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Complex Variational Calculus with Mean of $(\min, +)$-analysis

M. GONDRAN[1], A. KENOUFI[2*] and A. GONDRAN[3]

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ABSTRACT. One develops a new mathematical tool, the complex $(\min, +)$-analysis which permits to define a new variational calculus analogous to the classical one (Euler-Lagrange and Hamilton Jacobi equations), but which is well-suited for functions defined from $\mathbb{C}^n$ to $\mathbb{C}$. We apply this complex variational calculus to Born-Infeld theory of electromagnetism and show why it does not exhibit nonlinear effects.

Keywords: Variational Calculus, Lagrangian, Hamiltonian, Action, Euler-Lagrange and Hamilton-Jacobi equations, complex (min, +)-analysis, Maxwell's equations, Born-Infeld theory.

1 INTRODUCTION

As it was said by famous Isaac Newton, Nature likes simplicity. Physicists and mathematicians have tried to express this metaphysic statement through equations: this is the purpose of the Least Action Principle, elaborated in the middle of the 18$^{\text{th}}$ century by Pierre-Louis Moreau de Maupertuis and Leonhard Euler [5, 6, 8]. This principle leads to the so-called Euler-Lagrange equations set [18], which is the kernel of past and future laws in physics.

What is the optimal shape of a house with a fixed volume in order to get a minimal surface which optimizes the heat loss? Eskimos have known the solution of this problem since a long time: it's the hemisphere. Why the angles between three soap films common lines are two by two equal to 120°? The same for bees honeycombs', many rocks and dried soils configurations. Legend says that queen Didon, founder of the Carthage's city (around 814 B-C), was allowed to delimit the largest area she could circumscribe using strips cut in taurus' skin: she drew a circle. Those cases are examples of the economy of means, responding to a metaphysical ideal of simplicity.

At the end of the 17$^{\text{th}}$ century, Newton (1643–1727) and Leibniz (1646–1716) developed independently from each other, the elements of infinitesimal calculus, allowing in particular, function's extrema calculations. At the 18$^{\text{th}}$ century, three Swiss mathematicians, the brothers Jacques

*Corresponding author: Dr Abdelouahab Kenoufi – E-mail: kenoufi@s-core.fr
[1] European Interdisciplinary Academy of Sciences (EIAS), Paris, France. E-mail: michel.gondran@polytechnique.org
[2] Scientific COnsulting for Research & Engineering (SCORE), Strasbourg, France; European Interdisciplinary Academy of Sciences (EIAS), Paris, France.
[3] Ecole Nationale de l'Aviation Civile (ENAC), Toulouse, France. E-mail: alexandre.gondran@enac.fr

(1654–1705) and Jean Bernoulli (1667–1748), Leonhard Euler (1707–1783) who was Jacques's student, and the French mathematician Joseph-Louis Lagrange (1736–1813) used it to build a new domain of mathematics dedicated to the research of functional's extrema: the variational calculus.

Soon, practical concerns underlying the calculus of variations, created polemical philosophical and metaphysical discussions. In 1710, Leibniz developed the idea that the world had been created as the best of all thinkable world in *Essays on the Goodness of God, the Freedom of Man and the Origin of Evil*. Certainly, the creation has been flawed in so far as evil exists in the world, but this does not prevent anybody from considering that there is an almighty Creator, omniscient and infinitely good. God is to be the smartest and most powerful, it follows that our world is the best of all possible worlds: the grace of God and the kingdom of Nature are one and the same. Voltaire (1694–1778) used all the derision he mastered in *Candide*, published in 1758 to attack another great scholar of the time, Pierre-Louis Moreau de Maupertuis (1698–1759), with whom he had few complaints. Appointed President of the new Prussian Academy of Sciences by Frederick II (1712–1786) (Voltaire refused before this responsibility), to celebrate his starting in Berlin in 1746, Maupertuis had published the laws of motion and rest, deduced from a metaphysical principle, which was derivated from the least action principle. According to Maupertuis, this universal axiom is able to describe and explain all physical phenomena. Quite in the spirit of Leibniz, he postulated that the Nature had ruled the world with excellence, in an optimal way, and that God, as a "good" father and manager had ensured the profitability and efficiency of their businesses. Accordingly, he established that the Nature proceeded always with the maximal possible economy. Leibniz's pre-established harmony could only please Frederick II, known for his sense of savings. Maupertuis stated his metaphysical principle in the following way: *"If a change occurs in nature, the necessary amount of action to accomplish it, must be the smallest possible"*. Leibniz had already defined the mechanical action as $m \cdot v \cdot s$, where m, v, s, are respectively the mass, velocity and position; this quantity having the same dimensions as the product of energy $m \cdot v^2$ with time t. There remained, however, with mean of variational calculus to consolidate mathematically the principle of least action. As said Maupertuis: *"My only crime was to have discovered a principle that made some noise"*. Maupertuis did not have the competences to build a stronger mathematical theory for that, but the presence in the Berlin Academy of Leonhard Euler foreshadowed fruitful cooperation. Euler was a master in the calculus of variations and he wrote the first treaty, where he showed that the least action principle was able to describe the motion of a point mass in a central field, for example trajectory of a planet around the sun. In the early 1750s, Maupertuis was involved in a violent controversy: his accusers, among whom there was Voltaire, reproached him the validity of his argument and contested the paternity of the least action principle. We know now that Leibnitz was the first to have formulated and explained this principle in several letters. However, Euler created the corresponding mathematical structure, which served as a model for all the principles of variation subsequently incurred. From the variational principle follows a set of differential equations named Euler-Lagrange's equations. The least action principles, and those of virtual work and powers, are nowadays the most important mathematical tools to formulate elegantly and under invariant form the fundamental equations in physical and engineering sciences. Richard Feynman (1918-1988) showed with the path integral

formalism that the least action principle in classical mechanics is not a first principle, but a deductible result from quantum principles at the semi-classical limit, *ie* when the Planck's constant $h \to 0$ [10, 31]. This helped to build other field theories, such as electroweak theory (which unifies weak and electromagnetic interaction), or quantum chromodynamics (strong interactions between quarks), and to find new particles through the system symmetries. Since the 17$^{\text{th}}$ century, theoretical physics paradigm has been based on this approach and from philosophical and metaphysical point of view, it has needed a mathematical approach which has been based on variational calculus.

Classical variational calculus is devoted to functionals applied to functions defined from $\mathbb{R}^n$ to $\mathbb{R}$. One develops in this article a new mathematical tool, the complex variational calculus which is based on the definition of complex valued function extrema and the complex (min, +)-analysis introduced in [13,25]. This gives a complex analytical mechanics with complex Euler-Lagrange and Hamilton-Jacobi equations. This complex (min, +)-analysis [13, 14] can be considered as a generalization for functions defined from $\mathbb{C}^n$ to $\mathbb{C}$ of the real (min, +)-analysis previously defined and developed in several articles and books [11, 12, 23, 24, 26].

One reminds in section (2) the basis of (min, +)-analysis and introduces the (min, +) Path Integral. In section (3) the minimum of a complex valued function is defined and one explores the variational calculus for functionals of such functions, yielding thus to complex Hamilton-Jacobi equations. Those results are developed in section (4) to Lagrangian densities in order to derive a generalization of the Euler-Lagrange equations. We end with application of the complex variational calculation to Born-Infeld nonlinear theory of electromagnetism in section (5).

2 (min, +) FRAMEWORK

One presents briefly in this section (min, +)-analysis and explains why it is a powerful tool which permits to treat nonlinear problems.

2.1 Non-linear analysis with (min, +)

(min, +)-analysis takes it roots from the shortest path research in a finite graph [11,12,24]. First authors, M. Gondran *et al.* [24] have shown that the optimality equation to determine the shortest path is a linear equation with fixed-point solution in a particular algebraic structure: the dioid $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \min, +)$ which is an idempotent semi-ring different from real numbers field $(\mathbb{R}, +, \times)$ [24, 27, 35]. One has just to replace the usual addition $+$ with min operator, and the product $\times$ with $+$.

They have demonstrated that the classical resolution methods of linear algebra on the real numbers field can be re-written into this dioid $\mathbb{R}_{\min}$ yielding to computation algorithms in order to find the shortest path. In the same spirit, if one uses the dioid $\mathbb{R}_{\max,\min} = (\mathbb{R} \cup \{+\infty\}, \max, \min)$, it is possible to solve other problems such as to find maximal capacity path in a graph. Almost all usual concepts used in analysis for $\mathbb{R}$ can be transfered and studied in dioids, in particular, in $\mathbb{R}_{\min}$, such as eigenvectors and eigenvalues calculations, linear dependence, determinants computations.

Remark 1. *In the* $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \min, +)$ *dioid, the neutral elements of* min *and* + *operators are respectively* $+\infty$ *and* 0. *The* $\geq_{(\min,+)}$ *operator in the dioid corresponds then to the usual* $\leq$ *defined in the field of real numbers* $(\mathbb{R}, +, \times)$.

When one tries to find the shortest path in a continuous space, optimality equation given by the the classical variational calculus is the well-known Hamilton-Jacobi equation which expresses mathematically the Least Action Principle (LAP). The action $S(\mathbf{x}, t)$ has to verify

$$\begin{cases} \dfrac{\partial S(\mathbf{x}, t)}{\partial t} + \dfrac{1}{2m}(\nabla S(\mathbf{x}, t))^2 + V(\mathbf{x}, t) = 0, \ \forall (\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ \\ S(\mathbf{x}, 0) = S_0(\mathbf{x}) \ \forall \mathbf{x} \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

It is a nonlinear partial differential equation on the real numbers field $(\mathbb{R}, +, \times)$. Maslov *et al.* [27, 35] have shown that this equation is linear in the dioid $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \min, +)$: thus, if $S_1(\mathbf{x}, t)$ et $S_2(\mathbf{x}, t)$ are solutions of Hamilton-Jacobi equation (2.1), then $\min\{\lambda + S_1(\mathbf{x}, t), \mu + S_2(\mathbf{x}, t)\}$ for all $\lambda, \mu \in \mathbb{R}$, is a solution too of the same equation (2.1).

After this statement, Maslov and Gondran [24, 27, 35] have introduced the so-called (min, +)-analysis. This one consists to replace in the scalar product definition of two real-valued functions f and g defined on a domain X, the real number field $(\mathbb{R}, +, \times)$ with the (min, +) dioid $(\mathbb{R} \cup \{+\infty\}, \min, +)$. The classical scalar product $\langle f, g \rangle = \int_{x \in X} f(x) \cdot g(x) \cdot dx$ becomes then the (min, +) scalar product [11]

$$\langle f, g \rangle_{(\min,+)} = \inf_{x \in X} \big\{ f(x) + g(x) \big\}.$$

One reminds below the demonstration that it is a scalar product within the (min, +) dioid is straightforward [23].

- **Symmetry**

  Obviously, $\langle f, g \rangle_{(\min,+)} = \langle g, f \rangle_{(\min,+)}$.

- **Positive-definiteness**

  According to Remark (1), $+\infty$ is the neutral element of the min operator, thus, if $\langle f, f \rangle_{(\min,+)} = +\infty$, then $f(x) = +\infty$ for all $x \in X$. Furthermore, since $\geq_{(\min,+)}$ in the dioid corresponds to $\leq$ in the field of real numbers, and all functions are bounded by $+\infty$, one has $\langle f, f \rangle_{(\min,+)} \geq_{(\min,+)} +\infty$.

- **Bilinearity**

  One has to show that $\langle f, g \rangle_{(\min,+)}$ is distributive according to min, which means $\langle f, \min\{g_1, g_2)\rangle_{(\min,+)} = \min\{\langle f, g_1 \rangle_{(\min,+)}, \langle f, g_2 \rangle_{(\min,+)})$, and linear according to the addition of a scalar λ: $\langle f(x), \lambda + g(x) \rangle_{(\min,+)} = \lambda + \langle f, g \rangle_{(\min,+)}$. The linearity is obvious since $\inf_{x \in X}\{f(x) + \lambda + g(x)\} = \lambda + \inf_{x \in X}\{f(x) + g(x)\}$. Distributivity is obtained

in two steps. One has first to prove this equality with mean of two inequalities. We start first with the simple relations

$$\langle f, g_1\rangle_{(\min,+)} \leqslant f(x) + g_1(x), \text{ and } \langle f, g_2\rangle_{(\min,+)} \leqslant f(x) + g_2(x), \ \forall x.$$

This gives $\min\{\langle f, g_1\rangle_{(\min,+)}, \langle f, g_2\rangle_{(\min,+)}\} \leqslant \min\{f(x) + g_1(x), f(x) + g_2(x)\}\ \forall x$.

And since

$$\min\{f(x) + g_1(x), f(x) + g_2(x)\} = f(x) + \min\{g_1(x), g_2(x)\},$$

one has $\min\{\langle f, g_1\rangle_{(\min,+)}, \langle f, g_2\rangle_{(\min,+)}\} \leqslant f(x) + \min\{g_1(x), g_2(x)\}\ \forall x$, which yields to the inequality

$$\min\{\langle f, g_1\rangle_{(\min,+)}, \langle f, g_2\rangle_{(\min,+)}\} \leqslant \langle f, \min\{g_1, g_2\}\rangle_{(\min,+)}. \tag{2.2}$$

In a second step, one can write

$$\langle f, \min\{g_1, g_2\}\rangle_{(\min,+)} \leqslant f(x) + \min\{g_1(x), g_2(x)\} \leqslant f(x) + g_1(x)\ \forall x,$$

which becomes

$$\langle f, \min\{g_1, g_2\}\rangle_{(\min,+)} \leqslant \langle f, g_1\rangle_{(\min,+)}. \tag{2.3}$$

and in the same manner

$$\langle f, \min\{g_1, g_2\}\rangle_{(\min,+)} \leqslant f(x) + \min\{g_1(x), g_2(x)\} \leqslant f(x) + g_2(x)\ \forall x,$$

giving now

$$\langle f, \min\{g_1, g_2\}\rangle_{(\min,+)} \leqslant \langle f, g_2\rangle_{(\min,+)}, \tag{2.4}$$

and then from (2.3) and (2.4)

$$\langle f, \min\{g_1, g_2\}\rangle_{(\min,+)} \leqslant \min\{\langle f, g_1\rangle_{(\min,+)}, \langle f, g_2\rangle_{(\min,+)}\}. \tag{2.5}$$

From relations (2.2) and (2.5), one deduces finally the equality and thus the distributivity.

With this (min, +) scalar product, one obtains a distribution-like theory: the operator is linear and continuous according the dioid structure $(\mathbb{R} \cup \{+\infty\}, \min, +)$, nonlinear and continuous according to the classical structure $(\mathbb{R}, +, \times)$. The nonlinear distribution $\delta_{(\min,+)}$ defined on $\mathbb{R}^n$ as

$$\delta_{(\min,+)}(\mathbf{x}) = \{0 \text{ if } \mathbf{x} = 0, +\infty \text{ else}\}$$

is similar in (min, +) analysis to the classical Dirac distribution. Then, one has

$$\langle \delta_{(\min,+)}, f\rangle_{(\min,+)} = \min_{\mathbf{x}\in X}\{\delta_{(\min,+)}(\mathbf{x}) + f(\mathbf{x})\} = \min\{f(0), +\infty\} = f(0).$$

This permits to define a distribution theory which is continuously nonlinear in the field of real numbers, but which is linear in the dioid structure. Therefore it is interesting to study anlog results developed in Hilbert spaces functional analysis such as Riesz theorems, Fourier transforms, spectral analysis, measure theory [24, 35].

For example in (min, +)-analysis, the Legendre-Fenchel transform is similar to the Fourier transform in the real field [24]. It is defined as

$$\widehat{f}(\mathbf{r}) = \min_{\mathbf{x}\in\mathbb{R}^n}\{f(\mathbf{x}) - \mathbf{r}\cdot\mathbf{x}\}.$$

It corresponds to the Fourier transform when one uses lower semi-continuous convex functions instead of $L^2(\mathbb{R}^n)$ ones. This transform is very important in physics since it permits to pass from Lagrangian to Hamiltonian and conversely, from microcospic scales to macroscopic ones in statistical physics, and is the keystone mathematical tool for fractal and multifractal analysis [23, 26]. Main property is the possibility of passage from macroscopic scales to microscopic ones, and this can be expressed through the Stationnary Phase Approximation which uses the min operator [31]

$$\int_X \epsilon^{f(x)}\ dx \simeq \epsilon^{\inf_{x\in X}\{f(x)\}} \text{ when } \epsilon \to 0^+.$$

2.2 (min, +) path integral, Hamilton-Jacobi and Euler-Lagrange actions

One can show that (min, +)-analysis has a great importance in both classical and quantum physics. It exists indeed in classical mechanics an analog of the Feynman path integral in (min, +)-analysis, the so-called (min, +) **path integral**, which relies Hamilton-Jacobi action $S(\mathbf{x}, t)$ to classical Euler-Lagrange action $S_{cl}(\mathbf{x}, t; \mathbf{x}_0)$ through the definition given below.

Definition 1.

$$S(\mathbf{x}, t) = \min_{\mathbf{x}_0}\{S_0(\mathbf{x}_0) + S_{cl}(\mathbf{x}, t; \mathbf{x}_0)\} \tag{2.6}$$

where the minimum is computed on all initial positions $\mathbf{x}_0$ *and* $S_0(\mathbf{x})$ *is Hamilton-Jacobi action at initial time.*

This action S is thus an integral in (min, +)-analysis and permits to get a better comprehension of the Least Action Principle.

Let's remind that Hamilton-Jacobi and Euler-Lagrange actions are solutions of problems with different boundary conditions

- the Euler-Lagrange action (or classical action) $S_{cl}(\mathbf{x}, t; \mathbf{x}_0)$, linking the initial position $\mathbf{x}_0$ and its position $\mathbf{x}$ at time t,
- the Hamilton-Jacobi action $S(\mathbf{x}, t)$, which relates a family of particles of initial action $S_0(\mathbf{x})$ to their various positions $\mathbf{x}$ at time t.

Remark 2. *While the Euler-Lagrange case entails an unknown initial velocity, the Hamilton-Jacobi case implies an unknown initial position.*

If $L(\mathbf{x}, \dot{\mathbf{x}}, t)$ is the Lagrangian of the system, when the two positions $\mathbf{x}_0$ and $\mathbf{x}$ are given, *the Euler-Lagrange action* $S_{cl}(\mathbf{x}, t; \mathbf{x}_0)$ is the function defined by

$$S_{cl}(\mathbf{x}, t; \mathbf{x}_0) = \min_{\mathbf{u}(s), 0\le s\le t}\left\{\int_0^t L(\mathbf{x}(s), \mathbf{u}(s), s)ds\right\}, \tag{2.7}$$

where the minimum (or more generally an extremum) is taken on the velocity $\mathbf{u}(s)$ which is the control variable, $s \in [0, t]$, with the state $\mathbf{x}(s)$ given by the equations

$$\begin{cases} \dfrac{d\mathbf{x}(s)}{ds} = \mathbf{u}(s) & \text{for } s \in [0, t], \\ \\ \mathbf{x}(0) = \mathbf{x}_0. \end{cases} \tag{2.8}$$

This is the principle of least action defined by Euler [8] in 1744 and Lagrange [18] in 1755.

The solution $(\widetilde{\mathbf{u}}(s), \widetilde{\mathbf{x}}(s))$ of (2.7), if the Lagrangian $L(\mathbf{x}, \dot{\mathbf{x}}, t)$ is twice differentiable, and satisfies the Euler-Lagrange equations on the interval $[0, t]$

$$\begin{cases} \dfrac{d}{ds}\dfrac{\partial L}{\partial \dot{\mathbf{x}}}(\mathbf{x}(s), \dot{\mathbf{x}}(s), s) - \dfrac{\partial L}{\partial \mathbf{x}}(\mathbf{x}(s), \dot{\mathbf{x}}(s), s) = 0 & \text{for } s \in [0, t], \\ \\ \mathbf{x}(0) = \mathbf{x}_0. \end{cases} \tag{2.9}$$

This is the principle of least action defined by Euler [8] in 1744 and Lagrange [18] in 1755.

The knowledge of the velocity at each time s $(0 \leq s \leq t)$ requires the resolution of the Euler-Lagrange equations (2.9) on the whole trajectory as illustrated in Figure 1. Let's give the example of a non-relativistic particle in a linear potential field $V(\mathbf{x}) = -\mathbf{K} \cdot \mathbf{x}$. Its Lagrangian becomes $L(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{1}{2}m\dot{\mathbf{x}}^2 + \mathbf{K} \cdot \mathbf{x}$, and the initial velocity is equal to $\widetilde{\mathbf{v}}_0 = \frac{\mathbf{x}-\mathbf{x}_0}{t} - \frac{t}{2m}\mathbf{K}$. Then, $\widetilde{\mathbf{v}}_0$ depends on the position $\mathbf{x}$ of the particle at the final time t.

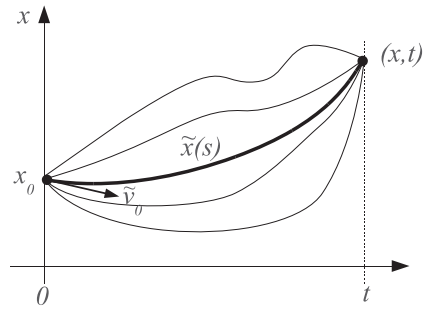

Figure 1: Illustration of the classical Euler-Lagrange equations boundaries and initial conditions for different trajectories $\mathbf{x}(s)$, $s \in [0, t]$ between initial position $(\mathbf{x}_0, 0)$ and final one $(\mathbf{x}, t)$, $\widetilde{\mathbf{x}}(s)$ being the optimal trajectory with initial velocity $\widetilde{\mathbf{v}}_0$. In the case of linear potential field, one gets $\widetilde{\mathbf{v}}_0 = \frac{\mathbf{x}-\mathbf{x}_0}{t} - \frac{t}{2m}\mathbf{K}$.

Equation (2.7) seems to show that, among the trajectories that can reach $(\mathbf{x}, t)$ from the initial position $\mathbf{x}_0$ as illustrated in figure (2), the principle of least action allows to choose the velocity at each time. In reality, the principle of least action used in this equation does not choose the velocity at each time s between 0 and t, but only when the particle arrives at $\mathbf{x}$ at time t.

This dependence of the "final causes" is general. This is Poincaré's main criticism of the principle of least action "*This molecule seems to know the point to which we want to take it, to foresee*

the time it will take to reach it by such a path, and then to know how to choose the most convenient path." One must conclude that, without knowing the initial velocity, the Euler-Lagrange action answers a problem posed by an observer "What would be the velocity of the particle at the initial time to attained $\mathbf{x}$ at time t?". The resolution of this problem implies that the observer solves the Euler-Lagrange equations (2.9) after the observation of $\mathbf{x}$ at time t. This is an *a posteriori* point of view.

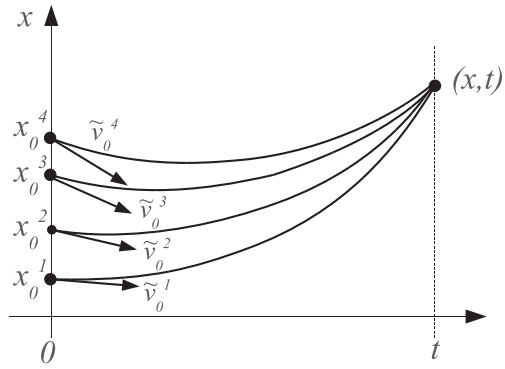

Figure 2: Illustration of the Hamilton-Jacobi equations boundaries and initial conditions for different classical trajectories $\widetilde{\mathbf{x}}(s), s \in [0, t]$ between different initial positions $(\mathbf{x}_0^i, 0)$ and $(\mathbf{x}, t)$ with different initial velocities $\widetilde{\mathbf{v}}_0^i$. In the case of linear potential field, one has directly $\widetilde{\mathbf{v}}_0 = \frac{\mathbf{x}-\mathbf{x}_0}{t} - \frac{t}{2m}\mathbf{K}$.

The Hamiton-Jacobi action will overcome this *a priori* lack of knowledge of the initial velocity in the Euler-Lagrange action. Indeed, at the initial time, the Hamilton-Jacobi action $S_0(\mathbf{x})$ is known. The knowledge of this initial action $S_0(\mathbf{x})$ involves the knowledge of the velocity field at the initial time that satisfies $\mathbf{v}_0(\mathbf{x}) = \frac{\nabla S_0(\mathbf{x})}{m}$. *The Hamilton-Jacobi action* $S(\mathbf{x}, t)$ at $\mathbf{x}$ and time t is then the function defined by

$$S(\mathbf{x}, t) = \min_{\mathbf{x}_0; \mathbf{u}(s), 0 \le s \le t} \left\{ S_0(\mathbf{x}_0) + \int_0^t L(\mathbf{x}(s), \mathbf{u}(s), s) ds \right\}. \tag{2.10}$$

where the minimum is taken on all initial positions $\mathbf{x}_0$ and on the controls $\mathbf{u}(s)$, $s \in [0, t]$, with the state $\mathbf{x}(s)$ given by the equations (2.8) as showed on Figure 2. This Hamilton-Jacobi action with its initial solution $S_0(\mathbf{x})$ is well known in the mathematical textbooks [9] for optimal control problems, but is ignored in physical ones [15, 19, 20], where there is no mention of the initial condition $S_0(\mathbf{x})$. It is often confused in the textbooks with the so-called principal function of Hamilton.

2.3 Physical interpretation

The introduction of the Hamilton-Jacobi action highlights the importance of the initial action $S_0(\mathbf{x})$, while texbooks do not well differentiate these two actions.

The initial condition $S_0(\mathbf{x})$ is mathematically necessary to obtain the general solution to the Hamilton-Jacobi equations (2.1) and (2.2). Physically, it is the condition that describes the preparation of the particles. We will see that this initial condition is the key to the least action principle understanding.

Nothing that because $S_0(\mathbf{x}_0)$ does not play a role in (2.10) for the minimization on $\mathbf{u}(s)$, we obtain equation (2.7) between the Hamilton-Jacobi action and Euler-Lagrange action

$$S(\mathbf{x}, t) = \min_{\mathbf{x}_0}\{S_0(\mathbf{x}_0) + S_{cl}(\mathbf{x}, t; \mathbf{x}_0)\}. \tag{2.11}$$

It is an equation that generalizes the Hopf-Lax and Lax-Oleinik formula [9],

$$S(\mathbf{x}, t) = \min_{\mathbf{x}_0}\left\{S_0(\mathbf{x}_0) + m\frac{(\mathbf{x} - \mathbf{x_0})^2}{2t}\right\},$$

which corresponds to the particular case of the free particle where the Euler-Lagrange action is equal to $m\frac{(x-x_0)^2}{2t}$. For the Lagrangian $L(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x}, t)$, one deduces that the velocity of a non-relativistic classical particle in a potential field is given for each point $(\mathbf{x},t)$ by

$$\mathbf{v}(\mathbf{x},t) = \frac{\nabla S(\mathbf{x},t)}{m}. \tag{2.12}$$

where $S(\mathbf{x},t)$ is the Hamilton-Jacobi action, solution to the Hamilton-Jacobi equations (2.1).

Equation (2.12) shows that the solution $S(\mathbf{x},t)$ of the Hamilton-Jacobi equations yields the velocity field for each point $(\mathbf{x}, t)$ from the velocity field $\frac{\nabla S_0(\mathbf{x})}{m}$ at initial time. In particular, if at initial time, one knows the initial position $\mathbf{x}_{init}$ of a particle, its velocity at this time is equal to $\frac{\nabla S_0(\mathbf{x}_{init})}{m}$. From the solution $S(\mathbf{x},t)$ of the Hamilton-Jacobi equations, one deduces the particle trajectories with (2.12). The Hamilton-Jacobi action $S(\mathbf{x},t)$ is then a field which "pilots" the particle. One can understand with equation (2.12) that among the trajectories that can reach $(\mathbf{x}, t)$ from an unknown initial position and a known initial velocity field, Nature chooses the initial position and at each time the velocity that yields the minimum (or the extremum) of the Hamilton-Jacobi action. Equations (2.1) and (2.12) confirm this interpretation. They show that the Hamilton-Jacobi action $S(\mathbf{x}, t)$ does not only solve a given problem with a single initial condition $\left(\mathbf{x}_0, \frac{\nabla S_0(\mathbf{x}_0)}{m}\right)$, but a set of problems with an infinity of initial conditions, all the pairs $\left(\mathbf{y}, \frac{\nabla S_0(\mathbf{y})}{m}\right)$. It answers the following question "If one knows the action (or the velocity field) at the initial time, can we determine the action (or the velocity field) at each later time?". This problem is solved sequentially by the (local) evolution equation (2.1). This is an *a priori* point of view. It is the problem solved by Nature with the principle of least action.

(min, +)-analysis can help us now to explain the differences between Hamilton-Jacobi and Euler-Lagrange actions.

The classical Euler-Lagrange action $S_{cl}(\mathbf{x}, t; \mathbf{x}_0)$ is the elementary solution to the Hamilton-Jacobi equations (2.1) in the (min, +)*-analysis with the initial condition*

$$S_0(\mathbf{x}) = \delta_{(\min,+)}(\mathbf{x} - \mathbf{x}_0) = \begin{cases} 0 & \text{if} \quad \mathbf{x} = \mathbf{x}_0, \\ +\infty & \text{otherwise} \end{cases} \tag{2.13}$$

The Hamilton-Jacobi action $S(\mathbf{x}, t)$ is then given by the (min, +) path integral (2.11).

3 COMPLEX VARIATIONAL CALCULUS

One introduces for complex valued functions, the definition of a minimum in a first step and develops variational calculus for functionals applied to such kind of functions in a second step.

3.1 Minimum of a complex valued function

Definition 2. *For a complex function* $f : \mathbb{C}^n \longrightarrow \mathbb{C}$, *with* $f(\mathbf{z}) \equiv f(\mathbf{x}+i\mathbf{y}) = P(\mathbf{x}, \mathbf{y}) + iQ(\mathbf{x}, \mathbf{y})$ *from* $\mathbb{C}^n$ *to* $\mathbb{C}$, *one defines the* min *operator if it exists for a closed set* $A = \{\mathbf{x}+i\mathbf{y} \,/ \mathbf{x} \in X \subset \mathbb{R}^n, \mathbf{y} \in Y \subset \mathbb{R}^n\} \subset \mathbb{C}^n$ *with* $\mathbf{z}_0 = \mathbf{x}_0 + i\mathbf{y}_0$ *as*

$$\mathbf{z}_0 = \arg\min_{\mathbf{z}\in A} f(\mathbf{z}) \in \arg\min_{\mathbf{x}\in X} \max_{\mathbf{y}\in Y} P(\mathbf{x}, \mathbf{y}) = \arg\max_{\mathbf{y}\in Y} \min_{\mathbf{x}\in X} P(\mathbf{x}, \mathbf{y}).$$

Example 1. *Let* $a \in \mathbb{R}$, *and* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *the function* $z = x + iy \mapsto (z-a)^2$, then $f(z) \equiv f(x, y) = (x^2 + a^2 - y^2) + 2i(xy - ay)$, and

$$\begin{cases} \arg\min\limits_{\mathbf{x}\in\mathbb{R}} \max\limits_{\mathbf{y}\in\mathbb{R}} \{x^2 + a^2 - y^2\} \equiv (a, 0) = a \\ \arg\max\limits_{\mathbf{y}\in\mathbb{R}} \min\limits_{\mathbf{x}\in\mathbb{R}} \{x^2 + a^2 - y^2\} \equiv (a, 0) = a, \end{cases} \tag{3.14}$$

which gives the minimum $z_0 = a$.

Remark 3. *Therefore* $(\mathbf{x}_0, \mathbf{y}_0)$ *has to be a saddle-point of* $P(\mathbf{x}, \mathbf{y})$. *If it is not unique, the complex part of* $\inf\{f(\mathbf{z}) \,/ \mathbf{z} \in \mathbb{C}^n\}$ *is a multivalued (set-valued) function.*

Definition 3. *A function* $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ *such as* $f(\mathbf{z}) = P(\mathbf{x}, \mathbf{y}) + iQ(\mathbf{x}, \mathbf{y})$ *will be called (strictly) convex if* $P(\mathbf{x}, \mathbf{y})$ *is (strictly) convex in* $\mathbf{x}$ *and (strictly) concave in* $\mathbf{y}$.

Remark 4. *If* f *is an holomorphic function, a necessary condition for* $\mathbf{z}_0$ *to be a minimum of* f *is that* $f'(\mathbf{z}_0) = 0$. *For such a function, its real part* $P(\mathbf{x}, \mathbf{y})$ *(strictly) convex in* $\mathbf{x}$ *is equivalent to* $P(\mathbf{x}, \mathbf{y})$ *(strictly) concave in* $\mathbf{y}$ *(Cauchy-Riemann conditions).*

Definition 4. *For all convex function* $f : \mathbb{C}^n \longrightarrow \mathbb{C}$, *one defines its Legendre-Fenchel transform* $\widehat{f} : \mathbb{C}^n \longrightarrow \mathbb{C}$ *as*

$$\widehat{f}(\mathbf{p}) = \min_{\mathbf{z}\in\mathbb{C}^n} \{f(\mathbf{z}) - \mathbf{p}\cdot\mathbf{z}\}.$$

Theorem 1. *The complex Legendre-Fenchel transform* $\widehat{f}(\mathbf{p})$ *of a function* f *from* $\mathbb{C}^n$ *to* $\mathbb{C}$, *which is holomorphic and strictly convex, is an involute and strictly convex transform as well and is defined by*

$$\widehat{\widehat{f}}(\mathbf{z}) = \min_{\mathbf{p}\in\mathbb{C}^n} \{\widehat{f}(\mathbf{p}) - \mathbf{p}\cdot\mathbf{z}\} = f(\mathbf{z}).$$

Definition 5. *Let* $L : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R}^+ \to \mathbb{C}$ *a functional* $L(\mathbf{p}, \mathbf{q}, \tau)$ *holomorphic in both* $\mathbf{p}$ *and* $\mathbf{q}$ *and derivable in* τ*, which will be called complex Lagrangian. For fixed* $(\mathbf{z}_i, \mathbf{z}_f) \in \mathbb{C}^n \times \mathbb{C}^n$ *and* $t \geq 0$*, one defines the complex action functional on the set of admissible functions* $A = \left\{\mathbf{w} \in C^2([0,t]; \mathbb{C}^n) \,/\, \mathbf{w}(0) = \mathbf{z}_i, \mathbf{w}(t) = \mathbf{z}_f\right\}$ *as*

$$J(\mathbf{w}) = \int_0^t L\left(\mathbf{w}(s), \frac{d\mathbf{w}}{ds}(s), s\right) ds. \tag{3.15}$$

Thus, the main goal of complex variational calculus is to find out a curve $z(\cdot) \in A$, verifying

$$J(\mathbf{z}) = \min_{\mathbf{w} \in A} J(\mathbf{w}) \tag{3.16}$$

where the min operator is considered in the sense of definition (2).

Theorem 2 (Complex Euler-Lagrange equation). *If the function* $\mathbf{z}(\cdot)$ *is an holomorphic solution of (3.16), then* $\mathbf{z}(\cdot)$ *verifies the equations system*

$$\frac{\partial L}{\partial \mathbf{z}}\left(\mathbf{z}(s), \frac{d\mathbf{w}}{ds}(s), s\right) - \frac{d}{ds}\left(\frac{\partial L}{\partial q}\left(\mathbf{z}(s), \frac{d\mathbf{w}}{ds}(s), s\right)\right) = 0, \forall t \in [0,t]. \tag{3.17}$$

Proof 1. The demonstration is similar to the classical one if $J(z) = \min_{w \in A} J(w)$ and if $\mathbf{z} \in B \subset A$, then $J(\mathbf{z}) = \min_{\mathbf{w} \in B} J(\mathbf{w})$. □

Definition 6. *One defines the complex action* $S(\mathbf{z}, t)$ *as the complex minimum of the integral*

$$S(\mathbf{z}(t), t) = \min_{\mathbf{v}(\tau), 0 \leq \tau \leq t} \left\{S_0(\mathbf{z}_0) + \int_0^t L(\mathbf{z}(\tau), \mathbf{v}(\tau), \tau) d\tau\right\} \tag{3.18}$$

where the complex minimum is calculated on all control variables $\mathbf{v}(\tau)$, $\forall \tau \in [0,t]$*. The state evolution* $z(\tau)$ *is given by system's evolution equation*

$$\frac{d\mathbf{z}}{dt}(\tau) = \mathbf{v}(\tau) \text{ and } \mathbf{z}(0) = \mathbf{z}_0.$$

Theorem 3. *The complex action* $S(\mathbf{z}, t)$ *verifies the complex Jamilton-Jacobi equations*

$$\begin{cases} \dfrac{\partial S}{\partial t} + H(\mathbf{z}, \nabla S, t) = 0 \quad \forall (\mathbf{z}, t) \in \mathbb{C}^n \times \mathbb{R}^+, \\ \\ S(\mathbf{z}, 0) = S_0(\mathbf{z}) \quad \forall \mathbf{z} \in \mathbb{C}^n, \end{cases} \tag{3.19}$$

where $H(\mathbf{z}, \mathbf{p}, t)$ *is the Legendre-Fenchel of* $L(\mathbf{z}, \mathbf{q}, t)$*.*

Proof 2. Analogous to the classical variational calculus with the assumption that the action $S(\mathbf{z}, t)$ is holomorphic in $\mathbf{z}$. □

3.2 Complex Hamilton-Jacobi equations

Let's write the action as $S(\mathbf{x}, t) = a(\mathbf{x}, t) + ib(\mathbf{x}, t)$ verifying the Hamilton-Jacobi equations with $a(\mathbf{x}, t)$ and $b(\mathbf{x}, t)$ from $\mathbb{R}^n \times \mathbb{R}^+$ to $\mathbb{R}$

$$\begin{cases} \dfrac{\partial S}{\partial t} + \dfrac{1}{2}(\nabla S)^2 = 0 \; \forall (\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ \\ S(\mathbf{x}, 0) = S_0(\mathbf{x}) \;\; \forall \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

with $S_0(x) = a_0(\mathbf{x}) + ib_0(\mathbf{x})$ where a_0 and b_0 are holomorphic (analytic) functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Those equations are equivalent to the partial derivative equations system

$$\begin{cases} \dfrac{\partial a}{\partial t} + \dfrac{1}{2}(\nabla a)^2 - \dfrac{1}{2}(\nabla b)^2 = 0 \quad \forall (\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ \\ \dfrac{\partial b}{\partial t} + \nabla a \cdot \nabla b = 0 \;\; \forall (\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ \\ a(\mathbf{x}, 0) = a_0(\mathbf{x}), \; b(\mathbf{x}, 0) = b_0(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^n. \end{cases} \tag{3.20}$$

If $a_0(\mathbf{x})$ is strictly convex, one can show that the solution $S(\mathbf{x}, t)$ of this system of equations (3.20) is equal to

$$S(\mathbf{x}, t) = \min_{\mathbf{z} \in \mathbb{C}^n} \left\{ S_0(\mathbf{z}) + \frac{(\mathbf{z} - \mathbf{x})^2}{2t} \right\}.$$

Remark 5. *In order to get the solution of those equations, it has been necessary to use complex variables. This is a general method. One can generalize the resolution of Hamilton-Jacobi equations for the complex ones. The following theorem gives a generalisation of Hopf-Lax formula.*

Theorem 4. *The solution $S(\mathbf{z}, t)$ of complex Hamilton-Jacobi equations*

$$\begin{cases} \dfrac{\partial S}{\partial t} + H(\nabla S) = 0, \quad \forall (\mathbf{z}, t) \in \mathbb{C}^n \times \mathbb{R}^+, \\ \\ S(\mathbf{z}, 0) = S_0(\mathbf{z}) \;\; \forall \mathbf{z} \in \mathbb{C}^n, \end{cases} \tag{3.21}$$

where H and S_0 are holomorphic and convex functions, is then equal to

$$S(\mathbf{z}, t) = \min_{\mathbf{z}' \in \mathbb{C}^n} \left\{ S_0(\mathbf{z}') + t\widehat{H}\left(\frac{\mathbf{z} - \mathbf{z}'}{t}\right) \right\}.$$

where $\widehat{H}$ is the complex Legendre-Fenchel of H.

4 LAGRANGIAN FIELDS

One develops in this sections a complex analytical mechanics from complex valued Lagrangian density. For a flat Minkowski space with metric tensor $\eta_{\mu\nu}$, one introduces usually a Lagrangian

density $\mathcal{L}(\varphi, \partial_\nu \varphi)$ depending on the functionnal $\varphi(\mathbf{x}^\mu)$ with $\mathbf{x}^\mu = (x^0, \mathbf{x}^i) = (t, \mathbf{x})$ and its first derivatives $\partial_\nu \varphi(\mathbf{x}^\mu) = \frac{\partial \varphi(\mathbf{x}^\mu)}{\partial \mathbf{x}^\nu}$. If one integrates it on the whole space or even on an open set $\Omega = \mathring{\Omega} \subset \mathbb{R}^3$, one gets the usual Lagrangian

$$L(\varphi(\cdot, t)) = \int_\Omega \mathcal{L}(\varphi(\cdot, t), \partial_\mu \varphi(\cdot, t)) d^3\mathbf{x}. \tag{4.22}$$

Let's consider the field $\varphi(\mathbf{x}^\mu)$ and its lagrangian density as complex valued. On assumes that $\mathcal{L}(\mathbf{z}, q)$ is holomorphic in $\mathbf{z}$ and $\mathbf{q}$, and strictly convex in $\mathbf{q}$.

One can then define the complex action between t_0 and $t \geq t_0$ by

$$S(\varphi) = \int_{[t_0, t]} L(\varphi(\cdot, s)) ds, \tag{4.23}$$

which can be written as well as

$$S(\varphi) = \int_{\Omega \times [t_0, t]} \mathcal{L}(\varphi, \partial_\mu \varphi) d^4\mathbf{x}. \tag{4.24}$$

One would like to generalize the Least Action Principle to this complex field in order to derive Euler-Lagrange-like equations. This principle states that system's physical fields $\widetilde{\varphi}(\mathbf{x}, t)$ correspond to the extrema of the action S.

One considers the action $S()\varphi_\varepsilon)$, with $\varphi_\varepsilon = \widetilde{\varphi} + \varepsilon \delta\varphi$, $\delta\varphi$ is an arbitrary field vanishing to 0 on $\partial\Omega$ the border of the integration volume Ω. Action stationnarity for $\widetilde{\varphi}$ means

$$\left[\frac{d}{d\varepsilon} S(\varphi_\varepsilon) \right]_{\varepsilon = 0} = 0.$$

After partial integration and using the fact that $\delta\varphi \equiv 0$ on $\partial\Omega$, it yields

$$\frac{d}{d\varepsilon} S(\varphi_\varepsilon) = \int d^4x \left[\frac{\partial}{\partial \varphi} \mathcal{L}(\varphi, \partial_\mu \varphi) - \partial_\mu \frac{\partial}{\partial \partial_\mu \varphi} \mathcal{L}(\varphi, \partial_\mu \varphi) \right]_{\varphi = \varphi_\varepsilon} \delta\varphi.$$

Since $\delta\varphi$ arbitrary in the integration volume, one gets the **Euler-Lagrange equations for complex Lagrangian**

$$\frac{\partial}{\partial \varphi} \mathcal{L}(\varphi, \partial_\mu \varphi) - \partial_\mu \frac{\partial}{\partial \partial_\mu \varphi} \mathcal{L}(\varphi, \partial_\mu \varphi) = 0 \quad in \quad \Omega \times]t_0, t[, \tag{4.25}$$

for which $\widetilde{\varphi}$ is the solution.

5 APPLICATIONS TO THE BORN-INFELD THEORY

We present in this section an application of our previous development about complex variational calculus to the Born-Infeld theory of electromagnetism. One explores here the possibility to replace the Faraday tensor $\mathcal{F}$ and its Hodge dual $\mathcal{F}^\star$ with a unique complex tensor $\mathcal{F}_C = \mathcal{F} + i \cdot \mathcal{F}^\star$. We show that this replacement coupled with an extension of least action principle to complex-valued Lagrangian densities, permits to deduce first the Maxwell's equations, and second to understand why experiments have never exhibited nonlinear Born-Infeld effects.

5.1 Tensorial and Lagrangian density formalism in electromagnetism

The answer to the question "What is the right tensor of the electromagnetic field?" seems obvious since the introduction of the Faraday tensor in 1908 by H. Minkowski [28]. In almost all texbooks on electromagnetism or relativity [3, 17, 20], one usually uses the Faraday tensor ($\mathcal{F}_{\mu\nu}$) and its Hodge-dual ($\mathcal{F}^*_{\mu\nu}$) to describe the electromagnetic field ($\mathbf{E}$, $\mathbf{B}$) with

$$\mathcal{F} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{5.26}$$

and

$$\mathcal{F}^\star = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{5.27}$$

We recall that the usual method for obtaining Maxwell's equations

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \text{ and } \mathbf{B} = \nabla \times \mathbf{A}, \tag{5.28}$$

which yields to

$$\nabla \cdot \mathbf{B} = 0 \text{ and } \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \tag{5.29}$$

is to find the extremum of the action $\int \mathcal{L}\, d^4\mathbf{x}$ according to the potential quadrivector (Φ, $\mathbf{A}$) with

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - j_\mu A^\mu \\ &= -\frac{1}{4}\mathcal{F}^*_{\mu\nu}\mathcal{F}^{*\mu\nu} - j_\mu A^\mu \\ &= \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) - \rho\Phi + \mathbf{j}\cdot\mathbf{A}. \end{aligned} \tag{5.30}$$

Remark 6. *This Lagrangian density uses only the first Lorentz invariant* $\mathbf{E}^2 - \mathbf{B}^2$ *and not simutaneously* $(\boldsymbol{E}\cdot\boldsymbol{B})^2$.

However, one can ask if $\mathcal{F}$ *and* $\mathcal{F}^*$ *are the right tensor of the electromagnetic theory. Why two electromagnetic tensors are not combined into only one as for other fields in physics? From a fundamental point of view, one can not define the Lagrangian density (5.30) with* $\frac{1}{2}\sqrt{-\det(\mathcal{F})} - j_\mu A^\mu$ *or* $\frac{1}{2}\sqrt{-\det(\mathcal{F}^*)} - j_\mu A^\mu$, *since* $-\det(\mathcal{F}) = -\det(\mathcal{F}^*) = (\boldsymbol{E}\cdot\boldsymbol{B})^2$, *which is the other Lorentz's invariant. Let us note that it is important to obtain the right electromagnetic tensor if*

one wants to combine it with another one such as the metric tensor. This was Born and Infeld's main idea, and also one of the research motivations for the B-branes theory.

In this paper, we show that a well-suited candidate for the electromagnetic tensor is the complex Faraday tensor

$$\mathcal{F}_C = \mathcal{F} + i \cdot \mathcal{F}^* = \begin{pmatrix} 0 & F_x & F_y & F_z \\ -F_x & 0 & -iF_z & iF_y \\ -F_y & iF_z & 0 & -iF_x \\ -F_z & -iF_y & iF_x & 0 \end{pmatrix} \tag{5.31}$$

which depends only on the complex vector $\mathbf{F} = \mathbf{E} + i\mathbf{B}$. *This vector* $\mathbf{F}$ *has a long history since its introduction in 1907 by L. Silberstein [32,33]. With mean of Maxwell's equations, one can write*

$$\mathbf{F} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi + i\nabla \times \mathbf{A}. \tag{5.32}$$

This tensor is unique and one can associate the following complex-valued Lagrangian density with the density quadrivector $(\rho, \mathbf{j})$

$$\mathcal{L}_C = \frac{1}{2}\sqrt{-\det(\mathcal{F}_C)} - j_\mu A^\mu = \frac{1}{2}\mathbf{F}^2 - \rho\Phi + \mathbf{j} \cdot \mathbf{A}, \tag{5.33}$$

whose real part is the Lagrangian density $\mathcal{L}$ *(5.30). We note that* $\mathbf{F}^2 = (\mathbf{E} + i\mathbf{B})^2 = (\mathbf{E}^2 - \mathbf{B}^2) + 2i\boldsymbol{E} \cdot \boldsymbol{B}$ *which exhibit the Lorentz invariants [17].*

One will propose below some new clues that allow to consider $\mathcal{F}_C$ *as the possible right electromagnetic tensor.*

5.2 Born-Infeld nonlinear equations

While seeking to identify a covariant action in order to build a nonlinear electrodynamics, Born and Infeld proposed to use a linear combination of the metric tensor $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and of the Faraday tensor $(\mathcal{F}^{\mu\nu})$. It is the following Lagrangian density [2,21,22]

$$\mathcal{L}_{BI} = -\frac{k^2}{2}\left(\sqrt{-\det\left(g^{\mu\nu} + \frac{1}{k}\mathcal{F}^{\mu\nu}\right)} - \sqrt{-\det(g^{\mu\nu})}\right) - \rho\Phi + \mathbf{j} \cdot \mathbf{A}, \tag{5.34}$$

where k is a large dimensional parameter. This uses both the two Lorentz's invariants, as can be shown on the following equivalent expression

$$\mathcal{L}_{BI} = -k^2\left(\sqrt{1 - \frac{\mathbf{E}^2 - \mathbf{B}^2}{k^2} - \frac{(\mathbf{E} \cdot \mathbf{B})^2}{k^4}} - 1\right) - \rho\Phi + \mathbf{j} \cdot \mathbf{A}. \tag{5.35}$$

When $k \to +\infty$, one recovers (5.30). This density describes a non-interacting gauge theory but has not been validated by experiments in order to demonstrate nonlinear classical effects [16].

However, it still remains a relevant and useful theory for membranes and superstrings theories [4, 30, 34].

If we consider that the relevant field-strength tensor in electrodynamics is the complex Faraday one, the Born-Infeld Lagrangian density should be rewritten by replacing $\mathcal{F}$ in (5.34) by $\mathcal{F}_C$

$$\mathcal{L}_{\mathcal{BIC}} = -\frac{k^2}{2}\left(\sqrt{-\det\left(g^{\mu\nu}+\frac{1}{k}\mathcal{F}_C{}^{\mu\nu}\right)}-\sqrt{-\det(g^{\mu\nu})}\right)-\rho\Phi+\mathbf{j}\cdot\mathbf{A}. \tag{5.36}$$

The calculation leads to

$$\begin{aligned}\det\left(g^{\mu\nu}+\frac{1}{k}\mathcal{F}_C{}^{\mu\nu}\right) &= \det\begin{pmatrix} 1 & -\frac{F_x}{k} & -\frac{F_y}{k} & -\frac{F_z}{k} \\ \frac{F_x}{k} & -1 & i\frac{F_z}{k} & -i\frac{F_y}{k} \\ \frac{F_y}{k} & -i\frac{F_z}{k} & -1 & i\frac{F_x}{k} \\ \frac{F_z}{k} & i\frac{F_y}{k} & -i\frac{F_x}{k} & -1 \end{pmatrix} \\ &= -\left(1-\frac{1}{k^2}\mathbf{F}^2\right)^2. \end{aligned} \tag{5.37}$$

We therefore obtain

$$\mathcal{L}_{BIC} = -\frac{k^2}{2}\left[\left(1-\frac{1}{k^2}\mathbf{F}^2\right)-1\right]-\rho\Phi+\mathbf{j}\cdot\mathbf{A} = \frac{1}{2}\mathbf{F}^2-\rho\Phi+\mathbf{j}\cdot\mathbf{A}. \tag{5.38}$$

Let's now apply the complex variational calculus that we have developed above complex Euler-Lagrange equations (5.39) with $\varphi = (\Phi, \mathbf{A})$

$$\frac{\partial}{\partial\varphi}\mathcal{L}_{BIC}(\varphi, \partial_\mu\varphi) - \partial_\mu\frac{\partial}{\partial\partial_\mu\varphi}\mathcal{L}_{BIC}(\varphi, \partial_\mu\varphi) = 0. \tag{5.39}$$

Using the expression of $\mathbf{F}$ in (5.32), the set of equations (5.39) yields directly to an equivalent form of Maxwell's equations $\mathbf{F}$

$$\nabla\cdot\mathbf{F} = \rho \quad \text{and} \quad \nabla\times\mathbf{F} - i\frac{\partial\mathbf{F}}{\partial t} = i\mathbf{j}. \tag{5.40}$$

This shows first that the right complex Born-Infeld Lagrangian has to be the complex Faraday one. Second, it shows that the complex Euler-Lagrange equations (Maxwell's equations) obtained from it yield to Maxwell's equations which are linear, and will thus not produce nonlinear effects, which concurs with the experiments [16]. It means that there are no nonlinear effects, excepted in the quantum treatment of electrodynamics. The difference between the Born-Infeld complex approach and the one presented here is due to the fact that the real part of $\mathcal{L}_{BIC}$ is different from the $\mathcal{L}_{BI}$ one. Indeed, the real part of the square root of a complex number is not equal to the square root of its real part.

6 CONCLUSION

One has proposed a new mathematical tool, the so-called complex (min, +)-analysis which is a generalization of the real approach one. This permits to develop a well-defined complex variational calculus, to generalize Hamilton-Jacobi and Euler-Lagrange equations to the complex case. Through this generalization and from a complex Lagrangian density, we have derived Maxwell's equations from the complex Faraday tensor $\mathcal{F}_C$. The analysis of the Born-Infeld theory through the complex Faraday tensor explains why experiments have never demonstrated nonlinear Born-Infeld effects and then confirms the Faraday complex tensor as a better candidate to represent the electromagnetic field. The complex vector $\mathbf{F} = \mathbf{E} + i\mathbf{B}$ has often been viewed as a possible wave function of the photon [1]. It is a pragmatic way to consider the existence of this wave function, as stated by P.A.M. Dirac in 1958 [29] *"The essential point is the association of each of the translational states of the photon with one of the wave functions of ordinary wave optics"*. The complex Lagrangian density proposed here is therefore an explicit functional of the wave function.

In order to unify General Relativity and Electrodynamics Theories, Einstein also defined a complex tensor which linked metric and electromagnetic tensors [7]. We have proved with equation (5.37) that the complex Faraday tensor is linked to the Minkowski metric tensor. One of the main challenge will be to define a covariant action by combining the metric tensor and a complex Faraday tensor to a curved space.

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RESUMO. Uma nova ferramenta matemática é desenvolvida, a análise complexa (min, +), que permite definir um novo cálculo variacional, análogo ao cálculo clássico (equações de Euler-Lagrange e Hamilton-Jacobi), que é adequado para funções definidas de $\mathbb{C}^n$ em $\mathbb{C}$. Aplicamos este cálculo variacional complexo à teoria de Born-Infeld de eletromagnetismo, e mostramos porque efeitos não lineares não são exibidos.

Palavras-chave: Cálculo Variacional, Lagrangeano, Hamiltoniano, Ação, Equações de Euler-Lagrange e Hamilton-Jacobi, análise complexa (min, +), Equações de Maxwell, Teoria de Born-Infeld.

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