolic. A first generalization of this characterization appeared in [2] by considering the existence of a particular type of non-hyperbolic equilibrium point, the so called type-zero saddle-node equilibrium point, on the stability boundary. In this paper, a further generalization of this characterization of the stability boundary is developed by allowing the presence of any type of saddle-node equilibrium point on the stability boundary. The characterization of the stability boundary in the presence of saddle-node equilibrium points is of fundamental importance for studying stability region bifurcations [3].

In this paper, a complete characterization of the stability boundary of nonlinear dynamical systems possessing saddle-node equilibrium points on it is presented. It is shown that the stability boundary consists of the stable manifolds of the hyperbolic equilibrium points on the stability boundary and the stable, stable center and center manifolds of the saddle-node equilibrium points on the stability boundary. Necessary and sufficient conditions for a saddle-node equilibrium point lying on the stability boundary are also developed.

2. Preliminaries

In this section, some classical concepts of the theory of dynamical systems are reviewed. In particular, an overview of the main features of the dynamic behavior of a system in the neighborhood of a specific type of non-hyperbolic equilibrium point, the saddle-node equilibrium point, is presented. More details on the content explored in this section can be found in [11, 21, 18].

Consider the nonlinear autonomous dynamical system

$$\dot{x} \quad = \quad f(x) \tag{2.1}$$

where $x \in \mathbb{R}^n$. One assumes that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field of class C^r with $r \geq 2$. The solution of (2.1) starting at x at time $t = 0$ is denoted by $\varphi(t, x)$. The map $t \to \varphi(t, x)$ defines in $\mathbb{R}^n$ a curve passing through x at $t = 0$ that is called *trajectory* or *orbit* of (2.1) through x. If M is a set of initial conditions, then $\varphi(t, M)$ denotes the set $\{\varphi(t, x), x \in M\} = \bigcup_{x \in M} \varphi(t, x)$. A set $S \in \mathbb{R}^n$ is said to be an *invariant set* of (2.1) if every trajectory of (2.1) starting in S remains in S for all t.

2.1. Hyperbolic equilibrium points

A point $x^* \in \mathbb{R}^n$ is an *equilibrium point* of (2.1) if $f(x^*) = 0$. An equilibrium point x^* is said to be *hyperbolic* if none of the eigenvalues of the Jacobian matrix $Df(x^*)$ of $f(x)$, calculated at the equilibrium point x^*, has real part equal to zero. Moreover, a hyperbolic equilibrium point x^* is of *type-k* if the Jacobian matrix $Df(x^*)$ possesses k eigenvalues with positive real part and $n - k$ eigenvalues with negative real part.

Let x^* be a hyperbolic equilibrium point of the nonlinear dynamical system (2.1). Then there exists a neighborhood U of x^* and *local stable* and *unstable manifolds* [13], $W^s_{loc}(x^*) = \{x \in U : \varphi(t,x) \to x^* \text{ as } t \to \infty\}$ and $W^u_{loc}(x^*) = \{x \in U : \varphi(t,x) \to x^* \text{ as } t \to -\infty\}$ with the following properties: (i) they have the same dimensions as those of the eigenspaces E^s and E^u of the linearized system $\dot{z} = Df(x^*)z$, therefore the sum of the dimension of $W^s_{loc}(x^*)$ and of $W^u_{loc}(x^*)$ equals the dimension of the state space; (ii) they are tangent to E^s and E^u at x^*; and (iii) they are as smooth as function f.

The *stable manifold* $W^s(x^*)$ and the *unstable manifold* $W^u(x^*)$, which are invariant sets, are obtained by letting the points in $W^s_{loc}(x^*)$ to flow backwards in time and the points in $W^u_{loc}(x^*)$ to flow forwards in time [22]:

$$W^s(x^*) = \bigcup_{t \leq 0} \varphi(t, W^s_{loc}(x^*)) \qquad W^u(x^*) = \bigcup_{t \geq 0} \varphi(t, W^u_{loc}(x^*)).$$

2.2. Saddle-Node equilibrium points

In this section, a specific type of non-hyperbolic equilibrium point, namely saddle-node equilibrium point, is studied. In particular, the dynamical behavior in a neighborhood of the equilibrium is investigated, including the asymptotic behavior of solutions in the invariant local manifolds.

Definition 2.1. *[21](**Saddle-Node Equilibrium Point**): A non-hyperbolic equilibrium point $p \in \mathbb{R}^n$ of (2.1) is called a saddle-node equilibrium point if the following conditions are satisfied:*
(i) $D_x f(p)$ has a unique simple null eigenvalue and none of the other eigenvalues have real part equal to zero.
(ii) $w(D^2_x f(p)(v,v)) \neq 0$,
with v as the right eigenvector and w the left eigenvector associated with the null eigenvalue.

Saddle-node equilibrium points can be classified in types according to the number of eigenvalues of $D_x f(p)$ with positive real part.

Definition 2.2. *(**Saddle-Node Equilibrium Type**): A saddle-node equilibrium point p of (2.1), is called a type-k saddle-node equilibrium point if $D_x f(p)$ has k eigenvalues with positive real part and $n-k-1$ with negative real part.*

If p is a saddle-node equilibrium point of (2.1), then there exist invariant local manifolds $W^s_{loc}(p)$, $W^{cs}_{loc}(p)$, $W^c_{loc}(p)$, $W^u_{loc}(p)$ and $W^{cu}_{loc}(p)$ of class C^r, tangent to E^s, $E^c \oplus E^s$, E^c, E^u and $E^c \oplus E^u$ at p, respectively [13]. These manifolds are respectively called stable, stable center, center, unstable and unstable center manifolds. The stable and unstable manifolds are unique, but the stable center, center and unstable center manifolds may not be.

If p is a saddle-node equilibrium point, then the following properties hold [21]:

(1) p is a type-0 saddle-node equilibrium point of (2.1):

(i) The $(n-1)$-dimensional local stable manifold $W^s_{loc}(p)$ of p exists, is unique, and if $q \in W^s_{loc}(p)$ then $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow +\infty$.

(ii) The unidimensional local center manifold $W^c_{loc}(p)$ of p can be splitted in two invariant submanifolds:

$$W^c_{loc}(p) = W^{c^-}_{loc}(p) \cup W^{c^+}_{loc}(p)$$

where $q \in W^{c^-}_{loc}(p)$ implies $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow +\infty$ and $q \in W^{c^+}_{loc}(p)$ implies $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow -\infty$. Moreover, $W^{c^+}_{loc}(p)$ is unique while $W^{c^-}_{loc}(p)$ is not.

(2) p is a type-k saddle-node equilibrium point of (2.1) with $1 \le k \le n-2$:

(i) The k-dimensional local unstable manifold $W^u_{loc}(p)$ of p exists, is unique, and if $q \in W^u_{loc}(p)$ then $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow -\infty$.

(ii) The $(n-k-1)$-dimensional local stable manifold $W^s_{loc}(p)$ of p exists, is unique, and if $q \in W^s_{loc}(p)$ then $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow +\infty$.

(iii) The $(n-k)$-dimensional local stable center manifold $W^{cs}_{loc}(p)$ of p can be splitted in two invariant submanifolds:

$$W^{cs}_{loc}(p) = W^{cs^-}_{loc}(p) \cup W^{cs^+}_{loc}(p)$$

where $q \in W^{cs^-}_{loc}(p)$ implies $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow +\infty$. The local stable center manifold $W^s_{loc}(p)$ is contained in $W^{cs^-}_{loc}(p)$, moreover, $W^{cs^-}_{loc}(p)$ is unique while $W^{cs^+}_{loc}(p)$ is not.

(iv) The $(k+1)$-dimensional local unstable center manifold $W^{cu}_{loc}(p)$ of p can be splitted in two invariant submanifolds:

$$W^{cu}_{loc}(p) = W^{cu^-}_{loc}(p) \cup W^{cu^+}_{loc}(p)$$

where $q \in W^{cu^+}_{loc}(p)$ implies $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow -\infty$. The local unstable center manifold $W^u_{loc}(p)$ is contained in $W^{cu^+}_{loc}(p)$, moreover, $W^{cu^+}_{loc}(p)$ is unique while $W^{cu^-}_{loc}(p)$ is not.

(3) p is a type-$(n-1)$ saddle-node equilibrium point of (2.1):

(i) The $(n-1)$-dimensional local unstable manifold $W^u_{loc}(p)$ of p exists, is unique, and if $q \in W^u_{loc}(p)$ then $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow -\infty$.

(ii) The unidimensional local center manifold $W^c_{loc}(p)$ of p can be splitted in two invariant submanifolds:

$$W^c_{loc}(p) = W^{c^-}_{loc}(p) \cup W^{c^+}_{loc}(p)$$

where $q \in W^{c^-}_{loc}(p)$ implies $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow +\infty$ and $q \in W^{c^+}_{loc}(p)$ implies $\varphi(t,q) \longrightarrow p$ as $t \longrightarrow -\infty$. Moreover, $W^{c^-}_{loc}(p)$ is unique while $W^{c^+}_{loc}(p)$ is not.

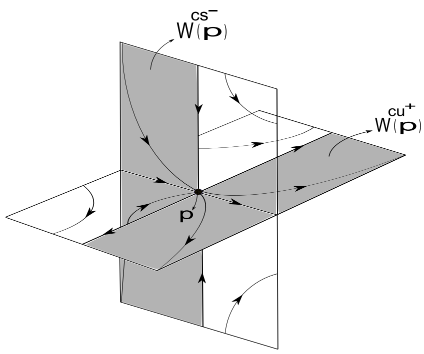

Figure 1: Manifolds $W_{loc}^{cs^-}(p)$ and $W_{loc}^{cu^+}(p)$ for a type-1 saddle-node equilibrium point p of system (2.1) in $\mathbb{R}^3$.

Although the stable and unstable manifolds of a hyperbolic equilibrium point are defined by extending the local manifolds through the flow, this technique cannot be applied to general non-hyperbolic equilibrium points. However, in the particular case of a saddle-node equilibrium point p, one still can define the global manifolds $W^s(p)$, $W^u(p)$, $W^{c^+}(p)$, $W^{c^-}(p)$, $W^{cs^-}(p)$ and $W^{cu^+}(p)$ extending the local manifolds $W_{loc}^s(p)$, $W_{loc}^u(p)$, $W_{loc}^{c^+}(p)$, $W_{loc}^{c^-}(p)$, $W_{loc}^{cs^-}(p)$ and $W_{loc}^{cu^+}(p)$ through the flow as follows:

$$W^s(p) := \bigcup_{t\leq 0} \varphi(t, W_{loc}^s(p)) \qquad W^u(p) := \bigcup_{t\geq 0} \varphi(t, W_{loc}^u(p))$$

$$W^{cs^-}(p) := \bigcup_{t\leq 0} \varphi(t, W_{loc}^{cs^-}(p)) \qquad W^{cu^+}(p) := \bigcup_{t\geq 0} \varphi(t, W_{loc}^{cu^+}(p))$$

$$W^{c^-}(p) := \bigcup_{t\leq 0} \varphi(t, W_{loc}^{c^-}(p) \quad \text{and} \quad W^{c^+}(p) := \bigcup_{t\geq 0} \varphi(t, W_{loc}^{c^+}(p)).$$

This extension is justified by the aforementioned invariance and the asymptotic behavior of the local manifolds $W_{loc}^s(p)$, $W_{loc}^u(p)$, $W_{loc}^{c^+}(p)$, $W_{loc}^{c^-}(p)$, $W_{loc}^{cs^-}(p)$ and $W_{loc}^{cu^+}(p)$, see items (1), (2) and (3) above. Figure 1 illustrates the manifolds $W_{loc}^{cs^-}(p)$ and $W_{loc}^{cu^+}(p)$ for a type-1 saddle-node equilibrium point p.

2.3. Stability Region

Suppose x^s is an asymptotically stable equilibrium point of (2.1). The *stability region* (or region of attraction) of x^s is the set

$$A(x^s) = \{x \in \mathbb{R}^n \ : \ \varphi(t, x) \to x^s \ as\ t \to \infty\},$$

of all initial conditions $x \in \mathbb{R}^n$ whose trajectories converge to x^s when t tends to infinity.

The stability region $A(x^s)$ is an open and invariant set. Its closure $\overline{A(x^s)}$ is invariant and the *stability boundary* $\partial A(x^s)$, the topological boundary of $A(x^s)$, is a closed and invariant set.

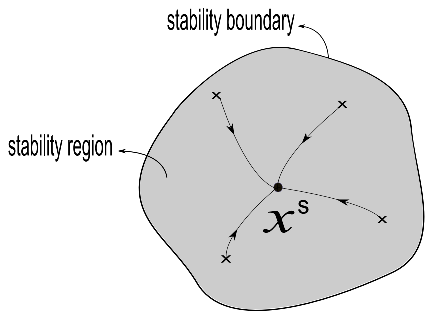

Figure 2: Stability region and stability boundary of an asymptotically stable equilibrium point x^s.

3. Hyperbolic Equilibrium Points on the Stability Boundary

In this section, an overview of the existing body of theory about the stability boundary characterization of nonlinear dynamical systems is presented. The unstable equilibrium points that lie on the stability boundary $\partial A(x^s)$ play an essential role in the stability boundary characterization.

Let x^s be a hyperbolic asymptotically stable equilibrium point of (2.1) and consider the following assumptions:
(**A1**) All the equilibrium points on $\partial A(x^s)$ are hyperbolic.
(**A2**) Every trajectory on $\partial A(x^s)$ approaches an equilibrium point as $t \to +\infty$.

Assumption (A1) is a generic property of dynamical systems in the form of (2.1). In other words, it is satisfied for almost all dynamical systems in the form of (2.1) and, in practice, does not need to be verified. On the contrary, assumption (A2) is not a generic property of dynamical systems and needs to be checked. In spite of that, many nonlinear dynamical systems satisfy this property. In particular, the existence of an energy function is a sufficient condition to guarantee the satisfaction of (A2) [6].

Under assumptions(A1)and(A2),the next theorem provides a complete characterization of the stability boundary $\partial A(x^s)$. It asserts that the stability boundary $\partial A(x^s)$ is the union of the stable manifolds of the unstable equilibrium points on $\partial A(x^s)$.

Theorem 3.1. ***(Stability boundary characterization)****[6] Let x^s be an asymptotically stable equilibrium point of (2.1) and $A(x^s)$ be its stability region. If as-*

sumptions $(A1)$ *and* $(A2)$ *are satisfied, then:*

$$\partial A(x^s) \subseteq \bigcup_i W^s(x^i)$$

where x^i, $i = 1, 2, ...$ *are the hyperbolic equilibrium points on the stability boundary* $\partial A(x^s)$. *If, in addition,* $W^u(x^i) \cap A(x^s) \neq \emptyset$, $i = 1, 2, ...$, *then*

$$\partial A(x^s) = \bigcup_i W^s(x^i).$$

Theorem 3.1 provides a complete stability boundary characterization of system (2.1) under assumptions $(A1)$ and $(A2)$. Sufficient conditions to guarantee that $W^u(x^i) \cap A(x^s) \neq \emptyset$ when a hyperbolic equilibrium point $x^i \in \partial A(x^s)$ are also provided in [6].

4. Saddle-Node Equilibrium Points on the Stability Boundary

In the presence of non hyperbolic equilibrium points on the stability boundary, Theorem 3.1 is not valid. In this section, a generalization of the results of Theorem 3.1 about stability boundary characterization is developed. We study the stability boundary characterization when assumption $(A1)$ is violated. In particular, a complete characterization of the stability boundary is developed when a particular type of non-hyperbolic equilibrium point, the so called saddle-node equilibrium point, lies on the the stability boundary $\partial A(x^s)$.

Next theorem offers necessary and sufficient conditions to guarantee that a saddle-node equilibrium point lies on the stability boundary in terms of the properties of its stable, unstable and center manifolds.

Theorem 4.1. ***(Saddle-Node Equilibrium Point on the Stability Boundary):*** *Let* p *be a saddle-node equilibrium point of (2.1). Suppose also, the existence of an asymptotically stable equilibrium point* x^s *and let* $A(x^s)$ *be its stability region. Then the following holds:*

(i) If p *is a type-*0 *saddle-node equilibrium point, then:*

$$p \in \partial A(x^s) \Leftrightarrow (W^{c^+}(p) - \{p\}) \cap \overline{A(x^s)} \neq \emptyset$$

$$p \in \partial A(x^s) \Leftrightarrow (W^s(p) - \{p\}) \cap \partial A(x^s) \neq \emptyset.$$

(ii) If p *is a type-*k *saddle-node equilibrium point,* $1 \leq k \leq n-2$, *then:*

$$p \in \partial A(x^s) \Leftrightarrow (W^{cu^+}(p) - \{p\}) \cap \overline{A(x^s)} \neq \emptyset$$

$$p \in \partial A(x^s) \Leftrightarrow (W^s(p) - \{p\}) \cap \partial A(x^s) \neq \emptyset.$$

(iii) If p is a type-$(n-1)$ saddle-node equilibrium point, then:

$$p \in \partial A(x^s) \Leftrightarrow (W^{cu^+}(p) - \{p\}) \cap \overline{A(x^s)} \neq \emptyset.$$

Proof. (i) The proof of item (i) can be found in [2].
(ii) $(\Longleftarrow)$ Suppose first that $(W^{cu+}(p) - \{p\}) \cap \overline{A(x^s)} \neq \emptyset$. Then there exists $x \in W^{cu^+}(p) \cap \overline{A(x^s)}$. Note that $\varphi(t,x) \longrightarrow p$ as $t \longrightarrow -\infty$. On the other hand, set $\overline{A(x^s)}$ is invariant, thus $\varphi(t,x) \in \overline{A(x^s)}$ for all $t \leq 0$. As a consequence, $p \in \overline{A(x^s)}$. Since $p \notin A(x^s)$, we have that $p \in (\mathbb{R}^n - A(x^s))$. Therefore, $p \in \partial A(x^s)$. Now if $(W^s(p) - \{p\}) \cap \partial A(x^s) \neq \emptyset$ then there exists at least a point $x \in (W^s(p) - \{p\}) \cap \partial A(x^s)$. Moreover, $\varphi(t,x) \to p$ as $t \to \infty$. Since $\partial A(x^s)$ is closed and invariant, then $p \in \partial A(x^s)$.
(ii) $(\Longrightarrow)$ Suppose that $p \in \partial A(x^s)$. Let D^{cu} be a neighborhood of p in $W^{cu}(p)$, whose boundary ∂D^{cu} is transversal to the vector field f on W^{cu+}, and define $D^{cu^+} := D^{cu} \cap W^{cu^+}(p)$. Consider $L_\epsilon^{cu^+} = \{x \in \mathbb{R}^n \ : \ d(x, \partial D^{cu^+}) < \epsilon\}$ for some $\epsilon > 0$. As a consequence of λ-lemma for non-hyperbolic equilibrium points [18], we can take a neighborhood U of p such that $\cup_{t\leq 0}\varphi(t, L_\epsilon^{cu^+}) \supset (U - \{W^{cs^-}(p)\})$. Since $p \in \partial A(x^s)$, we have that $U \cap A(x^s) \neq \emptyset$. On the other hand, as $W^{cs^-}(p) \cap A(x^s) = \emptyset$, we can affirm that $(U - \{W^{cs^-}(p)\}) \cap A(x^s) \neq \emptyset$. Thus, there exists a point $q^* \in L_\epsilon^{cu^+}$ and a time t^* such that $\varphi(t^*, q^*) \in A(x^s)$. Since $A(x^s)$ is invariant, we have that $q^* \in A(x^s)$. Since ϵ can be chosen arbitrarily small, we can find a sequence of points $\{q_i^*\}$ with $q_i^* \in A(x^s)$ for all $i = 1, 2, ...$ such that $d(q_i^*, \partial D^{cu^+}) \to 0$ as $i \longrightarrow +\infty$. By construction, this sequence is bounded, so has a convergent subsequence, that is, $q_{i_j}^* \to q$ as $i_j \to +\infty$. Thus, $d(q_{i_j}^*, \partial D^{cu^+}) \to d(q, \partial D^{cu^+})$ as $i_j \longrightarrow +\infty$ and consequently $q \in \overline{\partial D^{cu+}} \subset (W^{cu^+}(p) - \{p\})$. Therefore $(W^{cu^+}(p) - \{p\}) \cap \overline{A(x^s)} \neq \emptyset$. The proof that $p \in \partial A(x^s) \Leftrightarrow (W^s(p) - \{p\}) \cap \partial A(x^s) \neq \emptyset$ and of item (iii) is similar to the above proof and will be omitted. $\square$

Let x^s be an asymptotically stable equilibrium point of (2.1) and consider the following assumption:
$(\mathbf{A1}^{'})$ All the equilibrium points on $\partial A(x^s)$ are either hyperbolic or saddle-node equilibrium points.

Under assumptions $(A1^{'})$ and $(A2)$, next theorem offers a complete characterization of the stability boundary of nonlinear autonomous dynamical systems in the presence of saddle-node equilibrium points on the stability boundary $\partial A(x^s)$.

Theorem 4.2. ***(Stability Boundary Characterization)****: Let x^s be an asymptotically stable equilibrium point of (2.1) and $A(x^s)$ be its stability region. If assumptions $(A1^{'})$ and $(A2)$ are satisfied, then*

$$\partial A(x^s) \subseteq \bigcup_i W^s(x_i) \bigcup_j W^s(p_j) \bigcup_l W^{cs^-}(z_l) \bigcup_m W^{c^-}(q_m)$$

where x^i are the hyperbolic equilibrium points, p_j the type-0 saddle-node equilibrium points, z_l the type-k saddle-node equilibrium points, with $1 \leq k \leq n-2$ and q_m

the type-$(n-1)$ saddle-node equilibrium points on $\partial A(x^s)$, $i,j,l,m = 1,2,\ldots$. If, in addition, $W^{c^+}(p_j) \cap A(x^s) \neq \emptyset$, for all $j = 1,2,\ldots$ and the unstable manifolds of all other equilibrium points on the stability boundary $\partial A(x^s)$ intersect the stability region $A(x^s)$, then:

$$\partial A(x^s) = \bigcup_i W^s(x_i) \bigcup_j W^s(p_j) \bigcup_l W^{cs^-}(z_l) \bigcup_m W^{c^-}(q_m).$$

Proof. If $q \in \partial A(x^s)$, from assumptiom $(A2)$, we can affirm that $\varphi(t,q) \longrightarrow q^*$ for some equilibrium point $q^* \in \partial A(x^s)$. From assumptiom $(A1^{'})$, we can affirm that q^* is either a hyperbolic equilibrium point or a saddle-node equilibrium point, that is, $q^* = x_i$ or $q^* = p_j$ or $q^* = z_l$ or $q^* = q_m$ for some i,j,l,m. Since, the intersection $W^{c^-}(p_j) \cap \partial A(x^s)$ is empty [2], we can affirm that $q \in \bigcup_i W^s(x_i) \bigcup_j W^s(p_j) \bigcup_l W^{cs^-}(z_l) \bigcup_m W^{c^-}(q_m)$. Therefore, $\partial A(x^s) \subseteq \bigcup_i W^s(x_i) \bigcup_j W^s(p_j) \bigcup_l W^{cs^-}(z_l) \bigcup_m W^{c^-}(q_m)$. In order to prove the other inclusion, we explore the facts that $W^u(x_i) \cap A(x^s) \neq \emptyset$ for all $i = 1,2,\ldots$, $W^u(z_l) \cap A(x^s) \neq \emptyset$ for all $l = 1,2,\ldots$, $W^u(q_m) \cap A(x^s) \neq \emptyset$ for all $m = 1,2,\ldots$ and $W^{c+}(p_j) \cap A(x_s) \neq \emptyset$ for all $j = 1,2,\ldots$. If z_l is a type k saddle-node equilibrium point, with $1 \leq k \leq n-2$, on $\partial A(x^s)$, then, by assumption, $W^u(z_l) \cap A(x^s) \neq \emptyset$, then there is $w \in W^u(z_l) \cap A(x^s)$. Let $B(w,\epsilon)$ be an open ball with an arbitrarily small radius ϵ centered at w. Radius ϵ can be chosen arbitrarily small such that $B(w,\epsilon) \subset A(x^s)$. Let q_1 be an arbitrary point of $W^{cs^-}(z_l)$ and consider a disk D of dimension k that is transversal to $W^{cs^-}(z_l)$ at q_1. As a consequence of λ-lemma for non-hyperbolic equilibrium points [18], there exists an element $z \in D$ and a time $t^* > 0$ such that $\varphi(t^*, z) \in B(w,\epsilon)$. Since $A(x^s)$ is invariant, we have that $z \in A(x^s)$. Since ϵ and the disk D can be chosen arbitrarily small, then there exist points of $A(x^s)$ arbitrarily close to q_1. Therefore $q_1 \in \overline{A(x^s)}$. Since $W^{cs^-}(z_l)$ cannot contain points on $A(x^s)$, $q_1 \in \partial A(x^s)$. Exploring the fact that q_1 was arbitrarily taken in $W^{cs^-}(z_l)$, we can affirm that $W^{cs^-}(z_l) \subset \partial A(x^s)$. Similarly, it can be shown that $W^s(x_i) \subset \partial A(x^s)$, $W^s(p_j) \subset \partial A(x^s)$ and $W^{c^-}(q_m) \subset \partial A(x^s)$, consequently the theorem is proven. □

Theorem 4.2 is more general than Theorem 3.1, since assumption $(A1)$, used in the proof of Theorem 3.1, is relaxed. It also generalizes the results in [2] where only type-zero saddle-node equilibrium points were considered.

5. Example

The system of differential equations (5.1) was derived from problems of stability in power systems analysis [9]:

$$\begin{aligned} \dot{x} &= 1 - 2,84sen(x) - 2sen(x-y) \\ \dot{y} &= 1 - 3sen(y) - 2sen(y-x) \end{aligned} \tag{5.1}$$

The stability region and stability boundary of this system will be illustrated and the results of Theorem 4.2 will be verified. System (5.1) possesses an asymptoticaly stable equilibrium point $x^s = (0.35; 0.34)$ and two type-1 saddle-node equilibrium points on the stability boundary $\partial A(x^s)$, they are; $q_1 = (1,42; 3,39)$ and $q_2 = (2,12; -3,87)$. Moreover, eight unstable hyperbolic equilibrium points also belong to the stability boundary $\partial A(x^s)$. The stability boundary $\partial A(0,35; 0,34)$ is formed, according to Theorem 4.2, as the union of the manifolds $W^{c^-}(q_1)$, $W^{c^-}(q_2)$ and the stable manifolds of the unstable hyperbolic equilibrium points that belong to the stability boundary, see Figure 3.

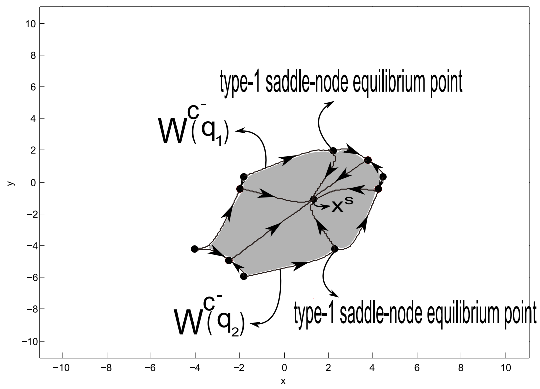

Figure 3: The gray area is the stability region of the asymptoticaly stable equilibrium point x^s. The stability boundary $\partial A(0,35; 0,34)$ is formed by the stable component of the center manifolds of the saddle-node equilibrium points q_1 and q_2 union with the stable manifolds of all the unstable hyperbolic equilibrium points that belong to the stability boundary.

6. Conclusions

This paper developed the theory of stability region of nonlinear dynamical systems by generalizing the existing results on the characterization of the stability boundary of asymptotically stable equilibrium points. The generalization developed in this paper considers the existence of a particular type of non-hyperbolic equilibrium point on the stability boundary, the so called saddle-node equilibrium point. Necessary and sufficient conditions for a saddle-node equilibrium point lying on the stability boundary were presented. A complete characterization of the stability boundary when the system possesses saddle-node equilibrium points on the stability boundary was developed for a large class of nonlinear dynamical systems. This characterization is an important step to study the behavior of the stability boundary and stability region under parameter variation.

Resumo. Uma caracterização dinâmica da fronteira da região de estabilidade para uma ampla classe de sistemas dinâmicos autônomos é desenvolvida neste artigo. Essa caracterização generaliza os resultados existentes na medida em que permite a existência de pontos de equilíbrio sela-nó na fronteira da região de estabilidade. Mostra-se que a fronteira da região de estabilidade de um ponto de equilíbrio assintoticamente estável consiste das variedades estáveis dos pontos de equilíbrio hiperbólicos na fronteira da região de estabilidade e das variedades estáveis, centro-estáveis e centrais dos pontos de equilíbrio sela-nó na fronteira da região de estabilidade.

Palavras-chave. Região de estabilidade, fronteira da região de estabilidade, ponto de equilíbrio sela-nó.

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