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Using the Interval Metric for Modeling Entities Geometrics in $\mathbb{R}^2$ – Case Study Interval Circumference

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ABSTRACT. The study of some distances provide science a way to separate two entities. It has applications in various fields such as remote sensing, data mining, pattern recognition and multivariate data analysis and others. If the distance is a Hausdorff metric, the guarantee is that all individuals are available. With the use of the distance of Trindade et al, we intend to extend the real topology to an interval topology, since the interval distance preserves the uncertainties and exits noise in the data. The present work proposes an interval circumference using an interval distance of a point to the center (pixel), like a set of pixels obeying certain distances to the center. With the interval circumference we intend to extend the notion of open ball and the concepts of neighborhood for the construction of the interval topology. A circumference separates a space into three regions, inner region, border region and outer region, where we construct our notion of neighborhood. In this work we will explore only the geometric properties of the interval circumference, we will extrapolate the notion from point to pixel by providing a differentiated frontier region for the clustering area.

Keywords: interval, interval distance, interval circumference, pixels.

1 INTRODUTION

The interval mathematics began to be diffused with the work of Ramon E. Moore [13] in the 1960's. Currently, it is a branch in mathematics with an interest in solving expressions that can

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be executed by computers. Therefore, it must be crucial that this language responds to questions of precision, efficiency, and consider inherent machine limitations. The interval mathematics is appropriate in contexts that involve uncertainties [19, 20].

Despite the success of interval mathematics in the area of scientific computing, the interval analysis did not obtain the same success of the variable theory with accurate real analysis. Perhaps it has not been successful as the basis of interval computing due to the insistence, for example, on a metric that was not essentially interval. The metric proposed by Moore was initially based on a distance between intervals represented by a real number. In the mid 2010, Trindade et al. [15], proposed an essentially interval metric. In this way, a distance was proposed that extends the concept of metric of real numbers to interval metric, where the distance between the elements of a set must be an interval metric. Distance study occupies a large amount of work in several areas of science such as: In [3] Bruce and Veloso, a metric is used for mobile robot trajectory planning. In 2017, Gomes et al. in [7], metric is used to represent sensor noises in a robot trajectory planning algorithm. Gligorič et al., 2018 [6] use interval arithmetic to model uncertainties in rock analysis. Amato et al., 1998 [1] also uses distance and probabilistic route methods. Kuffner in [10], rigid body path planning algorithms. In 2017, Hafezalkotob and Hafezalkotob [8] used interval distance in biomaterial selection methods and [9] uses interval distance to model decision-making processes. An interval platform for interval data clusters and a hybrid data solution that contains these data types by Silva in [4]. Waterm et al, 1976 [11], treats biological metrics, Robila [16] uses a spectral distance for spectral image processing and Menger proposed a statistical metric in 1945 [12]. Lopes [5] used interval math in the implementation of a light in robotics problem. Trindade [14] an essentially interval metric was proposed, opening up possibilities for extending several concepts from real mathematics to interval mathematics.

The interval metric developed in 2010, by Trindade et al. [15] was used in 2011 by Santana et al. [17, 18] for the development of signals and interval systems, thus opening a range of applications in the field of biomedicine. Then, in the year 2012 Takahashi et al. in [19, 20] used it as the basis for the development of an interval vector support machine and presented some applications in the computational area. In the year 2014, the same metric was used by Bhunia & Samanta [2] as theoretical foundation to define an interval metric and its application in multi-objective optimization with different objects.

With the notion of interval distance in hands, other mathematical concepts can be extended to the interval environment. For example, concepts that involve distances between real numbers can be extended to two-dimensional objects, the topological concepts of neighborhood, open ball, interior, boundary and exterior can also be extended to an interval topology. Based on this, this work aims to present an extension of the notion of circumference, having as notions: The cartesian plane with intervals, pixels and mainly distance between intervals, but keeping the intuitions of the notion of circumference as distance from each point (border) to the center equal to the radius, given the center and radius. The article is organized as follows. Section 2 presents the provisional definitions. In Section 3 we construct the intervalar circumference based on the intervalar metric developed by Trindade et al. [15] and finally Section 4 expresses the final considerations and future works.

2 PRELIMINARY NOTIONS

In this section we present the definitions necessary for a better understanding of an interval circumference: order, distance and interval metric. We will omit the basic definitions of interval arithmetic because we believe that this text is directed to the studies of intervals mathematic whose foundations can be found in [13]. For more details and demonstrations of the propositions, see Trindade et al. [15].

Definition 1. *(Kulisch-Miranker Order) Let X and $Y \in \mathbb{IR}$. X is least or equal to Y, denoted by $X \leq Y$, if $\underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$. If $X \leq Y$ and $X \cap Y = \emptyset$. Then we say that $X \prec Y$, which is equivalent to say that $Y \succ X$. An interval, X, is said to be positive, if $\underline{X} > 0$ and is negative if $\overline{X} < 0$*

With the Kulish-Miranker order, we will define the inverval metric.

Definition 2 (Interval Metric). *Let M be any set. A function $d : M \times M \to I$, is called a **interval metric** if it satisfies the following properties:*

1. *reflexivity:* $0 \in d(X,X)$;,
2. *triangular inequality:* $|d(X,Y)|_M \leq |d(X,Z)|_M + |d(Z,Y)|_M$;
3. *symmetry:* $d(X,Y) = d(Y,X)$;
4. *indiscernible identity : if* $0 \in d(X,Y) = d(X,X) = d(Y,Y)$ *then* $X = Y$.

Definition 3. *[An interval distance] Let X and $Y \in I$. An **interval distance** between X and Y, denoted by $m_{ei}(X,Y)$, is defined by*

$$m_{ei}(X,Y) = [\inf\{d_e(x,y) : x \in X \text{ and } y \in Y\}; \sup\{d_e(x,y) : x \in X \text{ and } y \in Y\}].$$

Proposition 1. *Let X and Y be two intervals, where $X \leq Y$ and $X \cap Y = \emptyset$. Then*

$$m_{ei}(X,Y) = [\underline{Y} - \overline{X}; \overline{Y} - \underline{X}].$$

Proposition 2. *Let X and Y be two intervals, where $X \leq Y$ and $X \cap Y \neq \emptyset$. Then,*

$$m_{ei}(X,Y) = [0; \overline{Y} - \underline{X}].$$

Proposition 3. *Let be two intervals X and Y, where $X \subseteq Y$, then*

$$m_{ei}(X,Y) = [0; \max\{\overline{X} - \underline{Y}, \overline{Y} - \underline{X}\}];$$

Corollary 4. *If $X \cap Y \neq \emptyset$, then $m_{ei}(X,Y) = [0; \max\{\overline{X} - \underline{Y}, \overline{Y} - \underline{X}\}]$.*

Proposition 5. *The distance m_{ei} coincides with the Euclidian distance d_e, when it is applied to degenerate intervals. So, if $X = [x;x]$ e $Y = [y;y]$, then*

$$m_{ei}(X,Y) = [d_e(x,y); d_e(x,y)].$$

Corollary 6. *A distance m_{ei}, restricted to degenerate intervals, is a metric interval.*

In a semantic field, where the intervals are used to represent uncertainties of certain systems, it is natural to expect that given two intervals X and Y, the distance between them is an uncertainty interval, which varies between $\min\{d_e(x,y) : x \in X \text{ and } y \in Y\}$ and $\max\{d_e(x,y) : x \in X \text{ and } y \in Y\}$.

Proposition 7. *A distance m_{ei} is an interval metric.*

Proposition 8. *Let X and $Y \in \mathbb{IR}$, $m_{ei}(X,Y) \leq [0; Diam(Y)]$ if only if $X \subseteq Y$, where $Diam(Y) = \overline{Y} - \underline{Y}$.*

Proposition 9. *Let X and $Y \in \mathbb{IR}$, such that $X \neq Y$, we have $m_{ei}(X,Y) \leq [0; Diam(Y) + Diam(X)]$ if only if $X \cap Y \neq \emptyset$.*

Proposition 10. *Let X and $Y \in \mathbb{IR}$, $[0; Diam(Y) + Diam(X)] \leq m_{ei}(X,Y)$ if only if $X \cap Y = \emptyset$.*

Proposition 11. *Let X and $Y \in \mathbb{IR}$, so we have $Diam(m_{ei}(X,Y)) \leq Diam(X) + Diam(Y)$.*

With this metric the notion of module can be redefined as follows.

Definition 4. *We call* ***the interval module*** *X, denoted by $|X|_I$, a distance $m_{ei}(X, [0;0])$.*

Theorem 12 (Interval module properties).

1. $|X|_I = \mathbf{0} \Leftrightarrow X = \mathbf{0}$;
2. $|X+Y|_I \leq |X|_I + |Y|_I$;
3. $|X \cdot Y|_I = |X|_I \cdot |Y|_I$.

3 INTERVAL CIRCUMFERENCE

In this section we will present an interval circumference based on the interval metric developed by Trindade et al. [15] as an extension of the classical circumference, where the main semantic consequence will be in the border region. We will also extend the notion of a point that will cease to be a dimensionless entity and will become a pixel in $\mathbb{R}^2$. So let's define some important concepts below.

Definition 5 (Interval point). *An* ***interval point*** *(or pixel), $P \in \mathbb{IR}^2$ is a pair (P_x, P_y) of intervals which are the interval cartesian coordinates of P. Therefore, an* ***interval point*** *is defined as a rectangular region of the cartesian plane, as can be seen in Figure 1*

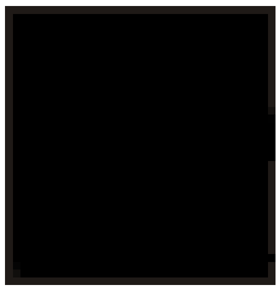

Figure 1: Representative image of an interval point or pixel.

Definition 6. *Let two pixels $P_1(P_{1x}, P_{1y})$ and $P_2(P_{2x}, P_{2y})$. The* ***interval distance between two pixels*** *$d(P_1, P_2)$ is given by:*

$$d(P_1, P_2) = \sqrt{(m_{ei}(P_{1x}, P_{2x}))^2 + (m_{ei}(P_{1y}, P_{2y}))^2},$$

where $m_{ei}(P_{1x}, P_{2x})$ and $m_{ei}(P_{1y}, P_{2y})$ are intervals distances which are Cartesian components of the distance between the pixels P_1, P_2.

Using the definition of interval distance between two pixels, we must consider that the following cases may occur, with $P_{1x} \leq P_{2x}$ e $P_{1y} \leq P_{2y}$:

Case 1 If $P_{1x} \cap P_{2x} = \varnothing$ and $P_{1y} \cap P_{2y} = \varnothing$, with $P_{1x} \leq P_{2x}$ e $P_{1y} \leq P_{2y}$, we will have:

$$\begin{cases} m_{ei}(P_{1x}, P_{2x}) = [\underline{P_{2x}} - \overline{P_{1x}}; \overline{P_{2x}} - \underline{P_{1x}}] \\ m_{ei}(P_{1y}, P_{2y}) = [\underline{P_{2y}} - \overline{P_{1y}}; \overline{P_{2y}} - \underline{P_{1y}}] \end{cases} \quad \text{by proposition 1}$$

Case 2 If $P_{1x} \cap P_{2x} = \emptyset$ and $P_{1y} \cap P_{2y} \neq \emptyset$, we will have:

$$\begin{cases} m_{ei}(P_{1x}, P_{2x}) = [\underline{P_{2x}} - \overline{P_{1x}}; \overline{P_{2x}} - \underline{P_{1x}}] \\ m_{ei}(P_{1y}, P_{2y}) = [0; \overline{P_{2y}} - \underline{P_{1y}}] \end{cases} \quad \text{by propositions 1,2}$$

Case 3 If $P_{1x} \cap P_{2x} \neq \emptyset$ and $P_{1y} \cap P_{2y} = \emptyset$, we will have:

$$\begin{cases} m_{ei}(P_{1x}, P_{2x}) = [0; \overline{P_{2x}} - \underline{P_{1x}}] \\ m_{ei}(P_{1y}, P_{2y}) = [\underline{P_{2y}} - \overline{P_{1y}}; \overline{P_{2y}} - \underline{P_{1y}}] \end{cases} \text{by propositions 1,2}$$

Case 4 If $P_{1x} \cap P_{2x} \neq \emptyset$ and $P_{1y} \cap P_{2y} \neq \emptyset$, we will have:

$$\begin{cases} m_{ei}(P_{1x}, P_{2x}) = [0; \overline{P_{2x}} - \underline{P_{1x}}] \\ m_{ei}(P_{1y}, P_{2y}) = [0; \overline{P_{2y}} - \underline{P_{1y}}] \end{cases} \text{by proposition 2}$$

Case 5 If $P_1 \subseteq P_2$, then we will have:

$$\begin{cases} m_{ei}(P_{1x}, P_{2x}) = [0; \max\{\overline{P_{1x}} - \underline{P_{2x}}, \overline{P_{2x}} - \underline{P_{1x}}\}] \\ m_{ei}(P_{1y}, P_{2y}) = [0; \max\{\overline{P_{1y}} - \underline{P_{2y}}, \overline{P_{2y}} - \underline{P_{1y}}\}] \end{cases} \text{by proposition 3}$$

Definition 7. ***The diameter of a point*** *$P \in \mathbb{IR}^2$ whose coordinates Intervals are P_x and P_y is defined as:*

$$D(P) = \sqrt{Diam^2(P_x) + Diam^2(P_y)}$$

With the definitions made, we will present some propositions that will establish a **relation between the distance between two pixels and their diameters**.

Proposition 1. *Let $P_1, P_2 \in \mathbb{IR}^2$. Then $d(P_1, P_2) \leq [0; D(P_2)]$ if and only if*

$$P_1 \subseteq P_2$$

Proof. If $d(P_1, P_2) = [\underline{d}, \overline{d}] \geq 0$ we will have, by hypothesis that $\underline{d} \geq 0$ and $\overline{d} \leq D(P_2)$.

And, therefore $d(P_1, P_2) \subseteq [0; D(P_2)]$.

$m_{ei}(P_{1x}, P_{2x}) \geq 0$ and $m_{ei}(P_{1y}, P_{2y}) \geq 0$ are components of $d(P_1, P_2)$.

As $d(P_1, P_2) \leq [0; D(P_2)]$, so it follows that:

$$m_{ei}(P_{1x}, P_{2x}) \leq [0; Diam(P_{2x})]$$

$$m_{ei}(P_{1y}, P_{2y}) \leq [0; Diam(P_{2y})].$$

By proposition 8

$$\begin{cases} m_{ei}(P_{1x}, P_{2x}) \leq [0; Diam(P_{2x})] \\ m_{ei}(P_{1y}, P_{2y}) \leq [0; Diam(P_{2y})] \end{cases} \Leftrightarrow \begin{cases} P_{1x} \subseteq P_{2x} \\ P_{1y} \subseteq P_{2y} \end{cases} \Leftrightarrow P_1 \subseteq P_2. \qquad \square$$

Proposition 2. *Let X and Y be two any intervals with $Y > X$. Suppose that:*

$$m_{ei}(X, Y) = M, \quad where \quad X \cap Y = \emptyset$$

$$m'_{ei}(X,Y) = N, \quad where \quad X \cap Y \neq \emptyset.$$

then $Diam(M) > Diam(N)$.

Proof. $X \cap Y = \varnothing \Rightarrow Diam(M) = Diam(X) + Diam(Y)$, *according to proposition 11.* $X \cap Y \neq \varnothing \Rightarrow N = m'_{ei}(X,Y) = [0, \overline{Y} - \underline{X}] = [0, Diam(X) + Diam(Y) - (\overline{X} - \underline{Y})]$. □

As $\overline{X} - \underline{Y} > 0$ by hypothesis, we see that $Diam(M) > Diam(N)$.

Proposition 3. *Let* $P_1(P_{1x}, P_{1y})$ *and* $P_2(P_{2x}, P_{2y})$ *be two any pixels, with* $P_{2x} > P_{1x}$ *and* $P_{2y} > P_{1y}$. *Then:*

$$a) \begin{cases} P_{1x} \cap P_{2x} = \emptyset \\ P_{1y} \cap P_{2y} = \emptyset \end{cases} \implies Diam^2(d(P_1, P_2)) \geq D^2(P_1) + D^2(P_2).$$

$$b) \begin{cases} P_{1x} \cap P_{2x} \neq \emptyset \\ \quad or \\ P_{1y} \cap P_{2y} = \emptyset \end{cases} \implies Diam^2(d(P_1, P_2)) < D^2(P_1) + D^2(P_2) + 2Z$$

where $Z = 2(Diam(P_{1x}) \cdot Diam(P_{2x}) + Diam(P_{1y}) \cdot Diam(P_{2y}))$

Proof. a) By proposition 11 we will have that:

$$\begin{cases} Diam(m_{ei}(P_{1x}, P_{2x})) = Diam(P_{1x}) + Diam(P_{2x}) \\ Diam(m_{ei}(P_{1y}, P_{2y})) = Diam(P_{1y}) + Diam(P_{2y}) \end{cases}$$

$$\begin{cases} Diam^2(m_{ei}(P_{1x}, P_{2x})) = (Diam(P_{1x}) + Diam(P_{2x}))^2 \\ Diam^2(m_{ei}(P_{1y}, P_{2y})) = (Diam(P_{1y}) + Diam(P_{2y}))^2 \end{cases}$$

$$Diam^2(m_{ei}(P_{1x}, P_{2x})) + Diam^2(m_{ei}(P_{1y}, P_{2y})) = Diam^2(d(P_1, P_2)) =$$

$$D^2(P_1) + D^2(P_2) + 2(Diam(P_{1x}) \cdot Diam(P_{2x}) + Diam(P_{1y}) \cdot Diam(P_{2y})).$$

Therefore,

$$Diam^2(m_{ei}(P_{1x}, P_{2x})) + Diam^2(m_{ei}(P_{1y}, P_{2y})) \geq D^2(P_1) + D^2(P_2).$$

b) By proposition 11 we can see that if there is an intersection between the components of P_1 or between the components of P_2, we will have

$$\begin{cases} Diam(m_{ei}(P_{1x}, P_{2x})) < Diam(P_{1x}) + Diam(P_{2x}) \\ Diam(m_{ei}(P_{1y}, P_{2y})) < Diam(P_{1y}) + Diam(P_{2y}) \end{cases} \text{and, therefore,}$$

$$Diam^2(d(P_1, P_2)) < D^2(P_1) + D^2(P_2) + 2Z.$$ □

3.1 Distance Between Pixels

The distances between pixels in $\mathbb{IR}^2$ vary according to the relative positions of pixels on the plane $\mathbb{IR}^2$. Therefore, we must distinguish:

Definition 8 (Parallels Pixels). *Two pixels C,P, are called* ***parallels*** *if some interval coordinate of C intersects with some interval coordinate of P. If there is no intersection of coordinates the pixels will be non-parallel.*

3.1.1 Shorter Distances

In order to define the shortest distance it is necessary to present the following proposition.

Proposition 4. *Let $C(C_x,C_y)$ and $P(X,Y)$ parallel pixels of $\mathbb{IR}^2$. Then the shortest distances between C and P will be zero or the minimum of the distances between the respective coordinates (differences between minor and major extreme of the respective coordinates).*

Proof. *Consider $C_x = [\underline{C_x},\overline{C_x}]$, $C_y = [\underline{C_y},\overline{C_y}]$, $X = [\underline{X},\overline{X}]$ and $Y = [\underline{Y},\overline{Y}]$. Then*

$$Min\{m_{ei}(C_x,X)\} = Min\{\left|[\underline{X}-\overline{C_x};\overline{X}-\underline{C_x}]\right|\} = \begin{cases} \underline{X}-\overline{C_x}\,,\ if\ X > C_x \\ \underline{C_x}-\overline{X},\ if\ X < C_x \end{cases}$$

for $C_x \cap X = \varnothing$.

$$Min\{m_{ei}(C_y,Y)\} = Min\{\left|[\underline{Y}-\overline{C_y};\overline{Y}-\underline{C_y}]\right|\} = \begin{cases} \underline{Y}-\overline{C_y},\ if\ Y > C_y \\ \underline{C_y}-\overline{Y},\ if\ Y < C_y \end{cases}$$

for $C_y \cap Y = \varnothing$.

If $C_x \cap X \neq \varnothing$, then $m_{ei}(C_x,X) = [0,\overline{X}-\overline{C_x}]$. Soon, $Min\{m_{ei}(C_x,X)\} = 0$.

If $C_y \cap Y \neq \varnothing$, then $m_{ei}(C_y,Y) = [\overline{Y}-\overline{C_y},0]$. Soon, $Min\{m_{ei}(C_y,Y)\} = 0$.

The minimum distance between pixels should be where the coordinates are minimum. Then,

$$Min\{d_{ei}(C,P)\} = \sqrt{Min^2\{m_{ei}(C_x,X)\}+Min^2\{m_{ei}(C_y,Y)\}}$$

If $C_x \cap X = \varnothing$ and $C_y \cap Y \neq \varnothing$, then

$$Min\{d_{ei}(C,P)\} = \begin{cases} \sqrt{(\underline{X}-\overline{C_x})^2+0^2} = \underline{X}-\overline{C_x}\ or \\ \sqrt{(\underline{C_x}-\overline{X})^2+0^2} = \underline{C_x}-\overline{X} \end{cases}$$

If $C_x \cap X \neq \varnothing$ and $C_y \cap Y = \varnothing$, then

$$Min\{d_{ei}(C,P)\} = \begin{cases} \sqrt{0^2+(\underline{Y}-\overline{C_y})^2} = \underline{Y}-\overline{C_y}\ or \\ \sqrt{0^2+(\underline{C_y}-\overline{Y})^2} = \underline{C_y}-\overline{Y} \end{cases}$$

If $C_x \cap X \neq \emptyset$ *and* $C_y \cap Y \neq \emptyset$, *then* $Min\{d_{ei}(C,P)\} = \sqrt{0^2+0^2} = 0.$ □

Remark 5. *In the last case above where there was intersection of both coordinates of C and P, there will also be intersection between the parallel pixels.*

For non-parallel pixels, let us consider the smallest distances between the coordinates, $Min\{m_{ei}(C_x,X)\}$, $Min\{m_{ei}(C_y,Y)\}$, to calculate

$$Min\{d_{ei}(C,P)\} = \sqrt{Min^2\{m_{ei}(C_x,X)\}+Min^2\{m_{ei}(C_y,Y)\}},$$

only for the cases where $C_x \cap X = \emptyset$ and $C_y \cap Y = \emptyset$.

$$Min\{d_{ei}(C,P)\} = \begin{cases} \sqrt{(\underline{X}-\overline{C_x})^2+(\underline{Y}-\overline{C_y})^2} \\ \sqrt{(\underline{X}-\overline{C_x})^2+(\underline{C_y}-\overline{Y})^2} \\ \sqrt{(\underline{C_x}-\overline{X})^2+(\underline{Y}-\overline{C_y})^2} \\ \sqrt{(\underline{C_x}-\overline{X})^2+(\underline{C_y}-\overline{Y})^2} \end{cases}$$

We see, in this case, that the minimum distances between pixels are distances between points of the plane $\mathbb{IR}^2$, from P and C, that is, the distances between the vertices of P and C that are 'the closest'. Besides that, in each case above, $Min\{d_{ei}(C,P)\} \in \mathbb{R}$ and each vertex are points of $\mathbb{R}^2$.

3.1.2 Longer Distances

To take the greatest distances between pixels C,P, even if they are parallel, we must take the greatest distances, the distances between the coordinates C_x and X, as well as between the coordinates C_y and Y (differences between major and minor extremes of their coordinates).

$$Max\{m_{ei}(C_x,X)\} = Max\{\left|[\underline{X}-\overline{C_x};\overline{X}-\underline{C_x}]\right|\} = \begin{cases} \overline{X}-\underline{C_x} \\ \overline{C_x}-\underline{X} \end{cases}$$

$$Max\{m_{ei}(C_y,Y)\} = Max\{\left|[\underline{Y}-\overline{C_y};\overline{Y}-\underline{C_y}]\right|\} = \begin{cases} \overline{Y}-\underline{C_y} \\ \overline{C_y}-\underline{Y} \end{cases}$$

The largest distances will be calculated as the greatest distances between vertices. So:
$Max\{d_{ei}(C,P)\} = \sqrt{Max^2\{m_{ei}(C_x,X)\}+Max^2\{m_{ei}(C_y,Y)\}}$. Then:

$$Max\{d_{ei}(C,P)\} = \begin{cases} \sqrt{(\overline{X}-\underline{C_x})^2+(\overline{Y}-\underline{C_y})^2} \text{ or} \\ \sqrt{(\overline{X}-\underline{C_x})^2+(\overline{C_y}-\underline{Y})^2} \text{ or} \\ \sqrt{(\overline{C_x}-\underline{X})^2+(\overline{Y}-\underline{C_y})^2} \text{ or} \\ \sqrt{(\overline{C_x}-\underline{X})^2+(\overline{C_y}-\underline{Y})^2} \end{cases}$$

3.2 Circumference in $\mathbb{IR}^2$

We will start the section with some definitions.

Definition 9. *Let $C_I(C_{Ix}, C_{Iy})$ be one point of $\mathbb{IR}^2$ and $R_I \geq \mathbf{0}$ any interval (radius). We call* ***interval circumference*** *the set of points $P(X,Y)$ such that $d(C_I, P) = R_I$.*

Definition 10 (Quasi-concentric Circumference). ***Quasi-concentric Circumference*** *are non-outer circumferences of space $\mathbb{R}^2$, whose centers belong to a single pixel. If their centers are a single point, they will be concentric.*

- A point $P(X,Y)$ will be a generic point of space $\mathbb{IR}^2$, where $X = [\underline{X}, \overline{X}]$ and $Y = [\underline{Y}, \overline{Y}]$.
- The pixel C_I is the center of interval circumference with $C_I = (C_{Ix}, C_{Iy})$ and $C_{Ix} = [\underline{C_{Ix}}, \overline{C_{Ix}}]$ and $C_{Iy} = [\underline{C_{Iy}}, \overline{C_{Iy}}]$.
- The interval R_I is the radius of interval circumference and $R_I = [\underline{R_I}, \overline{R_I}]$.

With defined interval distance, we can display 'interval' equations for interval circumference, considering the various cases involving the calculation of distances:

Case 1) $R_I > [0, D(C_I)]$

By the definition of interval distance, we can write, in general, that:

$$d_{ei}(C_I, P) = \sqrt{(m_{ei}(C_{Ix}, P_x))^2 + (m_{ei}(C_{Iy}, P_y))^2},$$

with $d_{ei}(C_I, P) = R_I$.

Since $R_I \geq \mathbf{0}$, we have to consider two cases:

(a) $R_I > \mathbf{0} \implies P \cap C_I = \emptyset$

We can have P belonging to the first, second, third or fourth quadrants. In each case, their distance to the center will be calculated, respectively, by the **interval equations**:

$$\text{i)} \begin{cases} X > C_{I_x} \\ Y > C_{I_y} \end{cases} \Rightarrow [\underline{R_I}, \overline{R_I}]^2 = [\underline{X} - \overline{C_{Ix}}, \overline{X} - \underline{C_{Ix}}]^2 + [\underline{Y} - \overline{C_{Iy}}, \overline{Y} - \underline{C_{Iy}}]^2$$

$$\text{ii)} \begin{cases} X < C_{I_x} \\ Y > C_{I_y} \end{cases} \Rightarrow [\underline{R_I}, \overline{R_I}]^2 = [\underline{C_{Ix}} - \overline{X}, \overline{C_{Ix}} - \underline{X}]^2 + [\underline{Y} - \overline{C_{Iy}}, \overline{Y} - \underline{C_{Iy}}]^2$$

$$\text{iii)} \begin{cases} X < C_{I_x} \\ Y < C_{I_y} \end{cases} \Rightarrow [\underline{R_I}, \overline{R_I}]^2 = [\underline{C_{Ix}} - \overline{X}, \overline{C_{Ix}} - \underline{X}]^2 + [\underline{C_{Iy}} - \overline{Y}, \overline{C_{Iy}} - \underline{Y}]^2$$

$$\text{iv)}\begin{cases} X > C_{I_x} \\ Y < C_{I_y} \end{cases} \Rightarrow [\underline{R_I},\overline{R_I}]^2 = [\underline{X}-\overline{C_{Ix}},\overline{X}-\underline{C_{Ix}}]^2 + [\underline{C_{Iy}}-\overline{Y},\overline{C_{Iy}}-\underline{Y}]^2$$

Consider also the cases where pixels can intersect with the regions A, B, C, D indicated in Figure 2. In these cases there will be intersections between the coordinates of the center and those of P. Accordingly, then, with corollary 4, we will have:

$$P \cap A \Longrightarrow [\underline{R_I},\overline{R_I}]^2 = [0, Max\{\overline{X}-\underline{C_{Ix}},\overline{C_{Ix}}-\underline{X}\}]^2 + [\underline{Y}-\overline{C_{Iy}},\overline{Y}-\underline{C_{Iy}}]^2$$

$$P \cap C \Longrightarrow [\underline{R_I},\overline{R_I}]^2 = [0, Max\{\overline{X}-\underline{C_{Ix}},\overline{C_{Ix}}-\underline{X}\}]^2 + [\underline{C_{Iy}}-\overline{Y},\overline{C_{Iy}}-\underline{Y}]^2$$

$$P \cap B \Longrightarrow [\underline{R_I},\overline{R_I}]^2 = [\underline{X}-\overline{C_{Ix}},\overline{X}-\underline{C_{Ix}}]^2 + [0, Max\{\overline{Y}-\underline{C_{Iy}},\overline{C_{Iy}}-\underline{Y}\}]^2$$

$$P \cap D \Longrightarrow [\underline{R_I},\overline{R_I}]^2 = [\underline{C_{Ix}}-\overline{X},\overline{C_{Ix}}-\underline{X}]^2 + [0, Max\{\overline{Y}-\underline{C_{Iy}},\overline{C_{Iy}}-\underline{Y}\}]^2$$

$$\textbf{(b) } R_I \geq \mathbf{0} \implies P \cap C_I \neq \emptyset \implies R_I^2 = \begin{cases} [\overline{X}-\underline{C_{Ix}},0]^2 + [0,\overline{Y}-\underline{C_{Iy}}]^2 \\ [\underline{X}-\overline{C_{Ix}},0]^2 + [0,\overline{Y}-\underline{C_{Iy}}]^2 \\ [\underline{X}-\overline{C_{Ix}},0]^2 + [0,\underline{Y}-\overline{C_{Iy}}]^2 \\ [\overline{X}-\underline{C_{Ix}},0]^2 + [0,\underline{Y}-\overline{C_{Iy}}]^2 \end{cases}$$

Remark 6. *In this case, the interval circumference is a 'circle' of pixels $P \in \mathbb{RI}^2$ limited by quasi-concentric circumferences.*

Case 2) $R_I < [0, D(C_I)]$

This case is analogous to the previous one and the circumference will also be a circular region limited by circumferences, but will intersect with the center of the circumference, as we shall see in figures 2 and 3.

3.3 Geometric Representation

An interval circumference (non-degenerate) at the plane $\mathbb{IR}^2$ will be bounded lower by:

1. Four segments of straight lines in the regions A, B, C, D according to Figure 2.

$$\underline{R_I} = \begin{cases} x - \overline{C_x} \Rightarrow x = \overline{C_x} + \underline{R_I} \text{ (B)} \\ \underline{C_x} - x \Rightarrow x = \underline{C_x} - \underline{R_I} \text{ (D)} \\ y - \overline{C_y} \Rightarrow y = \overline{C_y} + \underline{R_I} \text{ (A)} \\ \underline{C_y} - y \Rightarrow y = \underline{C_y} - \underline{R_I} \text{ (C)} \end{cases}$$

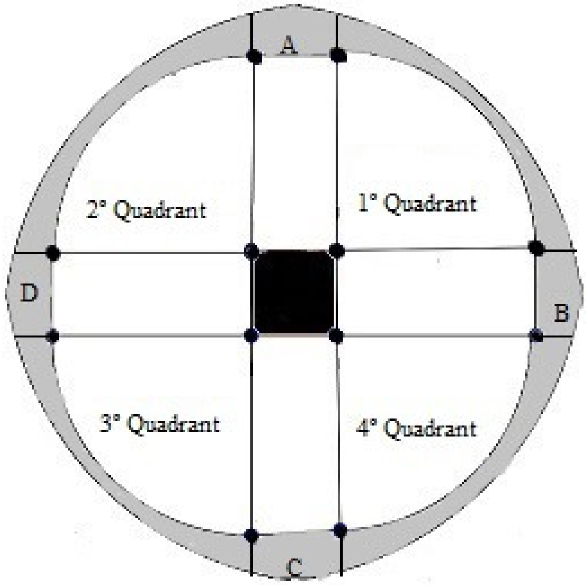

Figure 2: Representative image of interval circumference.

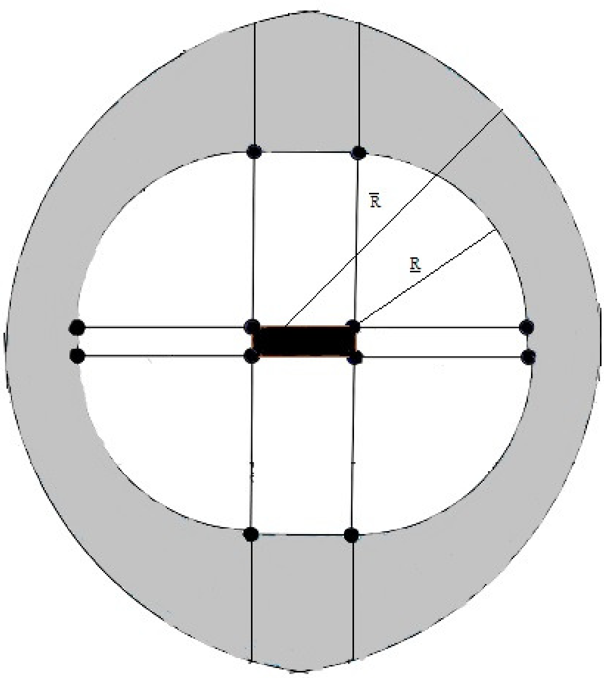

Figure 3: Representative image of interval circumference.

2. Four arcs of circumference in $\mathbb{R}^2$ with 90^o each and centered at the vertices of C_I and radius $\underline{R_I}$

The interval circunference in the plane $\mathbb{IR}^2$ possesses **equations** that depend on the angle where the vertex is centered, among other details.

(a) In the first quadrant the arch has center in $(\overline{C_x},\overline{C_y})$ and radius of the circumference

$$\underline{R_I}^2 = (x-\overline{C_x})^2 + (y-\overline{C_y})^2;$$

(b) In the second quadrant the arc has a center in $(\overline{C_x},\underline{C_y})$ and radius of the circumference

$$\underline{R_I}^2 = (x-\overline{C_x})^2 + (\underline{C_y}-y)^2.$$

(c) In the third quadrant the arc has a center in $(\underline{C_x},\overline{C_y})$ and radius of the circumference

$$\underline{R_I}^2 = (\underline{C_x}-x)^2 + (y-\overline{C_y})^2.$$

(d) In the fourth quadrant the arc has a center in $(\underline{C_x},\underline{C_y})$ and radius of the circumference

$$\underline{R_I}^2 = (\underline{C_x}-x)^2 + (\underline{C_y}-y)^2.$$

3. The largest distances to the pixels, P, are limited by four quasi-concentric arcs of circumferences $\mathbb{R}^2$, centered also on each of the vertices of C_I. Then, for generic points $P(x,y) \in \mathbb{R}^2$ that are vertices of pixels P of $\mathbb{IR}^2$ whose distances to C_I are $\overline{R}$, we will have:

$$Max\{d_{ei}(C,P)\} = \overline{R_I} = \begin{cases} \sqrt{(x-\underline{C_x})^2 + (y-\underline{C_y})^2} \text{ and} \\ \sqrt{(x-\underline{C_x})^2 + (\overline{C_y}-y)^2} \text{ and} \\ \sqrt{(\overline{C_x}-x)^2 + (y-\underline{C_y})^2} \text{ and} \\ \sqrt{(\overline{C_x}-x)^2 + (\overline{C_y}-y)^2} \end{cases} \tag{3.1}$$

Remark 7. *The four quasi-concentric circumference with centers at the vertices of C_I and described by equations (1) will limit superiorly the set of pixels contained in the center of the interval circumference C_I and radius R_I.*

Proposition 8. *There is a constant relation between $\overline{R_I}$ and $D(C)$.*

Proof. *Let $C((C_x,C_y)$ be the center and $\overline{R_I}$ the radius of a circumference centered on one of the vertices C.*

Let AB be a sector of circumference limited by $\overline{R_I}$ with center at the same vertex of C.

Let α be the angle determined by AB and, θ and ω the angles between $\overline{R_I}$ and the sides of C adjacent to the angle α according to Figure 4.

The points A and B (extrems of arc AB) are the bisectors of sides of C.

Then we can write:

$$sen\theta = \frac{Diam(C_x)}{2\overline{R_I}} \text{ and } sen\omega = \frac{Diam(C_y)}{2\overline{R_I}}.$$

$$\frac{Diam^2(C_x)}{4\overline{R_I}^2} + \frac{Diam^2(C_y)}{4\overline{R_I}^2} = sen^2\theta + sen^2\omega \Rightarrow \frac{D^2(C)}{4\overline{R_I}^2} = sen^2\theta + sen^2\omega.$$

So

$$\frac{D(C)}{\overline{R_I}} = 2\sqrt{sen^2\theta + sen^2\omega}.$$

$\square$

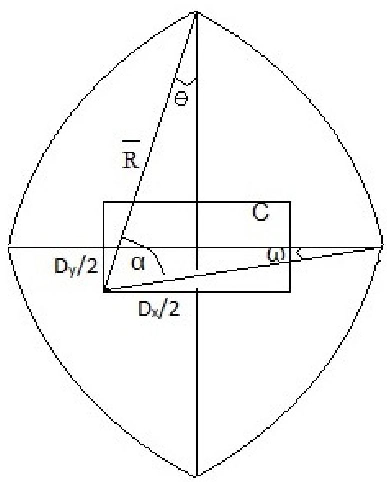

Figure 4: Representative image of the angles of the interval circumference.

Proposition 9. *The radius $\overline{R_I}$ should not be less than half of the diameter C.*

Proof. *Let* $sen\theta = \frac{Diam(C_x)}{2\overline{R_I}}$, *with* $0 \leq \theta \leq 90^o$

Therefore,

$$0 \leq \frac{Diam(C_x)}{2\overline{R_I}} \leq 1 \Rightarrow \overline{R_I} \geq \frac{Diam(C_x)}{2} \tag{3.2}$$

In the same way,

$$\overline{R_I} \geq \frac{Diam(C_y)}{2} \tag{3.3}$$

If $d = \max\{Diam(C_x), Diam(C_y)\}$*, we take* $d = Diam(C_x) = Diam(C_y)$.

then, per (1) *and* (2) *we have*

$$2\overline{R_I}^2 \geq \frac{d^2+d^2}{4} \Rightarrow \overline{R_I}^2 \geq \frac{D(C)^2}{4}$$

So,

$$\overline{R_I} \geq \frac{D(C)}{2}$$

□

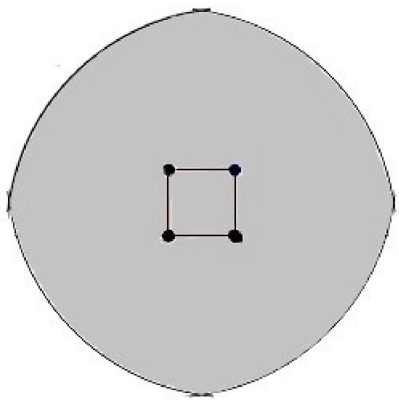

Figure 5: Representative image of degenerate interval circumference.

We present some special cases of interval circumference when $R > 0$ and $Diam(R) > D(C)$.

Case 1 $R = [\underline{R}, \overline{R}]$ represented by Figure 3.

Case 2 $R > 0, Diam(R) > D(C)$ with $R = [0, \overline{R}]$ represented by Figure 5.

Case 3 $R > 0, Diam(R) < D(C)$ with $R = [\underline{R}, \overline{R}]$ illustrated by Figure 6.

Case 4 $R \leq 0, \dfrac{D(C)}{2} < Diam(R) < D(C)$ and $R = [0, \overline{R}]$ which can be observed by Figure 7.

In the second Figure 7 the circumference can be located inside its center. It can be reduced to a point in the center of its center.

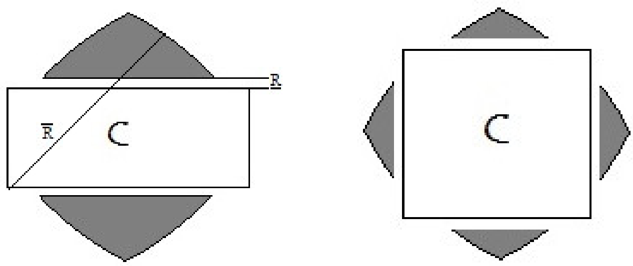

Figure 6: Representative images of circumference interval in Case 3.

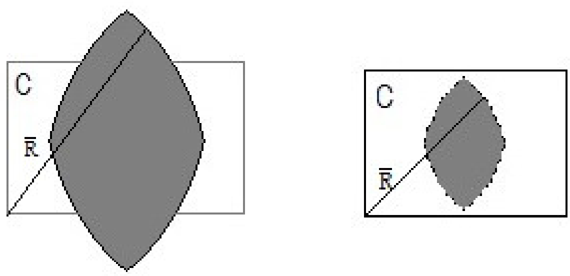

Figure 7: Representative images of circumference interval in Case 4.

4 FINAL CONSIDERATIONS

For the construction of the interval circumference we have maintained the intuition of circumference from a previously known center and radius. Using the previously defined distance and interval metric definition, we introduce the notion of pixel as center and we could define this circumference as a set of pixels, obeying certain distances to the center. A pixel can be interpreted as a point in $\mathbb{IR}^2$ and may have the semantics of representing a point in the $\mathbb{R}^2$ generated by two ranges of uncertainties. Similar interpretation can be given to interval, where it can represent the result of an experiment with fusion of sensors with noise, or a process of clusterization where the border range represents the similarity between the classes that are separated by the inner and outer region.

In this work we present the intervalar circumference and its characteristics, such as radius, center, equations, quasi-degenerate circumference, drawings and particular cases extending the notion of point and circumference. We believe that can extend all the entities of the Euclidean geometry to its interval version. As future works we can fix the size of pixels, compare images generated by different pixels, with overlays, among others. We intend to use the notion of interval circumference by extending the notion of open ball, closed ball, frontier and neighborhood to then extend the notion of topology. We also believe this work make way for new studies in various areas of science dealing with the representation of uncertainties.

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