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Quantum statistics: the indistinguishability principle and the permutation group theory

(Estatística quântica: o princípio da indistinguibilidade e a teoria do grupo de permutações)

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About two decades ago we have shown mathematically that besides bosons and fermions, it could exist a third kind of particles in nature that was named gentileons. Our results have been obtained rigorously within the framework of quantum mechanics and the permutation group theory. However, these papers are somewhat intricate for physicists not familiarized with the permutation group theory. In the present paper we show in details, step by step, how to obtain our theoretical predictions. This was done in order to permit a clear understanding of our approach by graduate students with a basic knowledge of group theory.
Keywords: quantum statistics; bosons, fermions and gentileons.

Há cerca de duas décadas passadas, mostramos matematicamente que, além de bósons e férmions, poderia existir na natureza um outro tipo de partículas que denominamos de gentileons. Os nossos resultados foram obtidos rigorosamente dentro do contexto da mecânica quântica e da teoria do grupo de permutações. Entretanto, os nossos artigos são difíceis de serem entendidos por físicos que não estão familiarizados com a teoria do grupo de permutações. Assim, no presente artigo nós vamos mostrar em detalhes, passo a passo, como obter as nossas previsões. Procedemos desse modo para permitir que estudantes de graduação de física possam entender claramente nossos cálculos com um conhecimento básico de teoria de grupos.
Palavras-chave: estatística quântica; bósons, férmions, gentileons.

1. Introduction

In preceding papers [1-8], we have performed a detailed analysis of the problem of the indistinguishability of N identical particles in quantum mechanics. It was shown rigorously, according to the postulates of quantum mechanics and the principle of the indistinguishability, that besides bosons and fermions it could mathematically exist another kind of particles that we have called gentileons. This analysis was performed using the irreducible representations of the permutation group (symmetry group) S_N in the Hilbert space. However, our first papers on the subject [1-3], that were taken as a point of departure to investigate the existence of a new kind of particles (gentileons) is somewhat intricate and complex from the mathematical point of view. We have used the group theory shown in the books of Weyl [9], Hamermesh [10] and Rutherford [11]. These papers are somewhat difficult for physicists not familiarized with the permutation group theory and its representations in the Hilbert space. So, in the present paper we intend to deduce our main results adopting a more simple and didactic mathematical approach. We will present our calculations in such way that graduate students with a basic knowledge of group theory would be able to understand our predictions.

In Sec. 2 the problem of the indistinguishability of identical particles in quantum mechanics is analyzed. In Sec. 3 we see how to connect the permutation of the particles with the eigenfunctions of the energy operator H using the Permutation Group. In Sec. 4 the calculation of the energy eigenfunctions of a system with $N = 3$ particles is shown in details. In Sec. 5 the essential results for the general case of systems with N identical particles are given. In Sec. 6 the Summary and Conclusions are presented.

2. The indistinguishability of identical particles in quantum mechanics

Identical particles cannot be distinguished by means of any inherent property, since otherwise they would not be identical in all respects. In classical mechanics, iden-

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tical particles do not lose their "individuality", despite the identity of their physical properties: the particles at some instant can be "numbered" and we can follow the subsequent motion of each of these in its paths. Hence, at any instant the particles can be identified.

In quantum mechanics [12-14], the situation is completely different since, due to the uncertainty relations, the concept of path of a particle ceases to have any meaning. Hence, by localizing and numbering the particles at some instant, we make no progress towards identifying them at subsequent instants: if we localize one of the particles, we cannot say which of the particles has arrived at this point. This is true, for instance, for electrons in a single atom, for neutrons in a single nucleus or for particles which interact with each other to an appreciable extent. However, electrons of different atoms or neutrons of different nucleus, to good approximation, are regarded as distinguishable because they are well separated from each other.

Thus, in quantum mechanics, there is in principle no possibility of separately following during the motion each one of the similar particles and thereby distinguishing them. That is, in quantum mechanics identical particles entirely lose their individuality, resulting in the complete indistinguishability of these particles. This fact is called Principle of Indistinguishability of Identical Particles and plays a fundamental role in the quantum mechanics of identical particles [12-14].

Let us consider an isolated system with total energy E composed by a constant number N of particles that is described by quantum mechanics. If H is the Hamiltonian operator of the system, the energy eigenfunction Ψ, obeys the equation $H\Psi = E\Psi$. The operator H and Ψ are functions of $\mathbf{x}_1,s_1,\ldots, \mathbf{x}_N,s_N$, where $\mathbf{x}_j$ and s_j denote the position coordinate and the spin orientation, respectively of the j^{th} particle. We abbreviate the pair $(\mathbf{x}_j,s_j)$ by a single number j and call 1, 2,..., N a particle configuration. The set of all configurations will be called the configuration space $\varepsilon^{(N)}$. So, we have simply $H = H(1,\ 2,\ldots,\ N)$ and $\Psi = \Psi(1,\ 2,\ldots\ N)$. These quantum states Ψ form a Hilbert space $L_2(\varepsilon^{(N)})$ of all square integrable functions [1-3] over $\varepsilon^{(N)}$.

Let us define P_i as the "permutation operator" ($i = 1,\ 2,\ldots\ N!$) which generate all possible permutations of the N particles in the space $\varepsilon^{(N)}$. Since the particles are identical the physical properties of the system must be invariant by permutations. In the next section we show how to use the permutation group S_N to describe the N-particles quantum system.

3. The permutation group and its representations in the configuration and Hilbert spaces

As seen above, P_i is the "permutation operator" ($i = 1,\ 2,\ldots\ N!$) which generate all possible permutations of the N particles in the space $\varepsilon^{(N)}$. The permutations P_i of the labels 1, 2,.., N constitute a symmetry group [9-11, 16-19], S_N of order $n = N!$

Because of the identity of the particles, H and Ψ obtained by merely permuting the particles must be equivalent physically, that is, $[P_i,\ H] = 0$ and $|P_i\ \Psi|^2 = |\Psi_i|^2 = |\Psi|^2$. This implies that the permutations are unitary transformations and that the energy E has a $N!$ degenerate energy spectrum. We assume that all the functions $\{\Psi_i\}_{i=1,2..,n}$ are different and orthonormal. To each operator P_i of the group S_N we can associate, in a one-to-one correspondence, an unitary operator $U(P_i)$ in the $L_2(\varepsilon^{(N)})$, [14, 15].

Now, let us put $n = N!$ and indicate by $\{\Psi_k\}_{k=1,2\ldots,n}$ the set of n-degenerate energy eigenfunctions, where $\Psi_k = U(P_k)\Psi$. It is evident that any linear combination of the functions Ψ_k is also a solution of the wave equation $H\Psi = E\Psi$. In addition, since $[U(P),\ H] = 0$ we see that $H\ U(P)\ \Psi_k = U(P)\ H\ \Psi_k = U(P)\ E\ \Psi_k = E\ U(P)\Psi_k$. This means that if Ψ_k is an eigenfunction of H, $U(P)\ \Psi_k$ is also an eigenfunction of H. Hence, it must be equal to linear combinations of degenerate eigenvectors, which is [14,15]

$$U(P)\Psi_k = \sum_{j=1,\ldots,n} \Psi_k D_{jk}(P), \tag{3.1}$$

where the $D_{jk}(P)$ are complex coefficients which depend on the group element.

According to Eq. (3.1) the n degenerate eigenfunctions of H thus span an n-dimensional subspace of the state-vector space of the system, and the operations of the group transform any vector which lies entirely in this subspace into another vector lying entirely in the same subspace, *i.e.*, the symmetry operations leave the subspace invariant.

Repeated application of the symmetry operations gives [14, 15]

$$U(Q)U(P)\Psi_k = \sum_{j=1,\ldots,n} U(Q)\Psi_k D_{jk}(P) = \sum_{j=1,\ldots,n}\ \sum_{i=1,\ldots,n} \Psi_i D_{ik}(Q) D_{jk}(P), \tag{3.2}$$

and also

$$U(QP)\Psi_k = \sum_{i=1,\ldots,n} \Psi_i D_{ik}(QP). \tag{3.3}$$

Since $U(QP) = U(Q)U(P)$ the left-hand sides of the Eqs. (3.2) and (3.3) are identical. Hence, comparing the right-hand sides of these same equations we get the basic equation:

$$D_{ik}(QP) = \sum_{j=1,\ldots,n} D_{ij}(Q) D_{jk}(P). \tag{3.4}$$

So, the permutation group S_N named "symmetry group" of the system, defined in the configuration space

$\varepsilon^{(N)}$, induces a group of unitary linear transformations U in the n-dimensional linear Hilbert space $L_2(\varepsilon^{(N)})$. We have shown (see Eqs. (3.1)-(3.3)) that the unitary operations defined by U can be translated into a matrix equations by introducing a complete set of basis vectors in the n-dimensional vector space of Ψ. This Hilbert space $L_2(\varepsilon^{(N)})$ is named "representation space". The set of $n \times n$ square matrices D form a group of *dimension* (*degree*) n equal to the order of S_N. The complete set of matrices D are said to form a "n-dimensional unitary representation of S_N".

The eigenfunctions $\{\Psi_i\}_{i=1,2..,n}$ are all different and orthonormal since they are solutions of the same Schrödinger equation. These functions can be used [11-19] with the Young tableaux, to determine the irreducible representations of the group S_N in the configuration space $\varepsilon^{(N)}$ and the Hilbert space $L_2(\varepsilon^{(N)})$. To do this the basis functions of the irreducible representations using the Young tableaux are constructed taking $\{\Psi_i\}_{i=1,2..,n}$ as an orthogonal unit basis. It is important to note that choosing this particular basis functions we are simultaneously determining the irreducible representations of S_N and eigenfunctions of the operator H which is given by linear combinations and permutations of the $\{\Psi_i\}_{i=1,2..,n}$. This method will be used in the Appendix A to determine the irreducible representations and the energy eigenfunctions for the trivial case of $N = 2$ and for the simplest non trivial case of $N = 3$.

In Sec. 4 using the method presented in Appendix A will be constructed the energy eigenfunctions of a system with $N = 3$ particles.

4. Systems with $N = 3$ particles

We will assume that a typical eigenfunction of energy E of the particles is written as $\Psi = \Psi(1, 2, 3) = u(1)v(2)w(3)$, where the single-particle functions (u, v, w) in the product are all different and orthogonal. According to our analysis in the Appendix A the 6-dimensional Hilbert space $L_2(\varepsilon^{(3)})$ spanned by the orthonormal unit vector basis (u, v, w) is composed by two 1d subspaces, $h([3])$ and $h([1^3])$, and one 4d subspace $h([2,1])$.

First let us consider the two 1d subspaces in the Hilbert space which are represented by following eigenfunctions ϕ_s and ϕ_a:

$$\phi_s = \frac{[u(1)v(2)w(3) + u(1)v(3)w(2) + u(2)v(1)w(3) + u(2)v(3)w(1) + u(3)v(1)w(3) + u(3)v(2)w(1)]}{\sqrt{6}}, \tag{4.1}$$

which is completely symmetric, associated to the horizontal Young tableaux [3],

$$\phi_a = \frac{[u(1)v(2)w(3) - u(1)v(3)w(2) - u(2)v(1)w(3) + u(2)v(3)w(1) + u(3)v(1)w(2) - u(3)v(2)w(1)]}{\sqrt{6}}, \tag{4.2}$$

completely antisymmetric, associated to the vertical Young tableaux $[1^3]$.

The 4d subspace $h([2,1])$, associated to the intermediate Young tableaux [2,1] is represented by the state function $Y([2,1])$. This subspace $h([2,1])$ breaks up into two 2d subspaces, $h_+([2,1])$ and $h_-([2,1])$, that are spanned by the basis vectors $\{Y_1, Y_2\}$, $\{Y_3, Y_4\}$ and represented by the wavefunctions $Y_+([2,1])$ and $Y_-([2,1])$, respectively. The state functions $Y([2,1])$, $Y_+([2,1])$ and $Y_-([2,1])$ are given respectively, by

$$Y([2,1]) = \frac{1}{\sqrt{2}}\begin{pmatrix} Y_+ \\ Y_- \end{pmatrix}, \quad Y_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad \text{and} \quad Y_- = \frac{1}{\sqrt{2}}\begin{pmatrix} Y_3 \\ Y_4 \end{pmatrix}, \tag{4.3}$$

where

$$Y_1 = \frac{[u(1)v(2)w(3) + u(2)v(1)w(3) - u(2)v(3)w(1) - u(3)v(2)w(1)]}{\sqrt{4}},$$

$$Y_2 = \frac{[u(1)v(2)w(3) + 2u(1)v(3)w(2) - u(2)v(1)w(3) + u(2)v(3)w(1) - 2u(3)v(1)w(2) - u(3)v(2)w(1)]}{\sqrt{12}},$$

$$Y_3 = \frac{[-u(1)v(2)w(3) + 2u(1)v(3)w(2) - u(2)v(1)w(3) - u(2)v(3)w(1) + 2u(3)v(1)w(2) - u(3)v(2)w(1)]}{\sqrt{12}}$$

and

$$Y_4 = \frac{[u(1)v(2)w(3) - u(2)v(1)w(3) - u(2)v(3)w(1) + u(3)v(2)w(1)]}{\sqrt{4}}.$$

As shown in the Appendix A, the functions Y_+([2,1]) and Y_-([2,1]) have equal symmetry permutation properties, that is, $P_i Y_\pm = D^{(2)}(P_i)$ $\mathrm{Y}_\pm$ where the (2 × 2) matrices $D^{(2)}(P_i)$ are unitary irreducible representations of the S_3 in the $\varepsilon^{(3)}$ and in the $L_2(\varepsilon^{(3)})$. The vectors Y associated with the 4d space h([2,1]) or with the two 2d subspaces, h_+([2,1]) and h_-([2,1]), will be indicated in what follows simply by Y([2,1]).

As well known [12-15], the totally symmetric function ϕ_s defined by Eq. (4.1) describes the bosons and the completely anti-symmetric function ϕ_a given by Eq. (4.4) describes the fermions. When two fermions occupy the same state we verify $\phi_a = 0$ which implies that two fermions are forbidden to occupy the same state. This kind of restriction does not exist for bosons since $\phi_a \neq 0$ when three Bosons occupy the same state.

We see from Eq. (4.3) that $Y_\pm \neq 0$ when 1 or 2 particles occupy the same state, however $Y_\pm = 0$ when 3 particles occupy the same state.

From these results we see that the functions Y([2,1]) must represent particles which are different from bosons or fermions. These new kind of particles was called gentileons [3]. This name was adopted in honor to the Italian physicist G. Gentile Jr. About six decades ago [20-22], he invented, without any quantum-mechanical or another type of justification, a parastatistics within a thermodynamical context. He obtained a statistical distribution function for a system of N weakly interacting particles assuming that the quantum states of an individual particle can be occupied by an arbitrary finite number d of particles. The Fermi and the Bose statistics are particular cases of this parastatistics for d = 1 and $d = \infty$, respectively. A recent detailed analysis of the d-dimensional ideal gas parastatistics was performed by Vieira and Tsallis [23].

Our analysis which gives support, within the framework of quantum mechanics and group theory, to the mathematical existence of news states Y([2,1]) associated with the intermediate Young tableaux [2,1], justifies, in a certain sense, the Gentile's hypothesis.

5. Systems composed by N identical particles. The statistical principle

In the Appendix A and in Section 3 we have studied in details the cases of systems composed by two and three particles. We have shown how to obtain the irreducible representations of the S_2 and S_3 in the configuration spaces $\varepsilon^{(2)}$ and $\varepsilon^{(3)}$ and in the Hilbert spaces $L_2(\varepsilon^{(2)})$ and $L_2(\varepsilon^{(3)})$. We have also constructed for these cases the eigenfunctions of the Hamiltonian operator.

In this Section we will give only the main results for the N-particles systems that have studied in details in preceding papers [1-3].

We have shown [1-3] that the dimensions $f(\alpha)$ of the irreducible f(α) × $f(\alpha)$ square matrices assume the values 1^2, 2^2,..., $(N - 1)^2$ and to each irreducible representation (α) is associated a subspace $h(\alpha)$ in the Hilbert space $L_2(\varepsilon^{(N)})$ with dimension $f(\alpha)$.

There are only two 1-dimensional ($f(\alpha) = 1$) irreducible representations given by the partitions $(\alpha) = [N]$ and $(\alpha) = [1^N]$. The first case is described by horizontal shape with N spaces. In the second case we have a vertical tableaux with N rows. The wavefunctions associated to them are, respectively, given by

$$\varphi_s = \frac{1}{\sqrt{N!}} \sum_{i=1}^{n} \Psi_i$$

and

$$\varphi_a = \frac{1}{\sqrt{N!}} \sum_{i=1}^{n} \delta_{Pi} \Psi_i \tag{4.4}$$

where $\delta_{Pi} = \pm\, 1$, if P_i is even or odd permutation.

The remaining representations have dimensions $f(\alpha)$ going from 2^2 up to $(N - 1)^2$ and are described by the various intermediate shapes [9-11, 17-19]. To each shape (α) there is an irreducible representation described by $f(\alpha) \times f(\alpha)$ square matrices $D_{ik}^{(\alpha)}$ with dimension $f(\alpha)$. The tableaux with the same shape (α) have equivalent representations and the different *shapes* cannot have equivalent representations. There is a one-to-one correspondence between each shape (α) and the irreducible matrices $D_{ik}^{(\alpha)}$.

To each shape (α) is associated a sub space $h(\alpha) \in L_2(\varepsilon^{(N)})$ with dimension $\tau = f(\alpha)$ spanned by the unit basis $\{Y_i\}_{i=1,2,\ldots,\tau}$. In this subspace $h(\alpha)$ the energy eigenfunction $Y(\alpha)$ is given by

$$Y(\alpha) = \frac{1}{\sqrt{\tau}} \begin{pmatrix} Y_1(\alpha) \\ Y_2(\alpha) \\ \vdots \\ Y_\tau(\alpha) \end{pmatrix} \tag{4.5}$$

where the functions $\{Y_i\}_{i=1,2,\ldots,\tau}$, that are constructed applying the Young operators to the functions $\{\Psi_i\}_{i=1,2..,n}$, obey the condition $< Y_i|Y_n > = \delta_{in}$.

Under the permutations $Y(\alpha) \in h(\alpha)$ is changed into $X(\alpha) \in h(\alpha)$ given by $X(\alpha) = U(P_i)$ Y(α), where $U(P_i)$ is an unitary operator. This permutation operator can also be represented by an unitary matrix $T(\alpha)$:$X(\alpha) = T(\alpha)\ Y(\alpha)$. Since the subspaces $h(\alpha)$ are equivalence classes [9-11, 19], different subspaces have different symmetry properties which are defined by the matrix $T(\alpha)$. This means that if $T(\alpha) \in h(\alpha)$ and $T(\beta) \in h(\beta)$, results $T(\alpha) \neq T(\beta)$ if $\alpha \neq \beta$.

Since $T(\alpha)^+T(\alpha) = 1$ the square modulus of $Y(\alpha)$ is permutation invariant, that is, $|Y|^2 = Y(\alpha)^+Y(\alpha) = X(\alpha)^+X(\alpha) = |X|^2$. So, the function $|\Phi(\alpha)|^2 = Y(\alpha)^+Y(\alpha) = \sum_i |Y_i|^2$ can be interpreted as the probability density function.

We note that for the 1d cases the symmetry properties of the state function $Y(\alpha)$ are very simple because $T = \pm 1$, whereas for the multidimensional $h(\alpha)$ the symmetry properties are not so evident because they are defined by a matrix $T(\alpha)$ which has τ^2 components. Moreover, the occupation number of the states by particles is not fermionic or bosonic.

To obtain the energy eigenfunction our basic hypothesis was that

$$[U(P_i), H] = 0.$$

Consequently, $[U(P_i),\, S(t)] = 0$, where $S(t)$ is the time evolution operator for the system.

The expectation values of an arbitrary Hermitean operator $A = A(1,\ 2, \ldots,\ N)$ for the energy statevectors $Y(\alpha)$ and $X(\alpha)$ are defined by

$$< A_y >=< Y(\alpha)|A|Y(\alpha) >= \\ \frac{1}{\tau}\sum_i < Y_i(\alpha)|A|Y_i(\alpha) >,$$

and

$$< A_x >=< X(\alpha)|A|X(\alpha) >= \\ \frac{1}{\tau}\sum_i < X_i(\alpha)|A|X_i(\alpha) >,$$

respectively. Since $X(\alpha) = T(\alpha)Y(\alpha)$ we see that

$$< A_x >=< X(\alpha)|A|X(\alpha) >= \\ < Y(\alpha)|T(\alpha)^{+}AT(\alpha)|Y(\alpha) >= \\ < Y(\alpha)|A|Y(\alpha) >=< A_y >,$$

implying that $[U(P_i), A(t)] = 0$. Moreover, if $U(P_i)$ commutes with $S(t)$ the relation $[U(P_i),\, A(t)] = [U(P_i),\, S^{+}(t)A(t)S(t)] = 0$ is satisfied. This means that $< A_y(t) > = < A_x(t) >$ at any time. This expresses the fact that since the particles are identical, any permutation of them does not lead to any observable effect. This conclusion is in agreement with the postulate of indistinguishability [12-15].

The occupation number of the states and the symmetries properties of the quantum energy eigenstates $Y(\alpha)$ associated with the intermediate Young shapes are completely different from the vertical (fermionic) and horizontal (bosonic) shapes. This lead us to propose the following statement which is taken as a principle (*Statistical Principle)*: "Bosons, fermions and gentileons are represented by horizontal, vertical and intermediate Young tableaux, respectively".

6. Summary and conclusions

We have shown that besides bosons and fermions it can exist mathematically a new kind of particles, named gentileons. Our theoretical analysis was done didactically using the basic group theory adopted in the graduate physics course.

Using the permutation group theory we studied in details the trivial case of systems formed by 2 particles and the simplest but non trivial case of systems formed by 3 particles. For the general N-particles systems only a brief review of the main results obtained in preceding papers [1-8] have been presented.

According to the amazing mathematical properties [2-8] of the intermediate representations of the permutation group theory the gentilionic systems cannot coalesce. Gentileons are always confined in these systems and cannot appear as free particles.

Based on these exotic properties we have conjectured [2-8] that quarks could be gentileons since we could explain, from first principles, quark confinement and conservation of the baryonic number.

Let us suppose that only bosons and fermions could exist in nature. In this case there remains the problem to discover the selection rules which forbid the existence of gentileons.

Finally, we must note that besides the "gentileons" there are other particles that do not obey the Fermi or Boson statistics, predicted by different theoretical approaches. We would like to mention first the "parabosons" and "parafermions" predicted by Green [3, 24]. A detailed analysis of the Green parastatistics can be seen, for instance, in the book of Ohnuki and Kamefuchi [25]. Besides the parabosons and parafermions we have, for instance, the "anyons" [26], and the "exclusons" [27]. The "anyons" and "exclusions" result from the interaction of the original bosons and fermions and could not be found asymptotically free in the nature, like the gentileons. Recently, according to Camino [28] the existence of anyons ("Laughlin particle" with fractionary charge) has been confirmed in the context of the fractional quantum Hall effect. More information about this experimental confirmation is given by Lindley [29].

Appendix A - Representations of the S_N group in the configuration space $\varepsilon^{(N)}$ and in the Hilbert space $L_2(\varepsilon^{(N)})$

We give here the basic ideas [16], concerning the representations of the S_N group in the configuration space $\varepsilon^{(N)}$. More detailed and complete analysis about this subject can be found in many books [9-11, 17-19].

If we can set up a homomorphic mapping

$$P_i : D^{(\mu)}(P_i) \tag{A.1}$$

between the elements $P_1,\ P_2, \ldots,\ P_n$ of the group S_N and a set of square $(\mu \times \mu)$ matrices $D^{(\mu)}(P_1)$, $D^{(\mu)}(P_2), \ldots,\ D^{(\mu)}(P_n)$ $(n = N!)$ such that

$$D^{(\mu)}(P_i)D^{(\mu)}(P_j) = D^{(\mu)}(P_iP_j), \qquad (A.2)$$

then the matrices $D^{(\mu)}(P_1)$, $D^{(\mu)}(P_2)$,..., $D^{(\mu)}(P_n)$ are said to be a μ-dimensional matrix representation of the group S_N in the configuration space $\varepsilon^{(N)}$. If the homomorphic mapping of S_N on $D(P_i)$ reduces to an isomorphism the representation is said to be faithful.

In general all matrices $D^{(\mu)}(P_i)$ of a μ-dimensional representation can be brought simultaneously to the form

$$D^{(\mu)}(P_i) = \begin{pmatrix} D^{(k)}(P_i) & A(P_i) \\ 0 & D^{(m)}(P_i) \end{pmatrix} \qquad (A.3)$$

where $D^{(k)}(P_i)$ and $D^{(m)}(P_i)$ are diagonal blocks with $k + m = \mu$. When, by a similarity transformation, all matrices $D^{(\mu)}(P_i)$ can be put in a diagonal form, that is, when $A(P_i) = 0$, the representation is named *reducible*. If the matrices cannot be written in a diagonal block structure the representation is said to be *irreducible*.

Let us consider, for instance, the simplest but non trivial case of the permutation group S_3 and define $P_1 = I =$ identity $= (123)$, $P_2 = (213)$, $P_3 = (132)$, $P_4 = (321)$, $P_5 = (312)$ and $P_6 = (231)$. We can show [16], that the S_3 has two 1-dimensional irreducible representations ($D_1^{(1)}$ and $D_2^{(1)}$) and only one 2-dimensional irreducible representation ($D^{(2)}(P_i)$).

For the two 1-dimensional representations the matrices $D^{(1)}(P_i)$ are given by

$$D_1^{(1)}(P_i) = 1 \quad (i = 1, 2, \ldots, 6); \qquad (A.4)$$

$$D_2^{(1)}(P_i) = 1 \quad (i = 1, 5 \text{ and } 6) \text{ and}$$

$$D_2^{(1)}(P_i) = -1 \quad (i = 2, 3 \text{ and } 4), \qquad (A.5)$$

which are homomorphic representations.

For the 2-dimensional representation the matrices $D^{(2)}(P_i)$ are given by

$$D^{(2)}(P_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$D^{(2)}(P_2) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$

$$D^{(2)}(P_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (A.6)$$

$$D^{(2)}(P_4) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

$$D^{(2)}(P_5) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

and

$$D^{(2)}(P_6) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$

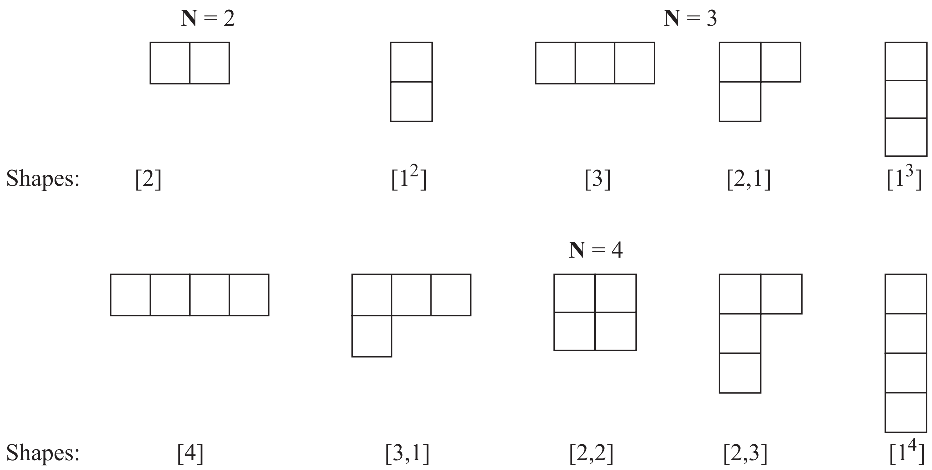
which is a faithful representation. Since the matrices shown in Eq. (6) are all orthogonal this irreducible representation is called *orthogonal*.

There are an infinite number of representations of a given group. We have obtained above the irreducible representations of the S_3 using the multiplication properties of the permutations P_i. Other two irreducible representations of S_3 can be obtained, for instance, taking into account (1)rotations of vectors in a 3d Euclidean space and (2)rotations of an equilateral triangle in the (x, y) plane [18].

A.1. Determination of the S_N representations by the Young tableaux

In the general case the determination of the S_N representations is performed by using more powerful methods developed by Young and Frobenius [9-11, 17-19]. They consider the *substitutional expression* $\Pi = a_1 P_1 + a_2 P_2 + \ldots + a_n P_n$, where $P_1, P_2, \ldots, P_n$ are the n distinct permutations of the S_N and $a_1, a_2, \ldots, a_n$ are numerical coefficients, and take into account the partitions of number N. Any partition of the number N denoted by $[\alpha_1, \alpha_2, \ldots, \alpha_k]$, where $\alpha_1 + \alpha_2 + \ldots + \alpha_k = N$, with $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_k$ will be represented simply by (α), when no confusion is likely to arise. The first work [17], using this approach have been done, around 1900, independently by Frobenius and by Young, that was a country clergyman. To each partition (α) of N is constructed a shape, named Young tableaux, denoted by α, having α_1 spaces in the first row, α_2 in the second row and so on [9-11, 17-19]. By the shape we mean the empty box, *i.e.*, the contour without the numbers. We show below all possible shapes associated with $N = 2$, 3 and 4 particles.

The N numbers 1, 2,..., N are arranged in the spaces of the shape α in $N! = n$ ways. Each such arrangement is called a *tableau* T and there are $N!$ *tableaux* with the same shape. The tableau T, for a given shape, is called *standard tableau* if the numbers increase in every row of T from left to right and in every column of T downwards.



The tableaux are constructed as follows: insert the numbers 1, 2, 3,. . . , N into the shape in any order to give a Young tableau. Once the tableau has been fixed, we consider two types of permutations [9]. Horizontal permutations p are permutations which interchange only numbers in the same row. Vertical permutations q interchange only numbers in the same column. Hence, we define the Young operator by $YO = PQ$ where the quantities P and Q are given by

$$P = \Sigma_p \; p \quad (\text{“symmetrizer”})$$

and

$$Q = \Sigma_q \; \delta_q q \quad (\text{“antisymmetrizer”}), \tag{A.7}$$

where the sums are over the horizontal and vertical permutations and δ_q is the parity of the vertical permutation q. The tableaux are obtained by the application of the Young operators on the initial standard tableau.

Note that if the arranged numbers increase in every row of T from left to right and in every column of T downwards, the tableau, for a given shape, is called *standard tableau.*

Let us indicate by T_1^α, T_2^α,. . . , T_n^α the different *tableaux* of the same shape α generated by the permutations defined by the operator YO. Any permutation applied to a tableau of shape α will produce another tableau of the same shape α.

Denoting by P_{ik}^α the permutations which changes T_k^α into T_i^α, we have $T_i^\alpha = P_{ik}^\alpha \; T_k^\alpha$. The matrices D_{ik} of an irreducible representation of degree f of S_N is calculated from the formula [10]

$$e_{ii} P e_{kk} = D_{ik} e_{ik},$$

where e_{ik} (i,k = 1, 2,..,f) are unit basis which satisfies the equations $e_{ij}e_{jk} = e_{ik}$ and $e_{ij}e_{hk} = 0$ ($h \neq j$). The parameter f, named *degree* of the irreducible representation, gives the dimension of the irreducible matrices.

The elements D_{ik} of the ($f \times f$) irreducible matrices can be determined adopting three different units e_{ik}: (1) *natural*, (2) *semi-normal* and (3) *orthogonal.* Note that the values found for the D_{ik} components depend on the choice of the unit basis [9-11, 17-19]. Of course these three irreducible representations are equivalent.

Let us present a brief review of the fundamental properties of the irreducible representations of the S_N in the configuration space $\varepsilon^{(N)}$:

(1) To each partition (α) there is an irreducible representation described by square matrices $D_{ik}^{(\alpha)}$ with f(α) dimension. So, the *tableaux* with the same *shape* (α) have equivalent representations and the different *shapes* cannot have equivalent representations. There is a one-to-one correspondence between each shape (α) and the irreducible matrices $D_{ik}^{(\alpha)}$.

(2) The dimensions $f(\alpha)$ of the irreducible square matrices assume the values 1^2, 2^2,. . . , $(N - 1)^2$.

(3) There are only two 1-dimensional irreducible representations given by the partitions (α) = [N] and (α) = [1^N]. The first case is described by horizontal *shape* with N spaces. In the second case we have a vertical *shape* with N rows. The remaining representations have dimensions going from 2^2 up to $(N - 1)^2$ and are described by the various shapes occupied by 3, 4,. . . , N particles, respectively [9-11, 17-19].

A.2. Systems with $N = 2$ and $N = 3$ particles: determination of the basis functions of their irreducible representations, their energy eigenvalues and their irreducible representations in the configuration and in the Hilbert spaces

We will show how to determine the irreducible representations for the trivial case $N = 2$ and the simplest but non-trivial case of $N = 3$ using the Young operators. This is done constructing the basis functions of the irreducible representations [11, 19]. using orthogonal unit basis. We will take as the unit basis the $n = N!$ degenerate energy orthonormal eigenfunctions $\{\Psi_i\}_{i=1,2..,n}$

which span an n-dimensional Hilbert space $L_2(\varepsilon^{(N)})$.

We will divide the process used to determine [11, 19] the irreducible representations in three parts **a**, **b** and **c**.

A.3. Construction of the Young operators

Following the recipes to construct the Young operators $Y = PQ$, defined by the Eq. (A.7) we obtain the following operators Y, associated with respective shapes [11, 19]:

$N = 2$

Shape [2]:

$$YO[2] = \frac{[I + P(1,2)]}{2}. \tag{A.8}$$

Shape [1^2]:

$$YO[1^2] = \frac{[I - P(1,2)]}{2}.$$

$N = 3$

Shape[3]:

$$YO[3] = \frac{[\sum_i P_i]}{6} = \frac{[I + P(132) + P(213) + P(231) + P(312) + P(321)]}{6}.$$

Shape [1^3]:

$$YO[1^3] := \frac{[\sum_i \delta_i P_i]}{6} = \frac{[I - P(132) - P(213) + P(231) + P(312) - P(321)]}{6}.$$

Shape [2,1]:

$$YO_{11}[2,1] = \frac{[I + P(213) - P(231) - P(321)]}{\sqrt{4}},$$

$$YO_{12}[2,1] = \frac{[P(132) - P(213) + P(231)/2 - P(312)]}{\sqrt{4}},$$

$$YO_{21}[2,1] = \frac{[P(132) - P(231) + P(312) - P(321)]}{\sqrt{4}},$$

$$YO_{22}[2,1] = \frac{[I - P(213) - P(312) + P(321)]}{\sqrt{4}}. \tag{A.9}$$

Let us indicate by $e_1 = \Psi(1,\ 2)$ and $e_2 = P(1,\ 2)\Psi(1,\ 2)$ the unit vector basis of the 2-dimension Hilbert space $L_2(\varepsilon^{(2)})$. Similarly, by $e_1 = \Psi(1,\ 2{,}3)$, $e_2 = \Psi(1{,}3{,}2)$, $e_3 = \Psi(2{,}1{,}3)$, $e_4 = \Psi(2{,}3{,}1)$, $e_5 = \Psi(3{,}1,\ 2)$ and $e_6 = \Psi(3{,}2{,}1)$ the unit vector basis of the 6-dimension Hilbert space $L_2(\varepsilon^{(3)})$ obtained by the permutations $P_i e_1$ ($i = 1$, 2,..,6).

A.4. Construction of the basis functions and the energy eigenfunctions

To construct the *basis functions for the various irreducible representations* [9, 19] of the S_2 and S_3 we apply the Young operators YO defined by Eqs. (A.8) and (A.9) to the function $\Psi = \Psi(1,\ 2)$ and $\Psi(1,\ 2{,}3)$ respectively. In these conditions we obtain:

For $N = 2$ the completely symmetric ϕ_s and antisymmetric ϕ_a normalized eigenfunctions of the two 1d subspaces are written as

$$\phi_s = \frac{(e_1 + e_2)}{\sqrt{2}} \quad \text{and} \quad \phi_a = \frac{(e_1 -_2)}{\sqrt{2}}. \tag{A.10}$$

For $N = 3$ we have the following eigenvectors:

Shape [3]:

$$\phi_s = \frac{(e_1 + e_2 + e_3 + e_4 + e_5 + e_6)}{\sqrt{6}}. \tag{A.11}$$

Shape [1^3]:

$$\phi_a = \frac{(e_1 - e_2 - e_3 - e_4 + e_5 + e_6)}{\sqrt{6}}.$$

Shape [2,1]:

$$Y_{11} = \frac{(e_1 + e_3 - e_4 - e_6)}{\sqrt{4}},$$

$$Y_{12} = \frac{(e_2 - e_3 + e_4 - e_5)}{\sqrt{4}}, \tag{A.12a}$$

$$Y_{21} = \frac{(e_2 - e_4 + e_5 - e_6)}{\sqrt{4}},$$

$$Y_{22} = \frac{(e_1 - e_3 - e_5 + e_6)}{\sqrt{4}}.$$

For $N = 3$ the unit vector basis $\{e_i\}_{i=1,2...,6}$ spans a 6-dimensional Hilbert space which is composed by two 1-dimensional subspaces, $h([3])$ and $h([1^3])$, and one 4-dimensional subspace $h([2{,}1])$. Since the functions $Y_{rs}(r{,}s = 1,\ 2,\ 3,\ 4)$ form a set of linearly independent functions in $h([2{,}1])$ we can construct by an orthonormalization process the base-vectors $\{Y_i\}_{i=1,..,4}$ of the subspace $h([2{,}1])$ that are given by

$$Y_1 = \frac{(e_1 + e_3 - e_4 - e_6)}{\sqrt{4}},$$

$$Y_2 = \frac{(e_1 + 2e_2 - e_3 + e_4 - 2e_5 - e_6)}{\sqrt{12}}, \tag{A.12b}$$

$$Y_3 = \frac{(-e_1 + 2e_2 - e_3 - e_4 + 2e_5 - e_6)}{\sqrt{12}},$$

$$Y_4 = \frac{(e_1 - e_3 - e_4 + e_6)}{\sqrt{4}}.$$

In these conditions the subspace $h([2{,}1])$ is spanned by the orthonormal vectors $\{Y_i\}_{i=1,2,...,4}$. and the eigenstate $Y([2,1])$ associated to this subspace is written as

$$Y([2,1]) = \frac{1}{\sqrt{4}} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} Y_+ \\ Y_- \end{pmatrix}, \quad \text{(A.13)}$$

where functions Y_+ and Y_- are defined by

$$Y_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \text{ and } Y_- = \frac{1}{\sqrt{2}} \begin{pmatrix} Y_3 \\ Y_4 \end{pmatrix}. \quad \text{(A.14)}$$

As can be easily verified, the functions ϕ_s, ϕ_a and $\{Y_i\}_{i=1,\dots,4}$ are orthonormal, that is, $< f_n|f_m > = \delta_{nm}$, where n, $m = s$, a, 1, 2, 3 and 4. From these orthonormal properties we can easily verify that

$$| < Y|Y > |^2 = \frac{(|Y_1|^2 + |Y_2|^2 + |Y_3|^2 + |Y_4|^2)}{4},$$

and that

$$| < Y_+|Y_+ > |^2 = \frac{(|Y_1|^2 + |Y_2|^2)}{2} = \frac{(|Y_3|^2 + |Y_4|^2)}{2} = | < Y_-|Y_- > |^2.$$

From the Eqs. (A.13)-(A.14) we see that the 4d subspace $h([2,1])$, which corresponds to the *intermediate Young shape* [2,1], breaks up into two 2d subspaces, $h_+([2,1])$ and $h_-([2,1])$, that are spanned by the basis vectors $\{Y_1, Y_2\}$ and $\{Y_3, Y_4\}$, respectively. To these subspaces are associated the wavefunctions $Y_+([2,1])$ and $Y_-([2,1])$ defined by Eq. (A.14). There is no linear transformation S which connects the vectors Y_+ and Y_-.

Note that the above functions ϕ_s and ϕ_a defined by Eqs. (A.10) are the energy eigenfunctions for the system with $N = 2$ particles. Similarly, the functions ϕ_s, ϕ_a and $\{Y_i\}_{i=1,\dots,4}$ seen in the Eqs. (A.11)-(A.14) are the energy eigenfunctions of the system with $N = 3$ particles.

A.5. Calculation of irreducible representations of the S_2 and S_3 groups

Finally, to calculate the *irreducible representations of the S_2 and S_3 groups* associated with the corresponding shapes it is necessary to apply the permutation operators P_i to the energy wavefunctions given by the Eqs. (A.10) and (A.11) and Eqs. (A.13) and (A.14).

$N = 2$ and 3:

Horizontal shapes [2] and [3]: $P_i\ \phi_s = (+1)\ \phi_s$, that is, $D[2] = D[3] = +1$.

Vertical shapes $[1^2]$ and $[1^3]$: $P_i\ \phi_a = (\pm 1)\ \phi_a$, that is, $D[1^2] = D[1^3] = \pm 1$,
showing that all the irreducible representations are 1-dimensional. To the shapes [2] and [3] are associated the matrix $D^{(1)} = 1$. To the shapes $[1^2]$ and $[1^3]$ are associated the matrix $D^{(1)} = \pm 1$.

$N = 3$, intermediate shape [2,1].

Applying the permutation operators P_i to Y_+ and Y defined by the Eqs. (A.14) and taking into account that $P_i\ e_j = e_m$, where i, j, m = 1, 2,3,..., 6, we can show that

$$\begin{aligned} P(123)Y_\pm &= P_1 Y_\pm = I Y_\pm, \\ P(132)Y_\pm &= P_2 Y_\pm = D^{(2)(}P_2)Y_\pm, \\ P(213)Y_\pm &= P_3 Y_\pm = D^{(2)(}P_3)Y_\pm, \\ P(321)Y_\pm &= P_4 Y_\pm = D^{(2)(}P_4)Y_\pm, \\ P(231)Y_\pm &= P_6 Y_\pm = D^{(2)(}P_6)Y_\pm, \\ P(312)Y_\pm &= P_5 Y_\pm = D^{(2)(}P_5)Y_\pm, \end{aligned} \quad \text{(A.15)}$$

where $D^{(2)}(P_i)$ (i = 1, 2,..., 6) are the same 2 × 2 matrices of the 2-dimensional irreducible representation of the S_3 given by Eq. (A.6). This implies that the 4 × 4 representation matrices associated with the shape [2,1] are broken into 2 × 2 irreducible matrices $D^{(2)}(P_i)$. These irreducible representations are equivalent. In this way the 4 × 4 representation matrices in the 4-dimensional subspace $h([2,1])$ can be written as the direct sum of two 2 × 2 equal irreducible matrices.

As pointed out above, adopting the particular unit vector basis $\{\Psi_i\}_{i=1,2,\dots,6}$ which are eigenvalues of the Hamiltonian H we have simultaneously determined the irreducible representations of the S_3 in configuration space $\varepsilon^{(3)}$ and in the Hilbert space $L_2(\varepsilon^{(3)})$ and constructed the eigenfunctions ϕ_s, ϕ_a, Y_+ and Y_- of the energy operator H. The 6d Hilbert space $L_2(\varepsilon^{(3)})$ which is spanned by the basis vectors $\{\Psi_i\}_{i=1,2\dots,6}$ is formed by tree subspaces $h(\alpha)$. Two of them, $h([3])$ and $h([1^3])$, are 1-dimensional. The 4d subspace $h[(2,1])$ which is spanned by the unit basis vectors $\{Y_i\}_{i\ =\ 1,\dots,4}$ is composed by two 2d subspaces, $h_+([2,1])$ and $h_-([2,1])$, spanned by the unit vectors $\{Y_1, Y_2\}$ and $\{Y_3, Y_4\}$, respectively.

Appendix B

Permutations in the $\varepsilon^{(3)}$ and the rotations of an equilateral triangle in an Euclidean space E_3

It will be shown in this Appendix that the permutations operations P_i on the state $Y([2,1])$ can be interpreted as rotations of an equilateral triangle in the Euclidean space E_3. To show this we will assume that in the E_3 the states u, v and w can occupy the vertices of an equilateral triangle taken in the plane (x, z) plane, as seen in Fig. 1. The unit vectors along the x, y and z axes are indicated by **i**, **j** and **k**. In Fig. 1 the unit vectors $\mathbf{m}_4$, $\mathbf{m}_5$ and $\mathbf{m}_6$ are given by $\mathbf{m}_4$ = -**k**, $\mathbf{m}_5$ = -($\sqrt{3}/2$) **i** +(1/2) **k** and $\mathbf{m}_6$ = ($\sqrt{3}/2$) **i** +(1/2) **k**, respectively.

We represent by $Y(123)$ the initial state whose particles 1, 2 and 3 occupy the vertices u, v and w, respectively. As is shown in details elsewhere [5, 7] the irreducible matrices $D^{(2)}(P_i)$ associated with the permutations $P_iY = D^{(2)}(P_i)\ Y$ can be represented by unitary operators $U = \exp[i\ \mathbf{j}.\vec{\sigma}(\theta/2)]$ and $V = i\ \exp[i\ \mathbf{m}_i.\vec{\sigma}(\phi/2)]$; $\theta = \pm\ 2\pi/3$ are rotations angles around the unit vector $\mathbf{j}$, $\phi = \pm\ \pi$ are rotations angles around the unit vectors $\mathbf{m}_4$, $\mathbf{m}_5$ and $\mathbf{m}_6$ and $\vec{\sigma}$ are the Pauli matrices.

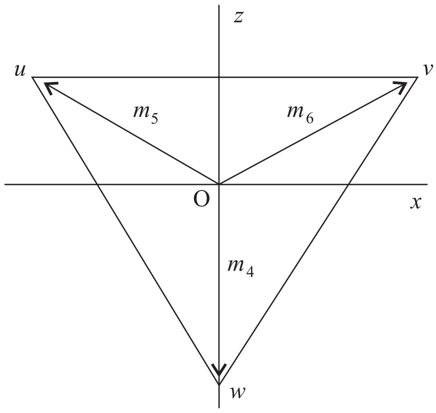


Figure 1 - The equilateral triangle in the Euclidean space (x, y, z) with vertices occupied by the states u, v and w.

From these results we see that: (a) the eigenvectors Y([2,1]] are spinors and (b) the permutation operators P_i in $\varepsilon^{(3)}$ are represented by linear unitary operators, U and V, in the Hilbert space $L_2(\varepsilon^{(3)})$.

According to a preceding paper [3], we have called AS_3 the algebra of the symmetric group S_3 .This algebra is spanned by 6 vectors, the irreducible matrices $\{D^{(2)}(P_i)\}_{i=1,2,..,6}$ that before Ref. [3], have been indicated by $\{\eta_i\}_{i=1,2,..,6}$. It was shown that associated to this algebra there is an algebraic invariant $K_{inv} = \eta_4 + \eta_5 + \eta_6 = (\mathbf{m}_4 + \mathbf{m}_5 + \mathbf{m}_6).\ \vec{\sigma} = 0$. From this equality results that K_{inv} can be represented geometrically in the (x, z) plane by the vector $\mathbf{M}$ identically equal to zero $\mathbf{M} = \mathbf{m}_4 + \mathbf{m}_5 + \mathbf{m}_6 = 0$. Usually, for continuous groups, we define the *Casimir invariants* which commute with all of the generators (in our case the generators are η_4 and η_6) and are, therefore, invariants under all group transformations. These simultaneously diagonalized invariants are the conserved quantum operators associated with the symmetry group. In our discrete case we use the same idea. So, the operator K_{inv} which corresponds to the genuine gentilionic representation of the AS_3 is identified with a quantum operator which gives a new conserved quantum number related to the S_3. Assuming that quarks are gentileons [3-8] and that the states u, v and w are the three $SU(3)$ color states we have interpreted the constant of motion $K_{inv} = 0$ as a color charge conservation which would imply consequently in *quark* confinement. In this case the AS_3 Casimir $K_{inv} = 0$ was called color Casimir.

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